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Cohen Reals and the Sequential Order of Groups

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Cohen reals and the sequential order of groups

Alexander Shibakov

2017

Sequential spaces

A metric space generalization.

Definition: Fréchet spaces

A space X is called *Fréchet* if for any $x \in \overline{A} \subseteq X$ there is a sequence $S \subseteq A$ such that $S \to x$.

Even more general definition.

Definition: sequential spaces

A space X is called *sequential* if for every $A \subseteq X$ such that $\overline{A} \neq A$ there is a $C \subseteq A$ such that $C \to x \notin A$.

Sequential closure and sequential order

Definition: sequential closure

Let $A \subseteq X$. Define *the sequential closure* $[A]' =$ "limits of convergent sequences of points of A". Now put $[A]_{\alpha+1} = [[A]_\alpha]'$ and $[A]_\alpha = \cup\{A_\beta : \beta < \alpha\}$ for limit α.

This leads to a useful ordinal invariant.

Definition: sequential order

Define the *sequential order* $\mathrm{so}(X)$ as the smallest $\alpha \leq \omega_1$ such that $[A]_\alpha = \overline{A}$ for every $A \subseteq X$.

Sequential spaces of sequential order ≤ 1 are exactly Fréchet spaces.

Building blocks

A convenient definition.

Definition: topologies determined by families

Let X be a space and $\mathcal{P}$ be a family of its subspaces. The topology of X is said to be *determined* by $\mathcal{P}$ if $U \subseteq X$ is open if and only if $U \cap P$ is open in P for every $P \in \mathcal{P}$.

A useful class of spaces.

Definition: k_ω-spaces

Let the topology of X be determined by a family $\mathcal{P}$ of its subspaces. If every element of $\mathcal{P}$ is compact and $\mathcal{P}$ is countable then X is called *a k_ω-space*.

Sequential spaces: simple examples

Sequential order: simple details

For a sequential X its sequential order $\mathrm{so}(X)$ is always defined and is always $\leq \omega_1$. Spaces of arbitrary sequential order exist in ZFC.

Arens' space has $\mathrm{so}(S_2) = 2$ (so is not Fréchet), S_n generalizes Arens' space and $\mathrm{so}(S_n) = n$, *Arkhangel'skii-Franklin space* S_ω is a homogeneous space of sequential order ω_1. The *sequential fan* $S(\omega)$ is Fréchet but is not first countable. All of the above are k_ω-spaces.

Sequential spaces: compact case

Compactness makes everything more complicated ....

Theorem

There exists a compact space of sequential order 2 (Mrówka's space). If MA holds, there exists a compact space of sequential order 4 (A. Dow).

But sometimes CH saves the day ....

Theorem (Bashkirov (CH))

There exists a compact space of every sequential order $\alpha \leq \omega_1$.

Fréchet groups

Nonmetrizable Fréchet group

A Σ-product of uncountably many circles is a countably compact nonmetrizable Fréchet group.

Nonmetrizable *countable* Fréchet groups

… exist under various set-theoretic assumptions (MA, adding ω_1 Cohen reals, $\mathfrak{p} > \omega_1$, …).

However,

(Hrušak, Ramos-García) Nonmetrizable countable Fréchet groups

… do not have to exist.

Sequential groups

(CH) A countable group G such that $\mathrm{so}(G) = \alpha$

... exists for every $\alpha < \omega_1$.

Other models

It is consistent that there are no sequential groups G such that $1 < \mathrm{so}(G) < \omega_1$.

Is convergence a purely countable phenomenon?

Question

Is it consistent that there are no countable sequential groups of intermediate sequential orders but there exists an uncountable group with this property?

Small sequential groups in Cohen extensions

Let $\mathbb{P}$ be the poset that adds ω_2 Cohen reals.

Theorem (CH)

in $V[\mathbb{P}]$ there are no countable sequential groups of intermediate sequential order.

Sketch of proof: Let M be an ω-closed elementary submodel of size ω_1. Let $\dot{G}$ be a name of a counterexample. Then $\dot{G}[M \cap \mathbb{P}]$ is a countable sequential group of intermediate sequential order in $V[M \cap \mathbb{P}]$. Its topology is a π-base of G. But G contains a copy of $S(\omega)$ so its character is ω_2. Contradiction.

