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A New Class of Dendrites Having Unique Second Symmetric Product

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A new class of dendrites having unique second symmetric product

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A *continuum* is a nonempty compact connected metric space.

The *second symmetric product of a continuum* X, $\mathcal{F}_2(X)$, is the hyperspace of all nonempty subsets of X having at most two elements.

The hyperspace $\mathcal{F}_2(X)$ is a continuum with Hausdorff metric.

A continuum X *has unique second symmetric product* provided that each continuum Y such that $\mathcal{F}_2(Y)$ is homeomorphic to $\mathcal{F}_2(X)$ must be homeomorphic to X.

Problem. Find condition on a continuum X, so that X has unique second symmetric product.

A locally connected continuum contains no simple closed curve is called *dendrite*.

Each element in following class of continua has unique second symmetric product.

- Dendrites whose set of end points is closed.
- Almost meshed dendrites.
- Meshed dendrites.

Each element in following class of continua has unique second symmetric product.

- Dendrites whose set of end points is closed.
- Almost meshed dendrites.
- Meshed dendrites.
- New class of dendrites.

A connected topological space X is said to be *unicoherent* provided that whenever A and B are connected closed subsets of X such that $X = A \cup B$, then $A \cap B$ is connected.

A point p in a unicoherent topological space Y *makes a hole* in Y, if $Y - \{p\}$ is not unicoherent.

Theorem (Anaya, Maya, Orozco-Zitli (2016))

Let X be a unicoherent locally connected continuum and $p, q \in X$. Then $\{p, q\}$ makes a hole in $\mathcal{F}_2(X)$ if and only if either $p = q$ and $X - \{p\}$ has at least three components or $p \neq q$ and both $X - \{p\}$ and $X - \{q\}$ are not connected.

$\mathcal{MH}(X) = \{A \in \mathcal{F}_2(X) : A$ makes a hole in $\mathcal{F}_2(X)\}$.

Corollary.

If X is a dendrite, then $\mathcal{MH}(X)$

$\mathcal{MH}(X) = \{A \in \mathcal{F}_2(X) : A$ makes a hole in $\mathcal{F}_2(X)\}$.

Corollary.

If X is a dendrite, then $\mathcal{MH}(X)$
$= \{A \in \mathcal{F}_2(X) - \mathcal{F}_1(X) :$
$A \cap E(X) = \emptyset\} \cup \mathcal{F}_1(R(X))$

$\mathcal{NMH}(X) = \{A \in \mathcal{F}_2(X) : A$ does not make a hole in $\mathcal{F}_2(X)\}$.

Corollary.

If X is a dendrite, then $\mathcal{NMH}(X)$

$\mathcal{NMH}(X) = \{A \in \mathcal{F}_2(X) : A$ does not make a hole in $\mathcal{F}_2(X)\}$.

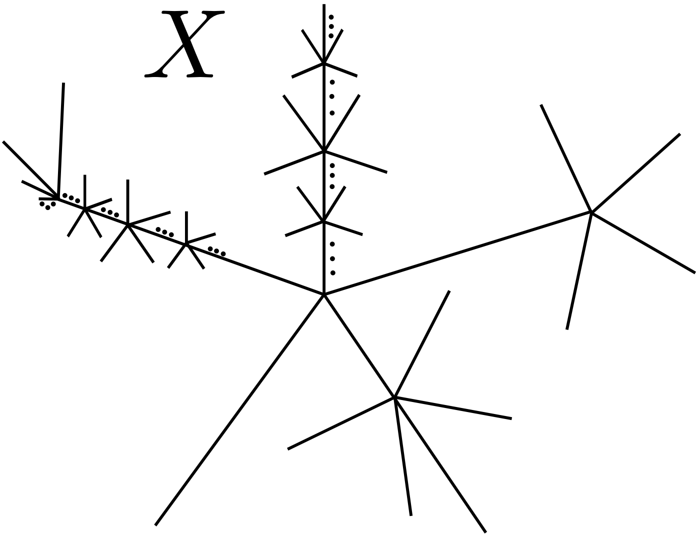
Corollary.

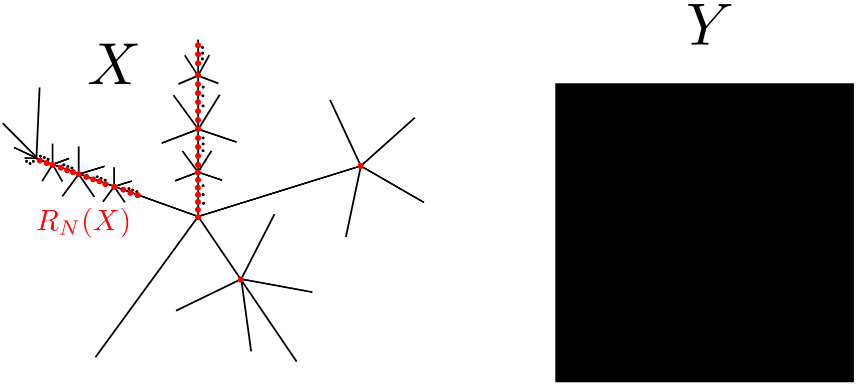
If X is a dendrite, then $\mathcal{NMH}(X)$
$= \mathcal{F}_1(O(X))$
$\cup \{A \in \mathcal{F}_2(X) : A \cap E(X) \neq \emptyset\}$

$$CR(X) = \cap\{A \in \mathcal{C}(X) : R(X) \subseteq A\}.$$

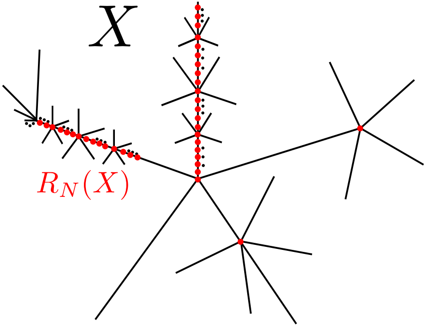
Theorem

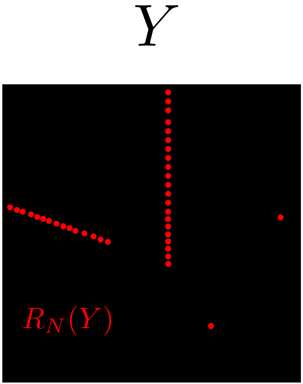
If X and Y are dendrites such that $CR(X) = \cap\{A \in \mathcal{C}(X) : R_N(X) \subseteq A\}$ and there exists a homeomorphism $h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ satisfying that $h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y))$, then X and Y are homeomorphic.


X

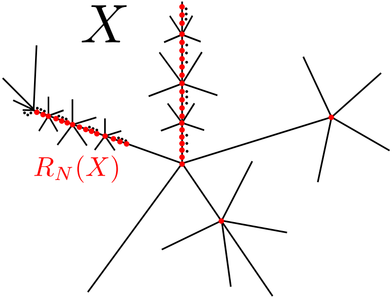


$h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ such that $h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y))$.

