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On Di-injective T0-Quasi-metric Spaces

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Contents

Introduction

A quasi-pseudometric space (Y, d_Y) is called di-injective if for any quasi-pseudometric space (X, d_X) and any subspace A of (X, d_X), a nonexpansive map $f : A \to (Y, d_Y)$ can be extended to a nonexpansive map $g : (X, d_X) \to (Y, d_Y)$ (compare [2, Definition 8]). Kemajou et al. in [2] proved that every T_0-quasi-metric space has a di-injective hull. In the same article, they showed that a T_0-quasi-metric space is q-hyperconvex if and only if it is di-injective in the category of T_0-quasi-metric spaces and nonexpansive maps. However their proof appealed to Zorn's Lemma. In this talk, we present, among other results, a constructive proof of the above result.

Preliminaries

- Let $X \neq \emptyset$ and $d : X \times X \to [0, \infty)$. Then d is called a **quasi-pseudometric** on X if
  (a) $d(x, x) = 0$ whenever $x \in X$, and
  (b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, x \in X$.

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  (a) $d(x, x) = 0$ whenever $x \in X$, and
  (b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, x \in X$.
- If d satisfies the additional condition that $d(x, y) = 0 = d(y, x)$ implies that $x = y$, we call d a T_0-quasi-metric.

Preliminaries

- If d is a quasi-pseudometric on a set X, then we define the conjugate quasi-pseudometric $d^{-1} : X \times X \to [0, \infty)$ by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$.

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- If d is a T_0-quasi-metric, then $d^s = \max\{d, d^{-1}\}$ is a metric.
- Let (X, d) be a quasi-pseudometric space. By an open ϵ-ball centered at a point $x \in X$ denoted $B_d(x, \epsilon)$, we mean $\{y \in X : d(x, y) < \epsilon\}$ for every $\epsilon > 0$.

Preliminaries

Example

(The usual T_0-quasi-metric on $\mathbb{R}$) *Given two real numbers a and b we shall write $a \dot{-} b$ for $\max\{a-b, 0\}$. Note that $u(x,y) = x \dot{-} y$ with $x, y \in \mathbb{R}$ defines a T_0-quasi-metric on the set $\mathbb{R}$ of the reals. Observe that $x \mapsto -x$ defines a bijective isometric map from $(\mathbb{R}, d)$ to $(\mathbb{R}, d^{-1})$.*

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Example

(The Sorgenfrey line). *For* $x, y \in \mathbb{R}$ *define a quasi-pseudometric* ρ *by* $\rho(x,y) = y - x$, *if* $x \leq y$ *and* $\rho(x,y) = 1$ *if* $x > y$. *Then* $(\mathbb{R}, \rho)$ *is a* T_0*-quasi-metric space also known as the Sorgenfrey line.*

Preliminaries

We note that for a quasi-pseudometric space (X, d):

- The collection $\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$ of all "open" balls yields a base for a topology $\tau(d)$. It is called the topology induced by d on X.

Nonexpansive function pairs

Nonexpansive function pairs

- Let (X, d) be a T_0-quasi-metric space and let f_i, $i = 1, 2$ be a map $f_i : X \to [0, \infty)$. We shall call $f = (f_1, f_2)$ a function pair on X. Recall from [2] that a function pair f on X is nonexpansive if and only if

$$f_1(x) \dot{-} f_1(y) \leq d(y, x)$$

and

$$f_2(x) \dot{-} f_2(y) \leq d(x, y)$$

whenever $x, y \in X$.

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whenever $x, y \in X$.

- We shall denote by $N(A, X)$ the set of all nonexpansive function pairs from a quasi-pseudometric space A into another quasi-pseudometric space X. Recall that X is said to be di-injective if for every quasi-pseudometric space B, every $A \subset B$ and every $f \in N(A, X)$ there is a $g \in N(B, X)$ such that $g|_A = f$. Note immediately that $A = \emptyset \neq B$ means that $X \neq \emptyset$.

Examples of di-injective function pairs

Example

Examples of di-injective quasi-metric spaces include $\mathbb{R}$ *and all nonempty closed intervals, with the quasi-metric defined by* $d(x, y) = x \dot{-} y$ *for every* $x, y \in \mathbb{R}$*. Let* $f \in N(A, \mathbb{R})$*, where* $\emptyset \neq A$ *is a subset of a quasi-pseudometric space* B*, then*

$$g_1(b) := \sup\{d(a, b) \dot{-} f_2(a) : a \in A\} \tag{4}$$

and

$$g_2(b) := \sup\{d(b, a) \dot{-} f_1(a) : a \in A\} \tag{5}$$

defines an extension $g \in N(B, \mathbb{R})$ *of* f*.*

Absolute nonexpansive retracts

- Recall from [3] that a quasi-pseudometric space (X, d) is called an absolute nonexpansive retract if, whenever $\varphi : X \to Y$ is an isometric embedding into another quasi-pseudometric space Y, then there exists a nonexpansive retraction of Y onto $\varphi(X)$.

