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On a Construction of Some Class of Metric Spaces

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- Motivations
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- Examples
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Definition

Definition (Aronszajn–Panitchpakdi, 1956)

We call a metric space hyperconvex, if each family of closed balls $\{\overline{B}(x_i, r_i)\}_{i\in J}$ such that

$$d(x_i, x_j) \leqslant r_i + r_j \quad \text{for } i, j \in J$$

has a nonempty intersection.

Remark

For normed spaces, hyperconvexity means that each pairwise intersecting family of closed balls has a nonempty intersection.

Examples

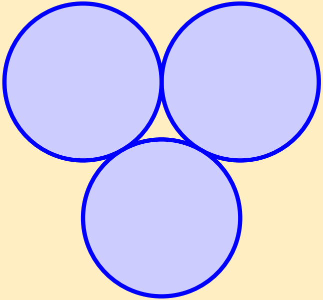
Examples

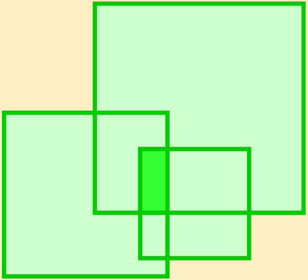
The following spaces are hyperconvex:

- $\mathbb{R}$;
- $\mathbb{R}^n$ with a "maximum" norm;
- l^∞ and $L^\infty(\mathbb{R})$;
- $\mathcal{C}_{\mathbb{R}}(K)$, where K is a compact and extremally disconnected Hausdorff topological space.

Examples (continuation)

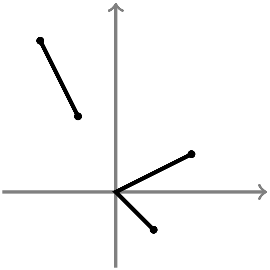
Examples



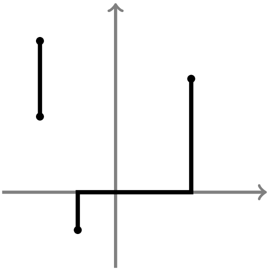


The space $\mathbb{R}^2$ with the euclidean norm is not hyperconvex.
However, the same space with the "maximum" norm is.

The radial metric

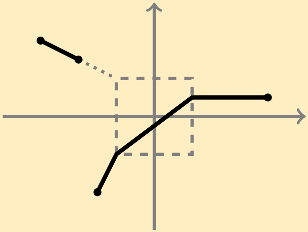


The "river" metric



The construction of certain metrics

Definition (pictorial; Borkowski, DB, Przybycień)



Linking construction

Definition (informal; Aksoy, Maurizi)

Let (X_c, d_c) and (X_i, d_i) for $i \in I$ be metric spaces. Assume that for every $i \in I$ the intersection $X_c \cap X_i$ consists of exaclty one point - let us denote it by x_i - and that for $i, j \in I$ the intersection $X_i \cap X_j$ is empty or it contains only the point $x_i = x_j$.

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Let (X_c, d_c) and (X_i, d_i) for $i \in I$ be metric spaces. Assume that for every $i \in I$ the intersection $X_c \cap X_i$ consists of exaclty one point - let us denote it by x_i - and that for $i, j \in I$ the intersection $X_i \cap X_j$ is empty or it contains only the point $x_i = x_j$. Let us denote $X := X_c \cup \bigcup_{i \in I} X_i$ and let us define the function $\varphi\colon X \times X \to [0, +\infty)$ by formulae

$$\psi(x, y) := \begin{cases} d_c(x, y), & \text{if } x, y \in X_c \\ d_i(x, y), & \text{if } x, y \in X_i \\ d_c(x, x_i) + d_i(x_i, y), & \text{if } x \in X_c \text{ i } y \in X_i \\ d_i(x, x_i) + d_c(x_i, x_j) + d_j(x_j, y), & \text{if } x \in X_i \text{ i } y \in X_j \end{cases}$$

Fact

The above defined function ψ is a metric on the set X.

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The definition of the metric φ

Definition (Száz)

Let (X, d) be a metric space, $T\colon X \to X$ be a mapping, g be a metric on $T(X)$ and let $\sim$ be such an equivalence relation on X that T is constant on its equivalence classes. Let $\varphi\colon X \times X \to [0, +\infty)$ be defined by the formula

$$\varphi(x,y) := \begin{cases} d(x,y), & \text{if } x \sim y, \\ d(x,T(x)) + g(T(x),T(y)) + d(T(y),y), & \text{otherwise.} \end{cases}$$

Fact

It is not difficult to check that the above defined function φ is a metric on the space X.

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The radial metric and the "river" metric

Example

Let $X := \mathbb{R}^2$, $d = g$ be the euclidean metric, $T((x_1, x_2)) := (0, 0)$ and $(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow$ the points $(0, 0)$, (x_1, x_2), (y_1, y_2) are colinear.
Then the metric φ is the well-known radial metric.

The radial metric and the "river" metric

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Then the metric φ is the well-known "river" metric.

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Properties

Fact

If $d = g$, the mapping T is continuous, the space (X, d) is complete and the equivlence classes of the relation "$\sim$" are closed, then the metric φ is complete.

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If the metrics d and g are bounded, then the metric φ is also bounded.

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Fact

If the relation "$\sim$" is the equality relation, then every cluster point of the space (X, φ) is a fixed point of the mapping T.