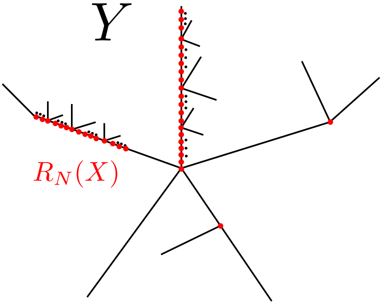




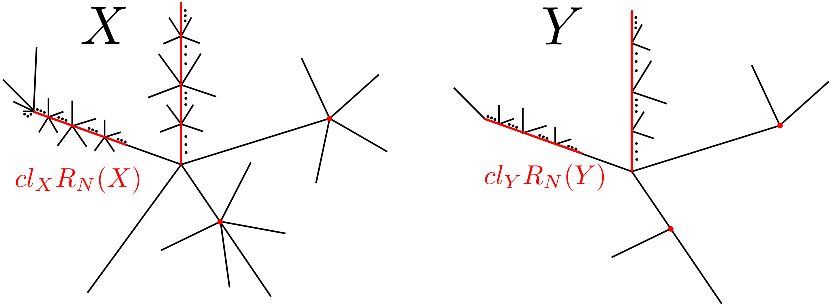
$$\varphi : R_N(X) \to R_N(Y) \text{ such that } h(\{p\}) = \{\varphi(p)\}.$$



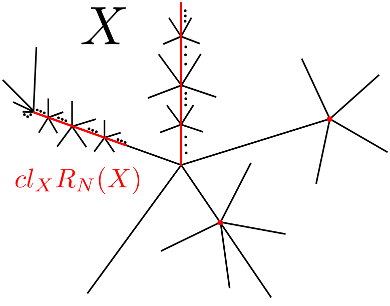
X
$R_N(X)$



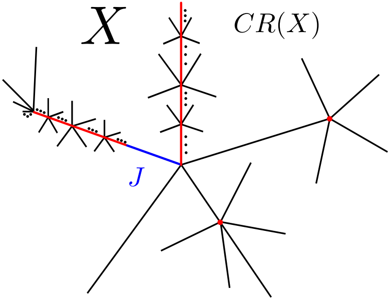
Y
$R_N(X)$



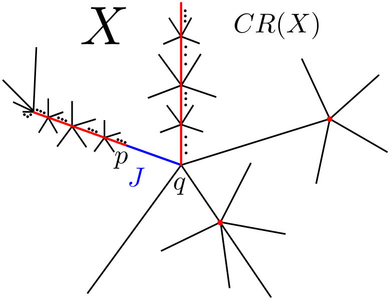
$$h(\mathcal{F}_1(cl_X R_N(X))) = \mathcal{F}_1(cl_Y R_N(Y)).$$

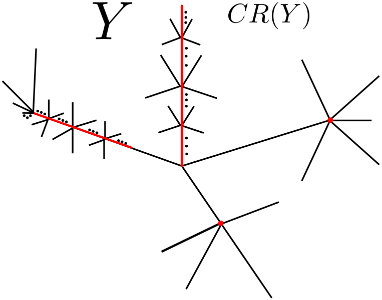


$\hat{\varphi} : cl_X R_N(X) \to cl_Y R_N(Y)$ such that $h(\{p\}) = \{\hat{\varphi}(p)\}$

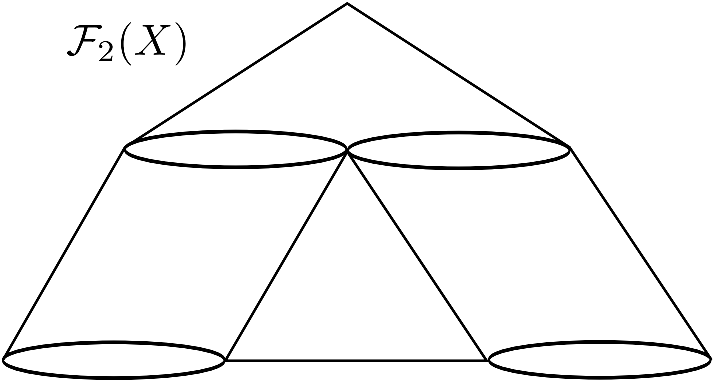


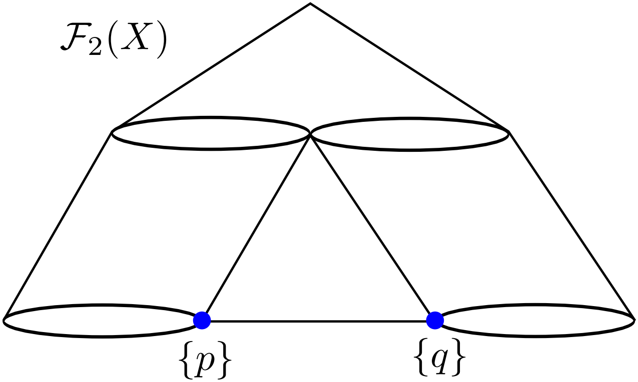
J is component of $X - cl_X R_N(X)$ such that $J \cap E(X) = \emptyset$.