Larger sequential groups in Cohen extensions

Theorem ($\diamondsuit$)

There exists a sequential group of sequential order 2 that stays a sequential group of order 2 in $V[\mathbb{P}]$.

The construction starts with the following result of K. Kunen

Theorem (K. Kunen (CH))

There exists a MAD family $\mathcal{K}$ on ω that remains MAD in $V[\mathbb{P}]$ (i.e. $\mathcal{K}$ is Cohen-indestructible*).*

Then proceeds by an induction on ω_1.

Simplifications: boolean and co-countable groups

Definition: boolean groups

An abelian group G is called *boolean* if $a + a = 0$ for every $a \in G$. Every boolean group can be naturally viewed as as vector space over the two element field $\mathbb{F}_2$.

To show that a topology on a boolean group is a group topology it is enough to show that the addition is continuous.

Definition: co-countable groups

An uncountable group G is called *co-countable* if every quotient of G that has a countable pseudocharacter is countable.

The 'precursor' group

Let ϕ be the one point compactification of the Mrówka space of $\mathcal{K}$. Start by embedding ϕ in 2^{ω_1}. Let G be the group generated by the copy of ϕ. Equip G with the finest group topology in which ϕ remains compact.

Lemma

In the topology above G is a co-countable sequential k_ω-group. Every compact subspace of G has a sequential order ≤ 2.

The following upper bound on the sequential order holds.

Lemma

In any topology coarser than the topology above G will have a sequential order ≥ 2 in $V[\mathbb{P}]$.

Inductive step: intrinsic k_ω-topology

An intrinsic version of a folklore lemma.

Lemma: topologies from compacts

Let G be a group, and $\langle F_n \mid n \in \omega \rangle$ be a cover of G. Suppose each F_n is given a compact Hausdorff topology such that for any $i, j \in \omega$ the set $F_i \cap F_j$ is closed in both F_i and F_j and the induced topologies are the same. Suppose further that the sums, inverses, and unions of any finite number of F_i's are contained in some (possibly different) F_n's and that the addition and algebraic inverse maps restricted to the corresponding (products of) compacts are continuous. Then the topology τ determined by $\langle F_n \mid n \in \omega \rangle$ on G is a T_2 group topology.

Inductive step: adding a sequence

Making some sequences converge.

Lemma: adding a sequence

Let G be an abelian k_ω group and $D \subseteq G$ be an infinite closed discrete subset of G. Let $a \in G$. Then there exists an infinite subset $C \subseteq D$ such that the finest group topology on G which is coarser than the original topology on G and such that $C \to a$ is a k_ω Hausdorff topology on G.

Now let $\{ C_\alpha \mid \alpha < \omega_1 \}$ be a $\diamondsuit$ sequence. View C_α as a code for an $\mathrm{Fn}(\omega_1, 2)$ name of a countable subset of G.

Remarks: other sequential orders

According to A. Dow, the compact spaces in the well known series constructed by A. Bashkirov can be made Cohen-indestructible. The group algebraically generated by such a compact space is co-countable since every quotient of countable pseudocharacter will be covered by countably many scattered metrizable compact subspaces. Thus the construction proceeds virtually unchanged to result in a sequential group of any sequential order $\alpha < \omega_1$. The group thus constructed has the property that $\overline{A} = \cup\{\, \overline{A \cap K} \mid K \in \mathcal{K}\,\}$ for any $A \subseteq G$ thus making τ "Fréchet (mod $\mathcal{K}$)" (see Kannan's book for a general discussion of ordinal invariants defined in this manner).

Questions

Question

Does there exist a sequential group $G \in V[\mathbb{G}]$ of sequential order 2 such that for any family $\mathcal{K}$ of compact subsets of G there exists a subset $A \subseteq G$ such that $\overline{A} \neq \cup\{\, \overline{A \cap K} \mid K \in \mathcal{K}\,\}$?

Additionally, it would be interesting to know if the intermediate sequential order "reflects" to a group of size ω_1 in $V[\mathbb{G}]$.

Question

Let $G \in V[\mathbb{G}]$ be a sequential group of intermediate sequential order. Does there exists a sequential group $G' \subseteq G$ of intermediate sequential order of size ω_1?

Questions

Question

Is it consistent that there exists a (countable) sequential group of any sequential order $\neq 2$ but no sequential group of sequential order 2?

Question

Is it consistent that there exists a countable non metrizable Fréchet group but not a boolean one?