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On the Chogoshvili Homology Theory of Continuous Maps of Compact Spaces

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Abstract

V. Baladze and A. Beridze investigated the $\check{C}$ech spectral and Chogoshvili projective homology groups of compact Hausdorff spaces. They axiomatically characterized these theories using the axioms of Hu and as well as continuity and partially continuity axioms, respectively [Bl-Br].

In this report we investigate the following

Problem (V. Baladze, A. Beridze): Construct and find an axiomatic characterization of $\check{C}$ech spectral and Chogoshvili projective homology groups on the category of maps of compact Hausdorff spaces $\mathbf{Mor_C}$.

Abstract

An exact homology functor from the category $\mathbf{Mor_C}$ of continuous maps of compact Hausdorff spaces to the category **LES** of long exact sequences of abelian groups is defined (cf. [Bal_1], [Bal_2], [Be-Tu]). This functor is extension of the homology theory in the Hu sense, which is uniquely defined on the category of finite CW complexes and is constructed without relative homology groups [Hu].

1 2 3 4

[1] [Bal_1] Baladze V. The (co)shape and (co) homological properties of continuous maps, Math. Vestnik, Belgrad. 66, 3, 235-247, 2014

[2] [Bal_2] Baladze V. On the spectral (co)homology exact sequences of maps, Georgian Math. J. 19,No.4,1-12, 2012

[3] [Be-Tu]5 Beridze A and Turmanidze L. Semi-Continuity of Strong Homology Groups of Continuitys map. J. Math. Sci. Vol.211 (2015), 13-30

[4] [Hu] Hu, S.-T., On axiomatic approach to homology theory without using the relative groups, Portugaliae mathematica 19(4), 211-225 (1960)

Abstract

For this aim we use the Chogoshvili construction of projective homology theory [Ch_1], [Ch_2]. For each continuous map $f : X \to Y$ of compact Hausdorff spaces, using the notion of the partition of spaces [Ch1], [Ch2] and approximations of continuous maps [Ba1], [Ba2], [Ba3], V. Baladze defined the inverse system $\mathbf{f} = \{f_\lambda, p_{\lambda\lambda'}, \Lambda\}$, where Λ is directed system of pairs $\lambda = (\alpha, \beta)$ of partitions, where α is refined in $f^{-1}(\beta)$ and $f_\lambda : X_\alpha \to Y_\beta$ is the simplicial map of nerves X_α and Y_β of closed coverings defined by partitions α and β.

5 6 7 8 9

[5] [Ch_1] Chogoshvili G. On the equivalence of functional and spcetral homology theories, Izv. Akad. Nauk SSR Ser. Mat. 15 (1951), 421-438

[6] [Ch_2] Chogoshvili G. On the homology of topological spaces, Mitt. Georg. Abt. Akad. Wiss. USSR 1 (1940), 337-340

[7] [Bal_1] Baladze V. The (co)shape and (co) homological properties of continuous maps, Math. Vestnik, Belgrad. 66, 3, 235-247, 2014

[8] [Bal_2] Baladze V. On the spectral (co)homology exact sequences of maps, Georgian Math. J. 19,No.4,1-12, 2012

[9] [Bl_3] Baladze V. Fiber shape theory, Proc. A. Razmadze Math. Inst.,

Abstract

Using the system $\mathbf{f} = \{f_\lambda, p_{\lambda\lambda'}, \Lambda\}$ we have defined inverse system of chain complexes $\mathbf{C}_*(\mathbf{f}) = \{C_*(f_\lambda), p^{\#}_{\lambda\lambda'}, \Lambda\}$ and the Chogoshvili projective and Čech spectral homology groups:

$$\bar{H}_*(f) \equiv H_*(\varprojlim\{C_*(f_\lambda), p^{\#}_{\lambda\lambda'}, \Lambda\})$$

$$\check{H}_*(f) \equiv \varprojlim(\{H_*(f_\lambda), p^{\#}_{\lambda\lambda'}, \Lambda\}).$$

A. Beridze shown that there exists the Milnor short exact sequence (cf. [Br-Tr], [In-Md], [Md]), which connects defined homology groups:

$$0 \to \varprojlim{}^1 H_{n+1}(f_\lambda) \to \bar{H}_*(f) \to \check{H}_*(f) \to 0.$$

10 11 12

[10] [Br-Tr] Beridze A and Turmanidze L. Semi-Continuity of Strong Homology Groups of Continuitys map. J. Math. Sci. Vol.211 (2015), 13-30

[11] [In-Mdz] Inasaridze, Kh. N.; Mdzinarishvili, L. D. On the connection between continuity and exactness in homology theory. (Russian) Soobshch. Akad. Nauk Gruzin. SSR 99 (1980), no. 2, 317–320.