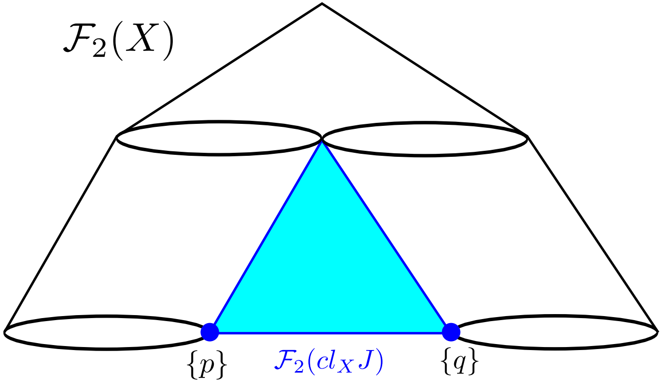


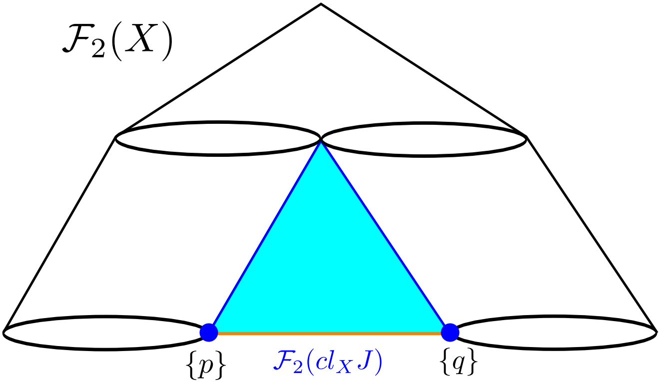


$E(cl_X J) = \{p, q\} \subseteq cl_X R_N(X)$ and

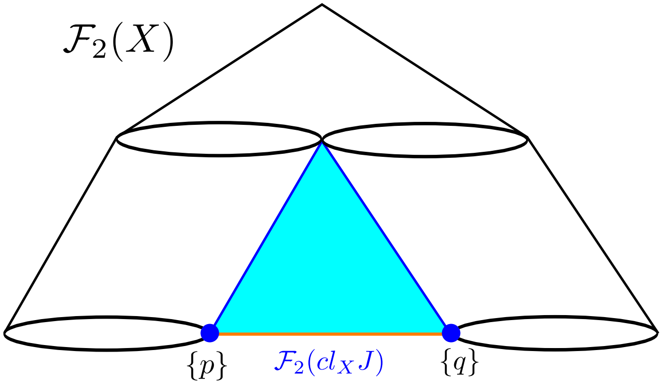

$\mathcal{F}_2(X)$


$\mathcal{F}_2(X)$
$\{p\}$
$\{q\}$

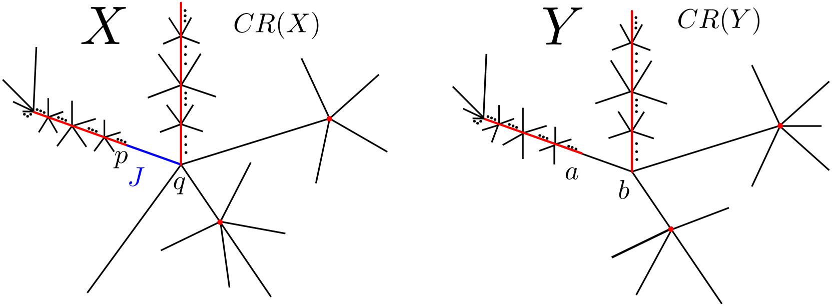

$\mathcal{F}_2(X)$
$\{p\}$
$\mathcal{F}_2(cl_X J)$
$\{q\}$



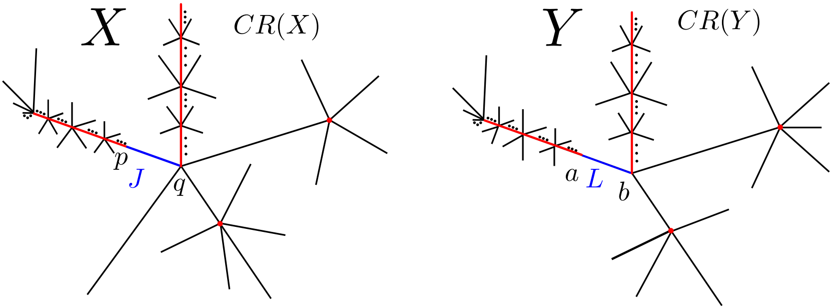
$\mathcal{J} = F_1(cl_X J)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{J}) = \{\{p\}, \{q\}\}$ and $\mathcal{J} - E(\mathcal{J}) \subseteq \mathcal{NMH}(X)$.



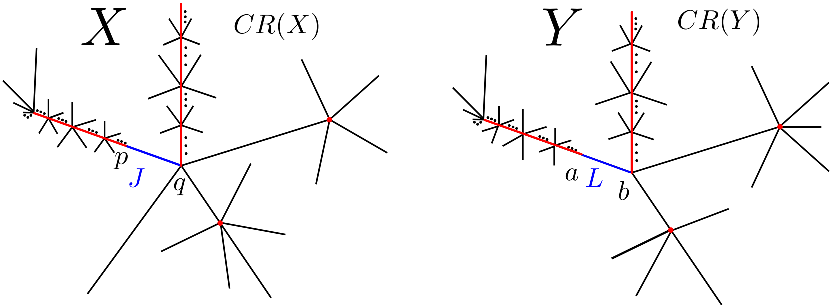
$h(\mathcal{J})$ is an arc in $\mathcal{F}_2(Y)$ such that $E(h(\mathcal{J})) = \{h(\{p\}), h(\{q\})\}$ and $h(\mathcal{J}) - E(h(\mathcal{J})) \subseteq \mathcal{NMH}(Y)$.



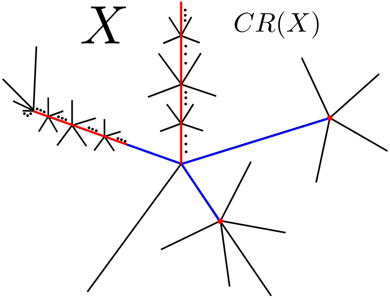
$a, b \in cl_Y R_N(Y)$ satisfying $h(\{p\}) = \{a\}$ and $h(\{q\}) = \{b\}$.

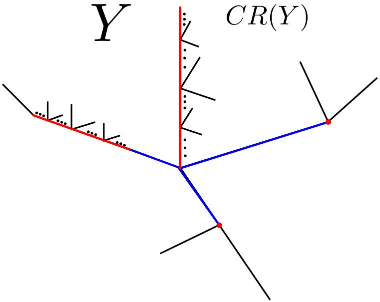


Then the unique component L of $Y - cl_Y R_N(Y)$ such that $E(cl_Y L) = \{a, b\}$ satisfies that $\mathcal{L} = F_1(cl_X L)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{L}) = \{\{a\}, \{b\}\}$ and $\mathcal{L} - E(\mathcal{L}) \subseteq \mathcal{NMH}(L)$.

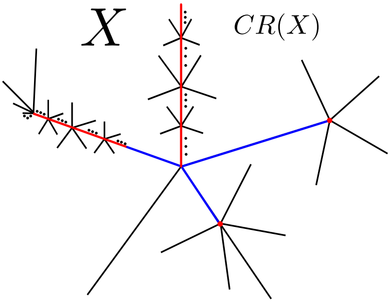


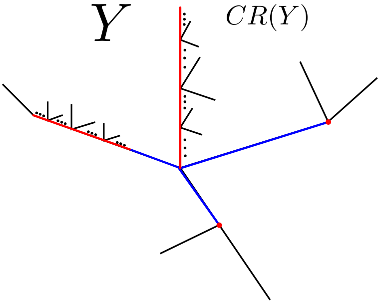
Then the unique component L of $Y - cl_Y R_N(Y)$ such that $E(cl_Y L) = \{a, b\}$ satisfies that $\mathcal{L} = F_1(cl_X L)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{L}) = \{\{a\}, \{b\}\}$ and $\mathcal{L} - E(\mathcal{L}) \subseteq \mathcal{NMH}(L)$. So, $h(\mathcal{J}) = \mathcal{L}$.