Absolute nonexpansive retracts

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- Observe that if X is di-injective and $\varphi : X \to Y$ is an isometric embedding, then $\varphi(X)$ is di-injective and hence the identity map on $\varphi(X)$ extends to a nonexpansive retraction $r : Y \to \varphi(X)$.

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- Observe that if X is di-injective and $\varphi : X \to Y$ is an isometric embedding, then $\varphi(X)$ is di-injective and hence the identity map on $\varphi(X)$ extends to a nonexpansive retraction $r : Y \to \varphi(X)$.
- We note by [1, Remark 3] that every di-injective T_0-quasi-metric space X is contractible.

q-hyperconvexity

Let (X, d) be a T_0-quasi-metric space.

- X is said to be q-hyperconvex if every family $(x_i)_{i \in I}$ of elements of X and nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ with the property that $d(x_i, x_j) \leq r_i + s_j$ for every $i, j \in I$ satisfies

$$\bigcap_{i \in I} \{C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)\} \neq \emptyset.$$

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- The next result was proved by Kemajou et al. in [2], but their proof appealed to Zorn's lemma. In the following we present a short proof that does not require the use of Zorn's lemma.

Result

Theorem

Every q-hyperconvex T_0-quasi-metric space (X, d) is di-injective.

Proof.

- Suppose that X is q-hyperconvex.

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- Note that if $f \in N(A, X)$, $\emptyset \neq A \subset B$, and $b \neq a$, then

$$d(a, b) + d(b, a_0) \geq d(a, a_0) \geq d(f_2(a), f_2(a_0)),$$

for all $a, a_0 \in A$. Thus we have that

$$f_2(a_0) \in C_d(f_2(a), d(a, b)).$$

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- Similarly we have that

$$d(a_0, b) + d(b, a) \geq d(a_0, a) \geq d(f_2(a_0), f_2(a)),$$

for all $a, a_0 \in A$, so that

$$f_2(a_0) \in C_{d^{-1}}(f_2(a), d(b, a)).$$

This shows that for every $a, a_0 \in A$,

$$C_d(f_2(a), d(a, b)) \cap C_{d^{-1}}(f_2(a), d(b, a)) \neq \emptyset,$$

hence by q-hyperconvexity we have that

Proof.

- $$\bigcap_{a \in A} (C_d(f_2(a), d(a,b)) \cap C_{d-1}(f_2(a), d(b,a))) \neq \emptyset.$$

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- $$\bigcap_{a \in A} (C_d(f_2(a), d(a,b)) \cap C_{d^{-1}}(f_2(a), d(b,a))) \neq \emptyset.$$
- Thus we get an extension $(f_b)_2 \in N(A \cup \{b\}, X)$ of f_2 by demanding that $(f_b)_2(b)$ be a point of this intersection.

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- Thus we get an extension $(f_b)_2 \in N(A \cup \{b\}, X)$ of f_2 by demanding that $(f_b)_2(b)$ be a point of this intersection.
- By following the same computations above, one can get an extension $(f_b)_1 \in N(A \cup \{b\}, X)$ of f_1 by demanding that $(f_b)_1(b)$ be a point of the intersection.

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- $$\bigcap_{a \in A} (C_d(f_2(a), d(a,b)) \cap C_{d^{-1}}(f_2(a), d(b,a))) \neq \emptyset.$$
- Thus we get an extension $(f_b)_2 \in N(A \cup \{b\}, X)$ of f_2 by demanding that $(f_b)_2(b)$ be a point of this intersection.
- By following the same computations above, one can get an extension $(f_b)_1 \in N(A \cup \{b\}, X)$ of f_1 by demanding that $(f_b)_1(b)$ be a point of the intersection.
- By iterating this process, one can infer that X is di-injective.

Katětov function pairs

Definition

Let (X, d) *be a* T_0*-quasi-metric space and* $f : X \to [0, \infty)$ *be a function pair. Then* f *will be said to be Katětov if*
(*a*) f *is ample,*
(*b*) $f \in N(X, [0, \infty))$.

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(a) f is ample,
(b) $f \in N(X, [0, \infty))$.

We denote by $Q(X, d)$ (or simply $Q(X)$ if there is no confusion with d) the set of all Katětov function pairs on (X, d). For each $f, g \in Q(X)$, set

$$D(f, g) = \sup_{x \in X}\{f_1(x) \dot{-} g_1(x)\} \vee \sup_{x \in X}\{g_2(x) \dot{-} f_2(x)\}.$$

Observe that $Q_X \subseteq Q(X)$.

Existence of retraction

In the following we will show that Q_X is di-injective without appealing to Zorn's lemma. First we recall the following result from [1].

Proposition

Let (X,d) be a T_0-quasi-metric space. There exists a retraction map $p : P_X \to Q_X$, i.e., a map that satisfies the conditions
(a) $D(p(f),p(g)) \leq D(f,g)$ whenever $f,g \in P_X$,
(b) $p(f) \leq f$ whenever $f \in P_X$.
(In particular $p(f) = f$ whenever $f \in Q_X$, since each $f \in Q_X$ is extremal, and thus p is indeed a retraction.)