[12] [Mdz]Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

Abstract

Using this property he have shown that the homology groups $\bar{H}_*(f)$ satisfies the universal coefficient formula:

$$0 \to Ext(\check{H}^{n+1}(f);G) \to \bar{H}_*(f) \to Hom(\check{H}^n(f);G) \to 0.$$

Consequently, using the methods developed in [Mdz], [Br-Md] and [In] we have shown that constructed functor is unique on the category Mor_C.

[13] [14] [15]

[13] [Mdz] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482.

[14] [Br-Mdz] Beridze A. and Mdzinarishvili L. On the Axiomatic Systems of Steenrod Homology Theory of Compact Spaces, arXiv:1703.05070

[15] [In] Inassaridze, Hvedri. On the Steenrod homology theory of compact spaces. Michigan Math. J. 38 (1991), no. 3, 323–338

Classical Homology Theories

- Singular Homology Theory

$$\mathrm{H}_* : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

- Chogoshvili (Steenrod, Strong, Total) Homology Theory

$$\bar{\mathrm{H}}_* : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

- Čech Homology Theory

$$\check{\mathrm{H}}_* : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

Singular homology theory

- It is homotopy invariant
- AD: It is an exact homology theory
- DIS: It is not continuous homology theory
- It satisfies only the week excision axiom

Advantage and Disadvantage

Čech homology theory

- AD: It is shape invariant
- DIS: It is not exact homology theory
- AD: It is a continuous homology theory
- AD: It satisfies the excision axiom

Advantage and Disadvantage

Chogoshvili (Steenrod, Strong, Total) homology theory

- AD: It is strong shape invariant
- AD: It is an exact homology theory
- It is a partially continuous homology theory
- AD: It satisfies the strong excision axiom

Homology Theories of Continuous Maps

- Singular Homology Theory of Continuous maps

$$\mathrm{H}_* : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

- Chogoshvili (Steenrod, Strong, Total) Homology Theory of Continuous maps

$$\bar{\mathrm{H}}_* : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

- Čech Homology Theory of Continuous maps

$$\check{\mathrm{H}}_* : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

Axioms of the Chogoshvili (Steenrod, Strong, Total) Homology Theory of Continuous Maps

- It should be homotopy invariant
- It should be an exact homology theory
- It should have the partially continuous property
- There is problem in formulating the excision axiom

Theorem (Mrozik):

A homology theory $\{H_*, \delta\}$ on the category $\mathbf{CM^2}$ is strong shape invariant if and only if it satifies the strong excision axiom (SE).
SE: for each compact metric pair (X, A) with $A \neq \emptyset$, the quotiont map

$$p : (X, A) \to (X/A, *)$$

induces an isomorphism:

$$p_* : H_*(X, A) \to H_*(X/A, *)$$

Axioms of the Chogoshvili (Steenrod, Strong, Total) Homology Theory of Continuous Maps Using the shape theory

- It should be strong shape invariant
- It should be an exact homology theory
- It should have the partially continuous property

Hu's Axioms of Homology theory (defined without using the relative groups)

Let $\mathscr{H} = \{H, *, \sigma\}$ be a collection [Hu], where:

- The first functor H assigns to each space X and an integer n, an abelian group $H_n(X)$;
- The second functor $*$ assigns to each map $f : X \to Y$ and an integer n, a homomorphism

$$f_* : H_n(X) \to H_n(Y);$$

- The third functor σ assigns to each space X and an integer n, an isomorphism

$$\sigma_* : H_n(X) \to H_{n+1}(SX),$$

where SX is suspension of X.