$$h(\mathcal{F}_1(CR(X))) = \mathcal{F}_1(CR(Y)).$$

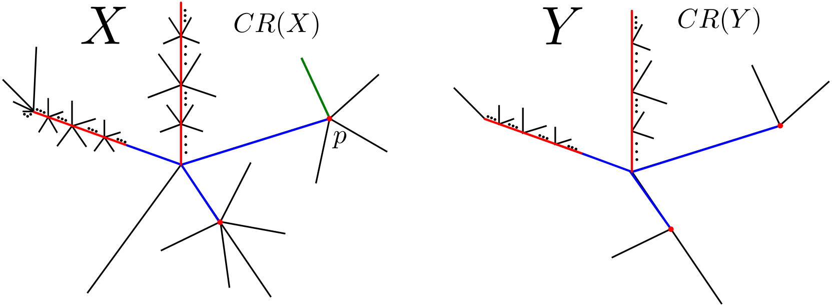




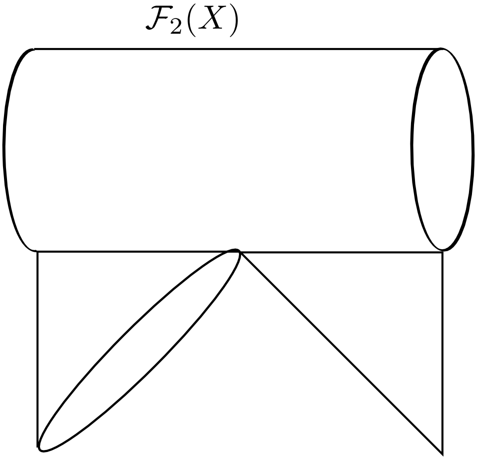
$h(\mathcal{F}_1(CR(X))) = \mathcal{F}_1(CR(Y)))$.
$\bar{\varphi} : CR(X) \to CR(Y)$ such that $h(\{x\}) = \{\bar{\varphi}(x)\}$.

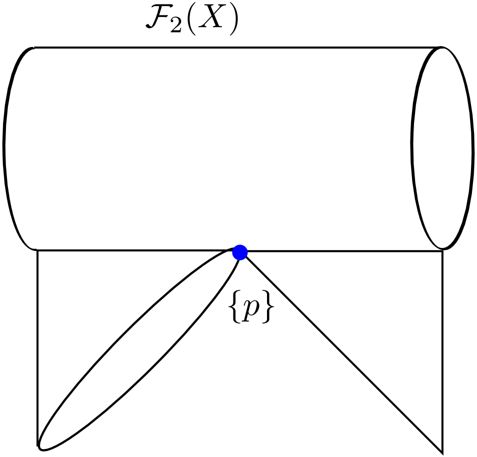
Theorem (Illanes, 2002)

If X *is a dendrite and* $Z \in \mathcal{C}(X)$ *is such that* $CR(X) \subseteq Z$ *and each component of* $X - CR(X)$ *intersects* Z*, then* Z *is homeomorphic to* X*.*

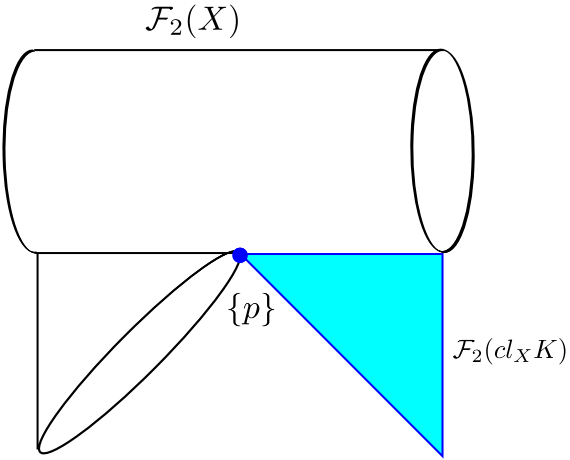


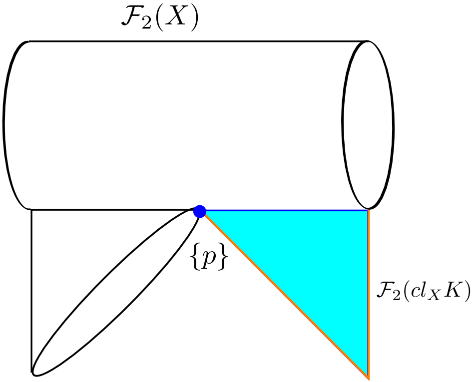
K is component of $X - CR(X)$,
$\{p\} = CR(X) \cap (cl_X K) = (cl_X R_N(X)) \cap (cl_X K)$ and
$K \cap E(X) = \{e\}$.


$\mathcal{F}_2(X)$

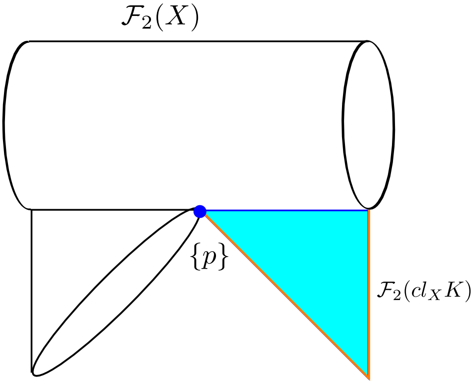


$\mathcal{F}_2(X)$
$\{p\}$

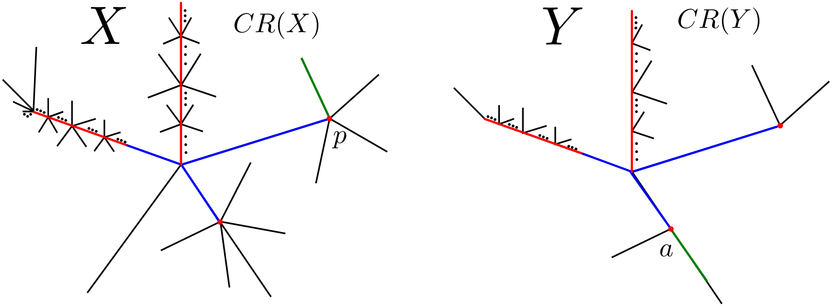

$\mathcal{F}_2(X)$
$\{p\}$
$\mathcal{F}_2(cl_X K)$



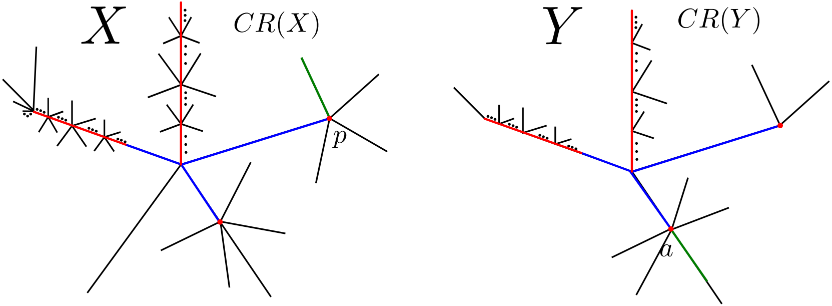
$\mathcal{K} = \mathcal{F}_1(cl_X K) \cup \{A \in \mathcal{F}_2(cl_X K) : e \in A\}$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{K}) = \{\{p\}, \{e, p\}\}$, $\mathcal{K} - E(\mathcal{K}) \subseteq \mathcal{NMH}(X)$ and $\mathcal{K} - E(\mathcal{K})$ does not contains ramification points of $\mathcal{NMH}(X)$.