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(In particular $p(f) = f$ whenever $f \in Q_X$, since each $f \in Q_X$ is extremal, and thus p is indeed a retraction.)

It is possible that the right hand side of (a) in the proposition above is infinite. However, if $f, g \in Q(X)$, then it is finite. Thus we have that the restriction of p to $Q(X)$ is a nonexpansive retraction onto Q_X. By this fact, to prove that Q_X is di-injective, we only show that $Q(X)$ is di-injective.

Main results

Proposition

For each T_0-quasi-metric space (X, d), Q_X and $Q(X)$ are di-injective.

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For each T_0-quasi-metric space (X, d), Q_X and $Q(X)$ are di-injective.

Proof.

Let $F : A \to Q(X) : a \mapsto f_a$ be a nonexpansive map, where $\emptyset \neq A \subset U$, U a T_0-quasi-metric space. For each $u \in U$, define

$$(g_u)_1(x) := \inf\{(f_a)_1(x) + d(a, u) : a \in A\}$$

and

$$(g_u)_2(x) := \inf\{(f_a)_2(x) + d(u, a) : a \in A\}.$$

Then one sees that g_u is nonnegative and $g_u \in N(X, [0, \infty))$ as the infimum of a family of such function pairs.

Proof continues

Let $a, a_0 \in A$ and $y \in X$. Then we have

$$(f_a)_1(y) \dot{-} (f_{a_0})_1(y) \leq D(f_{a_0}, f_a) = D(F(a_0), F(a)) \leq d(a_0, a),$$

and so

$$\begin{aligned}(g_u)_2(x) + (g_u)_1(y) &= \inf\{(f_a)_2(x) + d(u,a) : a \in A\} + \\ &\inf\{(f_{a_0})_1(y) + d(a_0, u) : a_0 \in A\} \\ &\geq \inf\{(f_a)_2(x) + (f_{a_0})_1(y) + d(a_0, a) : a, a_0 \in A\} \\ &\geq \inf\{(f_a)_2(x) + (f_a)_1(y) : a \in A\} \\ &\geq d(x, y).\end{aligned}$$

Thus we have that $g_u \in Q(X)$. Let $u, u_0 \in U$ and $x \in X$. Then

Proof continues

$$\begin{aligned}(g_u)_2(x) \dot{-} d(u,u_0) &= \inf\{(f_a)_2(x) + d(u,a) \dot{-} d(u,u_0) : a \in A\} \\ &\leq \inf\{(f_a)_2(x) + d(u_0,a) : a \in A\} \\ &= (g_{u_0})_2(x),\end{aligned}$$

so that $(g_u)_2(x) \dot{-} (g_{u_0})_2(x) \leq d(u,u_0)$. In a similar manner we can show that $(g_u)_1(x) \dot{-} (g_{u_0})_1(x) \leq d(u_0,u)$.
If $u \in A$, then $g_u(x) \leq f_u(x)$, i.e., $(g_u)_1(x) \leq (f_u)_1(x)$ and $(g_u)_2(x) \leq (f_u)_2(x)$. Moreover

$$\begin{aligned}(f_u)_2(x) &\leq (f_a)_2(x) + D(f_u, f_a) \\ &\leq (f_a)_2(x) + d(u,a) \\ &\leq \inf\{(f_a)_2(x) + d(u,a) : a \in A\} \\ &= (g_u)_2(x),\end{aligned}$$

for every $x \in X$, so that $(f_u)_2 \leq (g_u)_2$. Similarly we have $(f_u)_1 \leq (g_u)_1$. Hence $f_u = g_u$. This shows that the map $\varphi : u \mapsto g_u$ is a nonexpansive extension of F.

C.A. Agyingi, P. Haihambo, and H.-P.A. Künzi, Tight extensions of T_0-quasi-metric spaces, Ontos-Verlag: Logic, Computations, Hierarchies (2014), 9-22.

C.A. Agyingi, P. Haihambo, and H.-P.A. Künzi, Tight extensions of T_0-quasi-metric spaces, Ontos-Verlag: Logic, Computations, Hierarchies (2014), 9-22.

E. Kemajou, H.-P.A. Künzi and O.O. Otafudu, The Isbell-hull of a di-space, Topology and its Application **159**: 2463-2475, 2012.

- C.A. Agyingi, P. Haihambo, and H.-P.A. Künzi, Tight extensions of T_0-quasi-metric spaces, Ontos-Verlag: Logic, Computations, Hierarchies (2014), 9-22.
- E. Kemajou, H.-P.A. Künzi and O.O. Otafudu, The Isbell-hull of a di-space, Topology and its Application **159**: 2463-2475, 2012.
- O.O. Otafudu, Convexity in quasi-metric spaces, PhD thesis, University of Cape Town, 2012.

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Thank you for your attention.