16

[16] [Hu] Hu, S.-T., On axiomatic approach to homology theory without using the relative groups, Portugaliae mathematica 19(4), 211-225 (1960)

Hu's Axiom of Homology

A homology functor $\mathscr{H} = \{H, *, \sigma\}$ is called homology theory in the Hu sense if the following is fulfilled:

- **Homotopy Axiom:** If the maps $f, g : X \to Y$ are homotopic in $\mathscr{H}$, then
$$f_* = g_*.$$

- **Exactness Axiom:** For each map $f : X \to Y$ in $\mathscr{H}$ and each integer n, the following sequence is exact:

$$H_n(X) \xrightarrow{\quad f_* \quad} H_n(Y) \xrightarrow{\; P(f)_* \;} H_n(C(f)),$$

where $C(f)$ is geometric cone of continuous map $f : X \to Y$ and $P(f) : Y \to C(f)$ is canonical inclusion;

Hu's Axiom of Homology

- **Suspension Axiom:** For each map $f : X \to Y$ in $\mathscr{H}$ and each integer n, commutative holds in the following rectangle:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(Y) \\ \downarrow \sigma & & \sigma \downarrow \\ H_{n+1}(S(X)) & \xrightarrow{S(f)_*} & H_{n+1}(S(Y)), \end{array}$$

- **Dimension Axiom:** $H_n(S^0) = 0$ for every $n \neq 0$.

Uniqueness theorem of homology theory of Hu sense

Theorem (see [Hu])

There exists one and only one exact homology theory on the category of finite CW complexes with coefficients in an abelian group G, which satisfies the axioms of homotopy, exactness, suspension and dimension.

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[17] Hu, S.T., On axiomatic approach to homology theory without using the relative groups, Portugaliae mathematica 19(4), 211-225 (1960)

Our aim is:

- to construct an extension $\bar{\mathscr{H}} = \{\bar{H}, *, \sigma\}$ homology theory of the Hu sense $\mathscr{H} = \{H, *, \sigma\}$ from the category $\mathbf{Mor_{CW}}$ to the category $\mathbf{Mor_{C}}$ of continuous map of compact space
- to formulate axioms without using relative homology groups and prove the uniqueness theorem.

Construction of Chogoshvili system

Let X be a compact Hausdorff space. Consider the direct set $Part(X)$ of all partitions $\alpha = \{F_1, F_2, \cdots, F_n\}$ of X [Ch1], [Ch2]. Let $f : X \to X^{'}$ be a continuous map of compact Hausdorff spaces and $\Lambda = Part(f)$ of all pairs $\lambda = (\alpha, \beta)$, where $\alpha \in Part(X)$, $\beta \in Part(X^{'})$ and $f^{-1}\beta \leq \alpha$. Note that $\lambda^{'} = (\alpha^{'}, \beta^{'})$ is refinement of $\lambda = (\alpha, \beta)$ iff $\alpha \leq \alpha^{'}$ and $\beta \leq \beta^{'}$.

18 19

[18] [Ch1] Chogoshvili G. On the equivalence of functional and spcetral homology theories, Izv. Akad. Nauk SSR Ser. Mat. 15 (1951), 421-438

[19] [Ch1] Chogoshvili G. On the homology of topological spaces, Mitt. Georg. Abt. Akad. Wiss. USSR 1 (1940), 337-340

Construction of Chogoshvili system

The set Λ of all pairs $\lambda = (\alpha, \beta)$ defines the Chogoshvili inverse system $CH(f) = \{f_\lambda, p_{\lambda,\lambda'}, \Lambda\}$ [Bl-Br], where:

- $f_\lambda : X_\alpha \to Y_\beta$ is the simplicial map of the geometric realizations X_α an Y_β of the nerves of $N(\alpha)$ and $N(\beta)$ of closures $\bar{\alpha}$ and $\bar{\beta}$ of partitions α and β, respectively.
- $p_{\lambda,\lambda'} : f_\lambda \to f_{\lambda'}$ is a pairs $p_{\alpha,\alpha'}$ and $p_{\beta,\beta'}$ the geometric realizations of the simplicial maps

$$\pi_{\alpha,\alpha'} : N(\alpha^{'}) \to N(\alpha) \quad \text{and} \quad \pi_{\beta,\beta'} : N(\beta^{'}) \to N(\beta),$$

respectively.

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[20] [Bl-Br] Baladze V. Fiber shape theory, Proc. A. Razmadze Math. Inst., 132(2003), 1-70.