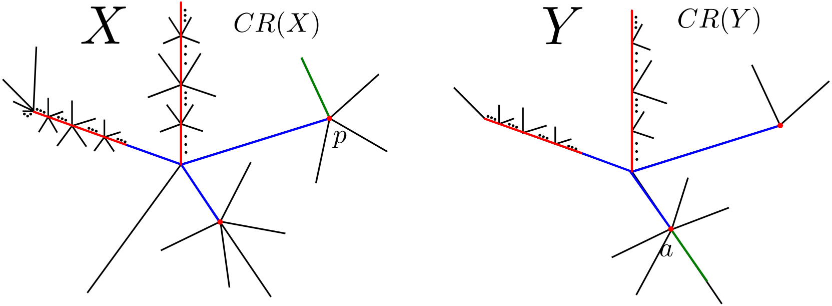
$h(\mathcal{K})$ is the unique arc in $\mathcal{F}_2(Y)$ such that
$E(h(\mathcal{K})) = \{h(\{p\}), h(\{e,p\})\}$, $h(\mathcal{K}) - E(h(\mathcal{K})) \subseteq \mathcal{NMH}(Y)$ and
$h(\mathcal{K}) - E(h(\mathcal{K}))$ does not contains ramification points of $\mathcal{NMH}(Y)$.



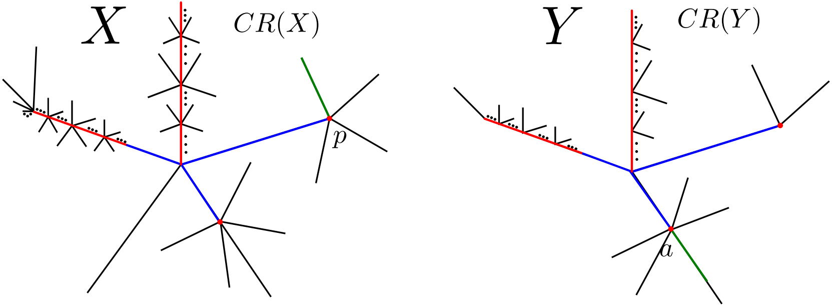
$a \in cl_Y R_N(Y)$ such that $h(\{p\}) = \{a\}$, then there exists a component G of $X - CR(X)$ such that $a \in cl_Y G$ and if $v \in (cl_Y G) \cap E(Y)$, then $h(\mathcal{K}) \subseteq \mathcal{F}_1(cl_Y G) \cup \{B \in \mathcal{F}_2(cl_Y G) : v \in B\} = \mathcal{G}$.



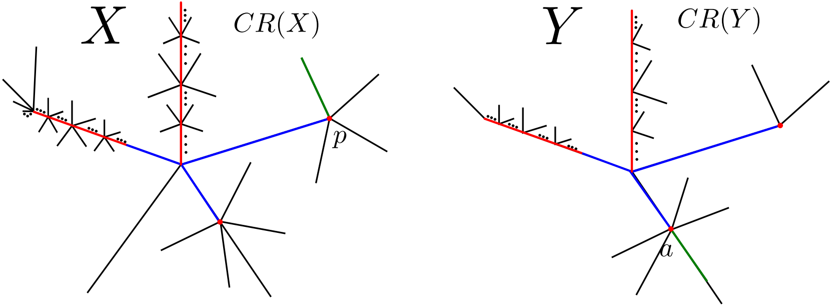
$h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)$ is an arc contained in $\mathcal{F}_1(cl_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G))$.



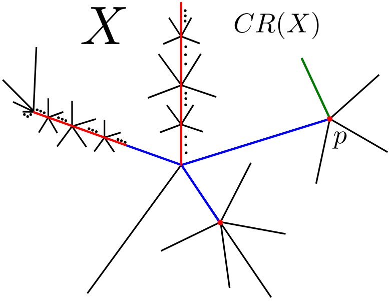
$h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)$ is an arc contained in $\mathcal{F}_1(cl_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G))$. Let $Y_K \in \mathcal{C}(cl_X G)$ such that $\mathcal{F}_1(Y_K) = h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y K)$.



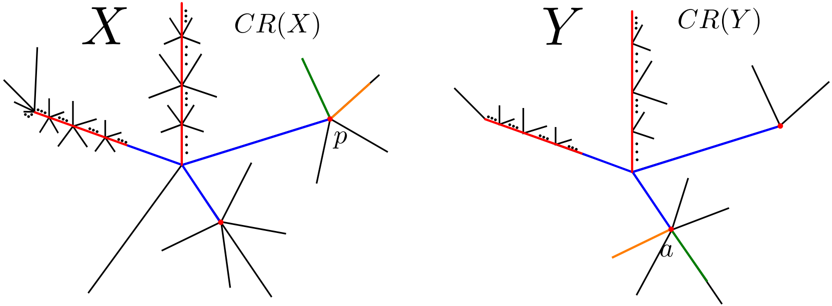
$h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)$ is an arc contained in $\mathcal{F}_1(cl_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G))$. Let $Y_K \in \mathcal{C}(cl_X G)$ such that $\mathcal{F}_1(Y_K) = h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y K)$. $Y_Z = CR(Y) \cup \bigcup\{Y_K : K$ is component of $X - CR(X)\}$.



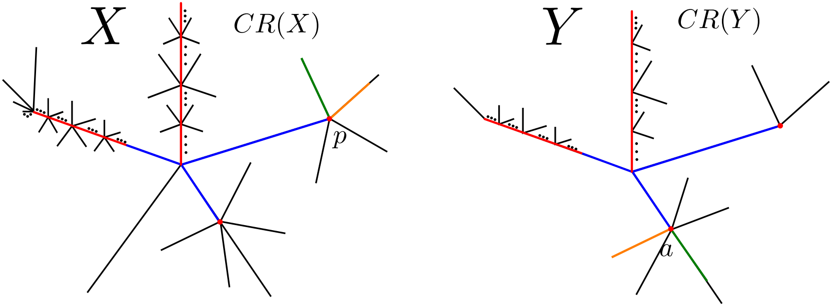
$h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)$ is an arc contained in $\mathcal{F}_1(cl_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G))$. Let $Y_K \in \mathcal{C}(cl_X G)$ such that $\mathcal{F}_1(Y_K) = h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y K)$. $Y_Z = CR(Y) \cup \bigcup\{Y_K : K$ is component of $X - CR(X)\}$. Thus, Y_Z and X are homeomorphic.



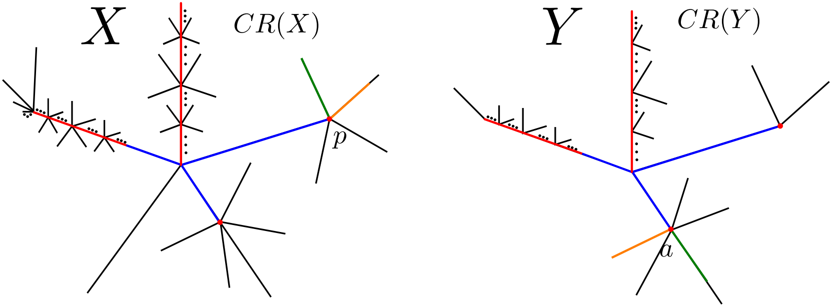
F is a component of $Y - CR(Y)$ such that $a \in cl_Y F$.