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The "floor" metric

Example

Let $X := \mathbb{R}$, $d = g$ be the euclidean metric, $T(x) := \lfloor x \rfloor$ and $x \sim y \Leftrightarrow x = y$. Then φ is a certain metric on the real line which we will denote by $\varphi_{\lfloor \cdot \rfloor}$.

The "floor" metric

Example

Let $X := \mathbb{R}$, $d = g$ be the euclidean metric, $T(x) := \lfloor x \rfloor$ and $x \sim y \Leftrightarrow x = y$. Then φ is a certain metric on the real line which we will denote by $\varphi_{\lfloor \cdot \rfloor}$.

Fact

The metric $\varphi_{\lfloor \cdot \rfloor}$ is neither translation invariant nor homogeneous. It is not metrically convex, however it is complete.

Shape of balls

Fact

Let $x \in \mathbb{Z}$ and $r > 0$. Then

$$B_{\varphi_{\lfloor \cdot \rfloor}}(x, r) = \left[x - \lfloor r \rfloor, x - \lfloor r \rfloor + r - \lfloor r \rfloor\right) \cup \left[x - \lfloor r \rfloor + 1, x + r\right).$$

Shape of balls

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Fact

Let $x \in \mathbb{R}$ *and* $r > 0$. *Then*
$B_{\varphi_{\lfloor\cdot\rfloor}}(x, r) = B_{\varphi_{\lfloor\cdot\rfloor}}\left(\lfloor x \rfloor, r - (x - \lfloor x \rfloor)\right) \cup \{x\}.$

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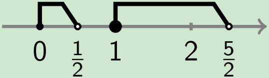
Remark

Analogous formulae hold for closed balls.

Shape of balls – examples

Examples

$$B_{\varphi_{\lfloor\cdot\rfloor}}\left(1, \tfrac{3}{2}\right) = \left[0, \tfrac{1}{2}\right) \cup \left[1, \tfrac{5}{2}\right)$$



Shape of balls – examples

Examples

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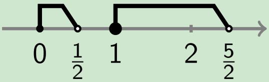
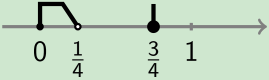


$$B_{\varphi_{\lfloor\cdot\rfloor}}\left(\tfrac{3}{4}, 1\right) = \left[0, \tfrac{1}{4}\right) \cup \left\{\tfrac{3}{4}\right\}$$



Form of open sets

Fact

A subset of the space $(\mathbb{R}, \varphi_{\lfloor \cdot \rfloor})$ is φ-open if and only if it is a union of pairwise disjoint maximal intervals with the property that if the right endpoint of any of these intervals in an integer, then this endpoint does not belong to that interval.

Form of open sets

Fact

A subset of the space $(\mathbb{R}, \varphi_{\lfloor \cdot \rfloor})$ *is* φ*-open if and only if it is a union of pairwise disjoint maximal intervals with the property that if the right endpoint of any of these intervals in an integer, then this endpoint does not belong to that interval.*

Corollary

A ball in the space $(\mathbb{R}, \varphi_{\lfloor \cdot \rfloor})$ *need not be a connected set. For example* $B\left(1, \frac{3}{2}\right) = \left[0, \frac{1}{2}\right) \cup \left[1, \frac{5}{2}\right)$ *is the union of two open sets.*

Measure of noncompactness

Fact

Let $A \subset [k, k+1)$ for some $k \in \mathbb{Z}$. The Kuratowski measure of noncompactness of the set A (in view of the metric $\varphi_{\lfloor \cdot \rfloor}$) expresses by the formula

$$\alpha(A) = 2\max\big(\{x - k \,|\, x \text{ is a cluster point of the set } A\} \cup \{0\}\big).$$

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$$\alpha(A) = 2\max\big(\{x - k \mid x \text{ is a cluster point of the set } A\} \cup \{0\}\big).$$

Remark

In the above formula *a cluster point* is considered in view of the euclidean metric.

Measure of noncompactness

Corollary

Let A be a bounded subset of $\mathbb{R}$. The Kuratowski measure of noncompactness of the set A (in view of the metric $\varphi_{\lfloor\cdot\rfloor}$) expresses by the formula

$$\alpha(A) = 2 \max_{k \in \mathbb{Z}} \max\left(\left\{ x - k \;\middle|\; \begin{array}{c} x \text{ is a cluster point} \\ \text{of the set } A \cap [k, k+1) \end{array} \right\} \cup \{0\}\right)$$

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References

A. G. Aksoy and B. Maurizi
Metric trees, hyperconvex hulls and extensions
Turk J. Math. 32 (2008), 219–234

M. Borkowski, D. Bugajewski and A. Burchardt
On topological properties of metrics defined via generalized "linking construction", submitted

M. Borkowski, D. Bugajewski, H. Przybycień
Hyperconvex spaces revisited
Bull. Austral. Math. Soc., 68 (2003), 191–203

D. Bugajewski, E. Grzelaczyk
A fixed point theorem for hyperconvex spaces
Arch. Math. (*Basel*), 75 (2000), 395–400

D. Bugajewski, E. Grzelaczyk
On the measures of noncompactness in some metric spaces
New Zealand J. Math., 27(2) (1998), 177–182

References

Á. Száz
A common generalization of the postman, radial, and river metrics
Rostock. Math. Kolloq. 67 (2012), 89–125

Thank you
for your attention.