Inverse sytem of chain Complex

Let $C_*(f_\lambda)$ be the chain cone of the chain map $f_\lambda^{\#} : C_*(X_\alpha) \to C_*(X_\beta)$ induced by $f_\lambda \in CH(f)$ and $p_{\lambda\lambda'}C_*(f_{\lambda'}) \to C_*(f_\lambda)$ be chain map induced by $p_{\lambda\lambda'} : f_\lambda \to f_\lambda$, respectively. For each $f \in \mathbf{Mor_C}$ consider the corresponding inverse system $\mathbf{C}(f) = \{C_*(f_\lambda), p^{\#}_{\lambda\lambda}, \Lambda)\}$ of chain complexes $C_*(f_\lambda)$.

Projective and Spectral homology groups

Definition

Projective homology group $\bar{H}_*(f)$ of continuous map $f \in \mathbf{Mor_C}$ is called the homology group of inverse limit of $\mathbf{C}(f)$:

$$\bar{H}^p_*(f) = H_*(\varprojlim\{C_*(f_\lambda, p^{\#}_{\lambda\lambda}, \Lambda)\})$$

Definition

Spectral homology group $\check{H}_*(f)$ of continuous map $f \in \mathbf{Mor_C}$ is called the inverse limit of inverse sytems of the homology groups $H_*(C_*(f_\lambda))$:

$$\check{H}_*(f) = \varprojlim\{H_*(C_*(f_\lambda), p^*_{\lambda\lambda}, \Lambda)\}$$

Homology groups of continuous map

Theorem (cf. [Hu]):

For each continuous maps $f : X \to Y$ of compact Hausdorff spaces there are an isomorphisms:

$$\bar{H}_n(f) \cong \bar{H}_n(C(f)),$$

$$\check{H}_n(f) \cong \check{H}_n(C(f)),$$

where $C(f)$ is the geometric cone of continuous map $f : X \to Y$.

Homology groups of continuous map

Theorem (cf. [Hu]):

For each continuous maps $f : X \to Y$ of compact Hausdorff spaces there are an isomorphisms:

$$\bar{\bar{H}}_n(f) \cong \bar{\bar{H}}_n(C(f)),$$

$$\check{H}_n(f) \cong \check{H}_n(C(f)),$$

where $C(f)$ is the geometric cone of continuous map $f : X \to Y$.

Corollary:

For each pair (X, A) of compact Housdorff spaces there are the an isomorphisms:

$$\bar{\bar{H}}_n(i) \cong \bar{\bar{H}}_n(X, A),$$

$$\check{H}_n(i) \cong \check{H}_n(X, A),$$

where $i : A \to X$ is the inclusion.

Suspension Axiom

Theorem (cf. [Hu]):

For each map $f : X \to Y$ in Mor_C and each integer n, there is an isomorpism

$$\sigma : \bar{H}_n(X) \to \bar{H}_{n+1}(S(X))$$

such the the following diagram is commutative:

$$\begin{array}{ccc}
\bar{H}_n(X) & \xrightarrow{f_*} & \bar{H}_n(Y) \\
\downarrow \sigma & & \sigma \downarrow \\
\bar{H}_{n+1}(S(X)) & \xrightarrow{S(f)_*} & \bar{H}_{q+1}(S(Y)),
\end{array}$$

Exactness Axiom

Theorem (cf.[Ba_2], [Be-Tu], [Hu])

For each continuous map $f : X \to Y$ of compact Hausdorff spaces there exists a long exact sequence $\mathbf{H}_*(f)$:

$$\cdots \longrightarrow \bar{H}_n(X) \longrightarrow \bar{H}_n(Y) \longrightarrow \bar{H}_n(f) \to \bar{H}_{n-1}(X) \longrightarrow \cdots$$

21 22 23

[21] [Bl_2] Baladze V. On the spectral (co)homology exact sequences of maps, Georgian Math. J. 19,No.4,1-12, 2012

[22] [Be-Tu] Beridze A and Turmanidze L. Semi-Continuity of Strong Homology Groups of Continuitys map. J. Math. Sci. Vol.211 (2015), 13-30

[23] [Hu] Hu, S.T., On axiomatic approach to homology theory without using the relative groups, Portugaliae mathematica 19(4), 211-225 (1960)

Induced homomorphism between long exact sequences

Theorem (cf. [Bl$_2$], [Be-Tu])