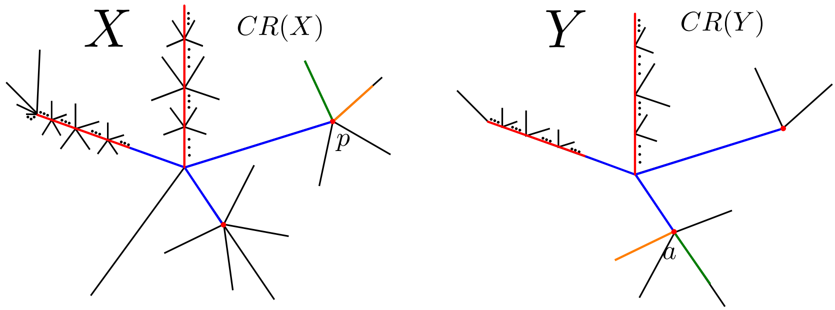
$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.



$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.
$X_Z = CR(X) \cup \bigcup \{X_F : F \text{ is a component of } Y - CR(Y)\}$.



$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.
$X_Z = CR(X) \cup \bigcup\{X_F : F$ is a component of $Y - CR(Y)\}$.
X_Z is homeomorphic to Y and X.



$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.
$X_Z = CR(X) \cup \bigcup\{X_F : F$ is a component of $Y - CR(Y)\}$.
X_Z is homeomorphic to Y and X.

Problem. $h(\mathcal{F}_1(R_N)) = \mathcal{F}_1(R_N(Y))$.

The *multicoherence degree* of a connected topological space Y, $r(Y)$, is defined by

$$\sup\left\{b_0(L\cap K) : \begin{array}{c} L \text{ and } K \text{ are connected} \\ \text{closed subset of } Y \\ \text{and } Y = L\cup K \end{array}\right\} - 1.$$

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$$\sup \left\{ b_0(L \cap K) : \begin{array}{c} L \text{ and } K \text{ are connected} \\ \text{closed subset of } Y \\ \text{and } Y = L \cup K \end{array} \right\} - 1.$$

$r(Y) = 0$ if and only if Y is *unicoherent*.

Theorem

If X is a dendrite and $p \in R_N(X)$ is such that ord$(p, X) = n$, then

$$r(\mathcal{F}_2(X) - \{\{p\}\}) = \frac{(n-1)(n-2)}{2}.$$

Theorem

If X is a dendrite and $p, q \in R_N(X) \cup O(X)$ are such that $p \neq q$, $ord(p, X) = n$ and $ord(q, X) = m$, then

$$r(\mathcal{F}_2(X) - \{\{p, q\}\}) = (n-1)(m-1).$$

For a dendrite X, set
$\Omega_X = \{\mathrm{ord}(p, X) : p \in R_N(X)\}$

For a dendrite X, set $\Omega_X = \{\text{ord}(p, X) : p \in R_N(X)\}$

Lemma

If X and Y are dendrites such that there exists an homeomorphism $h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ and $\Omega_X \subseteq \{5, 6, \ldots\}$, then $\Omega_Y \subseteq \{5, 6, \ldots\}$.

Theorem

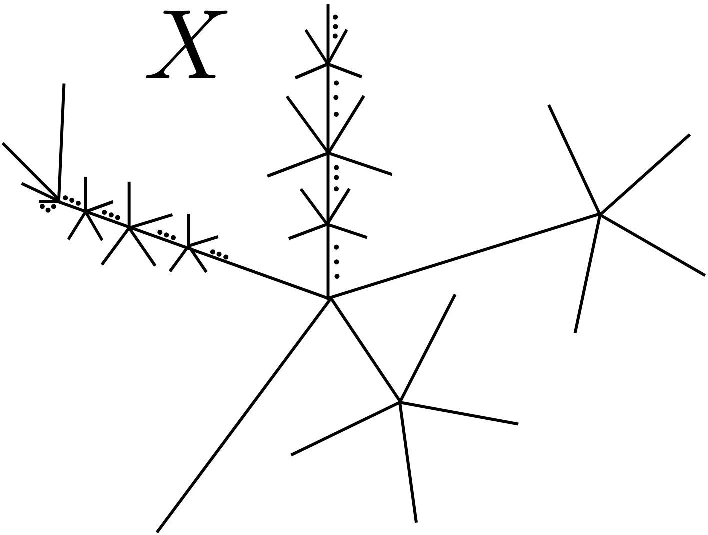
Let X and Y be dendrites. If $|\Omega_X| = 1$, $\Omega_X \subseteq \{5, 6, \ldots\}$ and $h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is a homeomorphism, then $h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y))$.

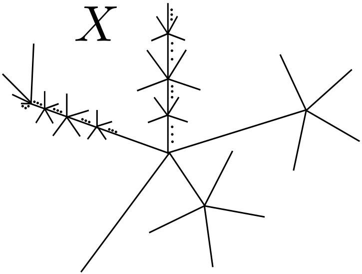
Theorem

Let X and Y be dendrites. If $h : \mathcal{F}_2 \to \mathcal{F}_2(Y)$ be a homeomorphism, $\Omega_X \subseteq \{5, 6 \ldots\}$ and

1. $\Omega_X \cap \left\{\frac{(j-1)(j-2)}{2} + 1 : j \geq 5\right\} = \emptyset,$
2. $\{(n-1)(m-1) : n, m \in \Omega_X\} \cap \left\{\frac{(j-1)(j-2)}{2} : j \geq 5\right\} = \emptyset,$

then $h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y))$.


X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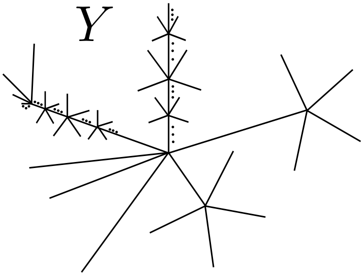
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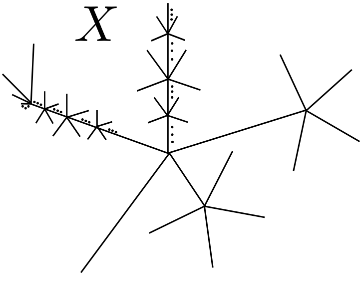
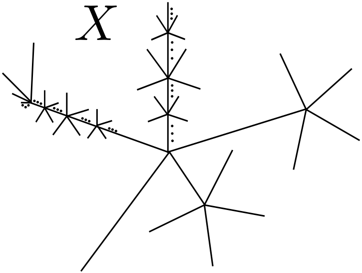
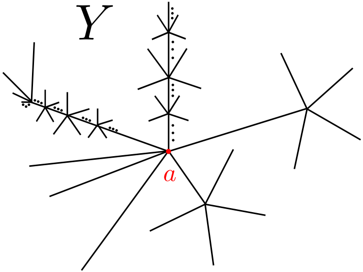
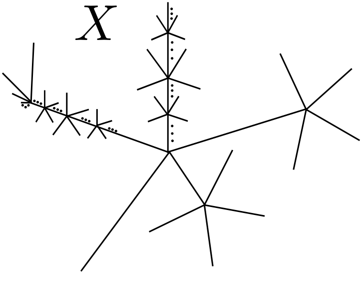


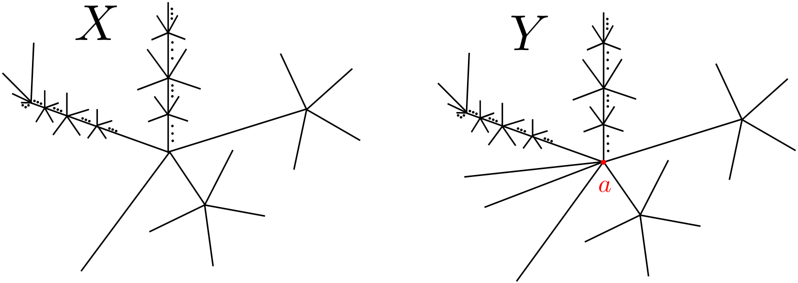
$h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is a homeomorphism


X

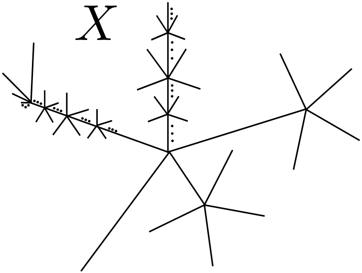

Y
a



$a \in R_N(Y)$

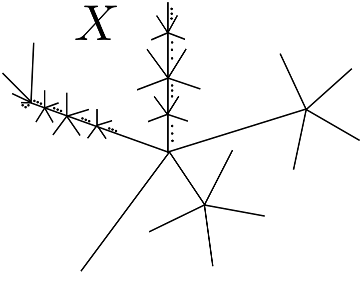


$a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{MH}(Y).$

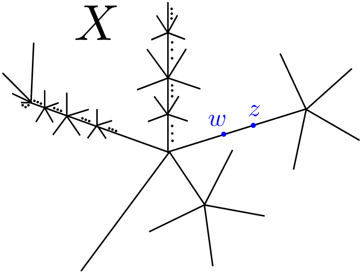


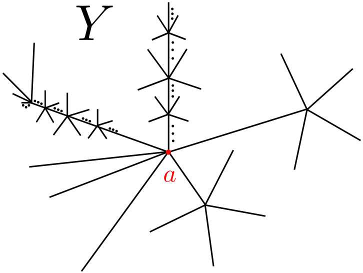
$a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{MH}(Y).$

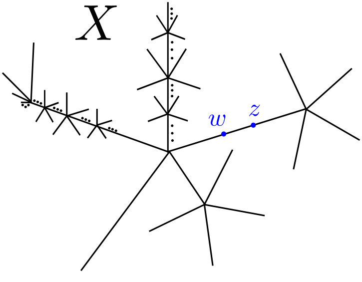
$ord(a, Y) = m \geq 5 \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2}.$



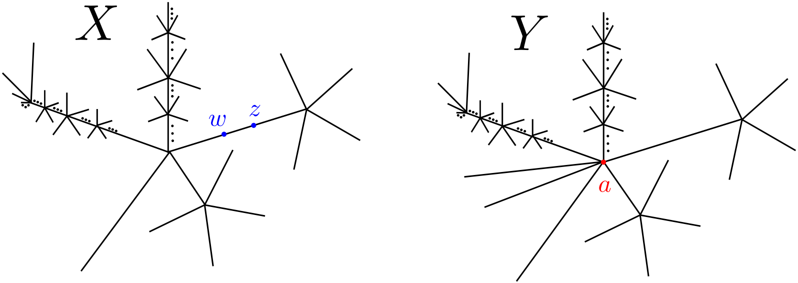
$a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{MH}(Y)$.
$ord(a, Y) = m \geq 5 \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2}$.
$w, z \in X$ such that $h(\{w, z\}) = \{a\} \Rightarrow \{w, z\} \in \mathcal{MH}(X)$


X
w
z

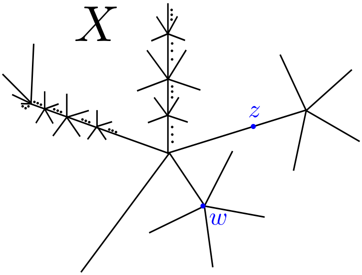

Y
a

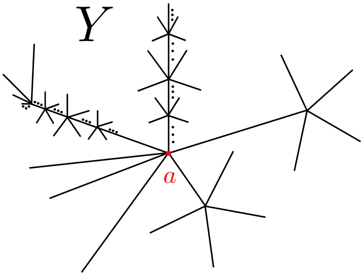


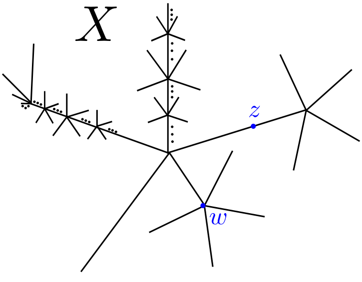
$$w, z \in O(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = 1.$$



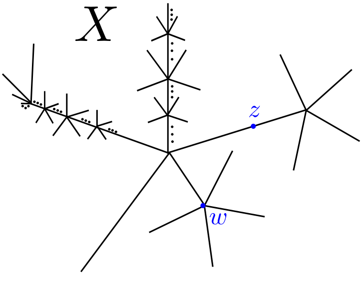
$w, z \in O(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = 1.$
$\Rightarrow \frac{(m-1)(m-2)}{2} = 1 \Rightarrow m = 3.$


X
z
w


Y
a

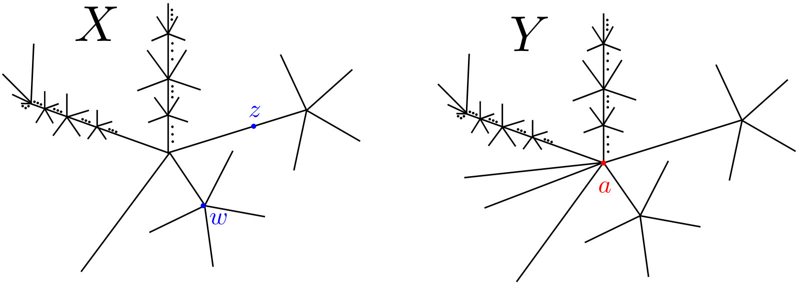


$w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = ord(w, X) - 1.$



$w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = ord(w, X) - 1.$

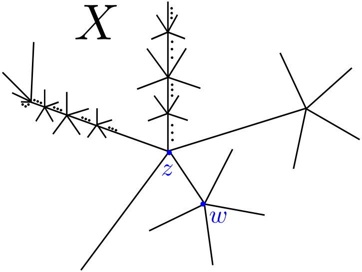
$\Rightarrow \frac{(m-1)(m-2)}{2} + 1 = ord(w, X) \in \Omega_X.$