Any morhism $(\phi, \psi) : f \to f'$ from the category Mor_C induces morphism $(\phi, \psi)_* : \mathbf{H}_*(f) \to \mathbf{H}_*(f')$ of exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \bar{H}_n(X) & \longrightarrow & \bar{H}_n(Y) & \longrightarrow & \bar{H}_n(f) & \longrightarrow & \bar{H}_{n-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow \phi_* & & \downarrow \psi* & & \downarrow (\phi,\psi)_* & & \downarrow \phi_* & & \\
\cdots & \longrightarrow & \bar{H}_n(X') & \longrightarrow & \bar{H}_n(Y') & \longrightarrow & \bar{H}_n(f') & \longrightarrow & \bar{H}_{n-1}(X') & \longrightarrow & \cdots
\end{array}$$

24 25

[24] [B1$_2$] Baladze V. On the spectral (co)homology exact sequences of maps, Georgian Math. J. 19,No.4,1-12, 2012

[25] [be-tu] Beridze A and Turmanidze L. Semi-Continuity of Strong Homology Groups of Continuitys map. J. Math. Sci. Vol.211 (2015), 13-30

Homotopy axiom

Theorem (cf. [Bl$_2$], [Be-Tu]):

If two morhisms $(\phi_1, \psi_1), (\phi_2, \psi_2) : f \to f'$ are homotopic in the category Mor_C then they induce the same morphisms between of coresponding homological sequences:

$$(\phi_1, \psi_1)_* = (\phi_2, \psi_2)_* : \mathbf{H}_*(f) \to \mathbf{H}_*(f')$$

Homology theory of Hu sense defined on the category Mor_C

Theorem:

Restriction of Homology theory $\bar{\mathscr{H}} = \{\bar{H}, *, \sigma\}$ defined on the category Mor_C on the subcategory Mor_{CW} coincides the homology theory Hu sense $\mathscr{H} = \{H, *, \sigma\}$

Partial continuity

Theorem (cf. [Be-Md], [Be-Tu], [In-Md], [Md])

For each continuous map $f : X \to Y$ of compact Hausdorff spaces there exists a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 H_{n+1}(f_\lambda) \longrightarrow \bar{H}_n(f) \longrightarrow \varprojlim H_n(f_\lambda) \longrightarrow 0$$

where $\mathbf{f} = \{f_\lambda\} : \mathbf{X} \to \mathbf{Y}$ is a mapping of inverse system of Polyhedron spaces such that $X = \varprojlim \mathbf{X}$, $Y = \varprojlim \mathbf{Y}$ and $f = \varprojlim \mathbf{f}$.

26 27 28 29

[26] [Be-Md] Beridze A. and Mdzinarishvili L. On the Axiomatic Systems of Steenrod Homology Theory of Compact Spaces, arXiv:1703.05070

[27] [Be-Tu] Beridze A and Turmanidze L. Semi-Continuity of Strong Homology Groups of Continuitys map. J. Math. Sci. Vol.211 (2015), 13-30

[28] [In-Md] Inasaridze, Kh. N.; Mdzinarishvili, L. D. On the connection between continuity and exactness in homology theory. (Russian) Soobshch. Akad. Nauk Gruzin. SSR 99 (1980), no. 2, 317–320.

[29] [Mz] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41)

Universal coefficient formula

Theorem (cf. [Ber]):

For each continuous map $f : X \to Y$ of compact Hausdorff spaces there exists a short exact sequence

$$0 \longrightarrow Ext(H^{n+1}(f); G) \longrightarrow \bar{H}_n(f) \longrightarrow Hom(H^n(f); G) \longrightarrow 0$$

where $\mathbf{f} = \{f_\lambda\} : \mathbf{X} \to \mathbf{Y}$ is a mapping of inverse system of Polyhedron spaces such that $X = \varprojlim \mathbf{X}$, $Y = \varprojlim \mathbf{Y}$ and $f = \varprojlim \mathbf{f}$.

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[30] [Ber] Berikashvili N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Uniqueness theorem for the Cogoshvili homology theory on the category Mor_C

Theorem:

There exists one and only one exact homology theory on the category of finite Mor_C with coefficients in abelian group G, which satisfies the axioms of homotopy, exactness, suspension, dimension and partially continuous property or universal coefficient formula.

Thank You!