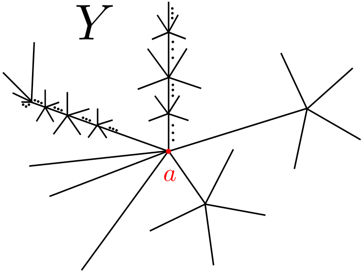


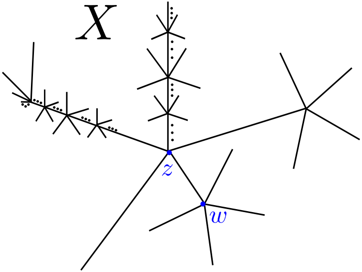
$w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = ord(w, X) - 1.$

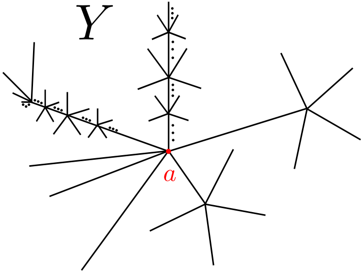
$\Rightarrow \frac{(m-1)(m-2)}{2} + 1 = ord(w, X) \in \Omega_X.$

(2) $\Omega_X \cap \left\{\frac{(j-1)(j-2)}{2} + 1 : j \geq 5\right\} = \emptyset.$

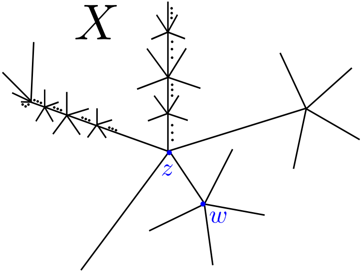

X
z
w

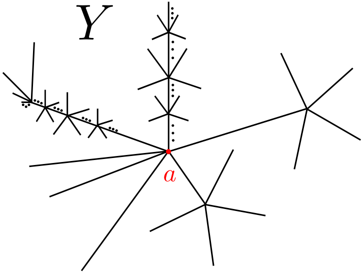

Y
a





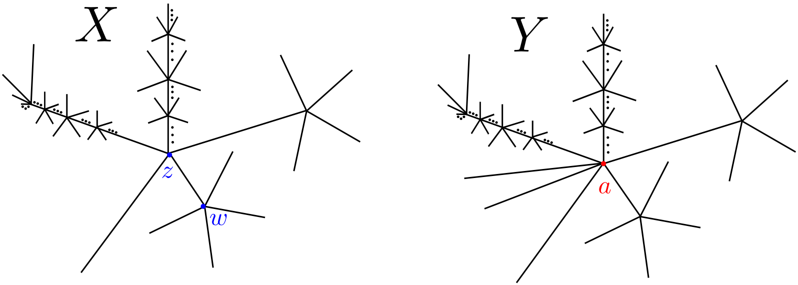
$w, z \in R_N(X), w \neq z$



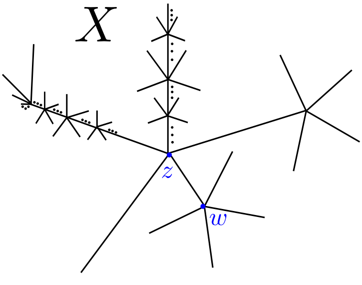


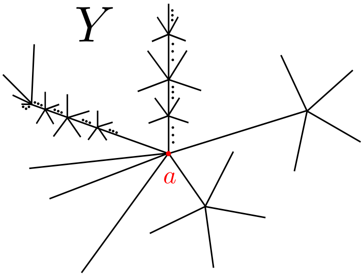
$w, z \in R_N(X), w \neq z$

$\Rightarrow r(\mathcal{F}_2(X) - \{\{w,z\}\}) = (ord(w,X)-1)(ord(z,X)-1)$



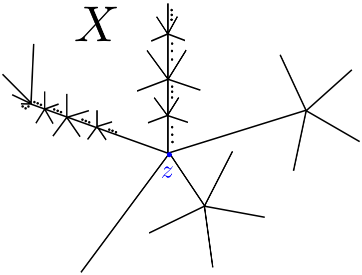
$w, z \in R_N(X), w \neq z$
$\Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = (ord(w, X) - 1)(ord(z, X) - 1)$
$\Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2} \in \{(n-1)(m-1) : n, m \in \Omega_X\}.$

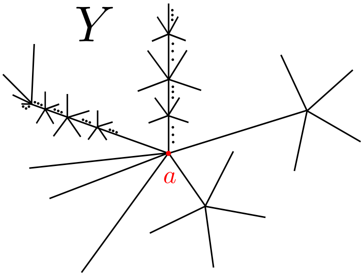




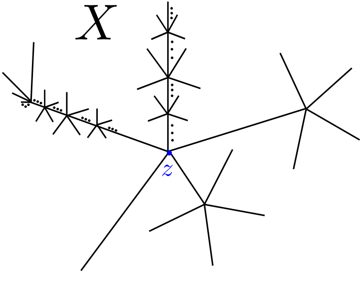
$w, z \in R_N(X), w \neq z$
$\Rightarrow r(\mathcal{F}_2(X) - \{\{w,z\}\}) = (ord(w,X)-1)(ord(z,X)-1)$
$\Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2} \in \{(n-1)(m-1) : n, m \in \Omega_X\}.$

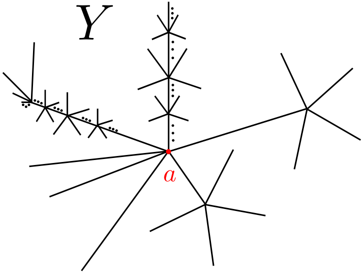
(I) $\{(n-1)(m-1) : n, m \in \Omega_X\} \cap \left\{ \frac{(j-1)(j-2)}{2} : j \geq 5 \right\} = \emptyset.$



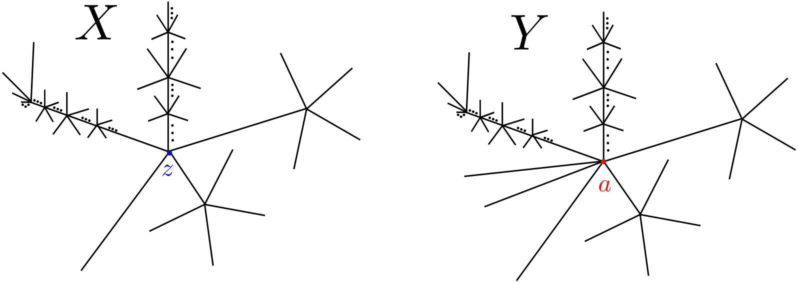


$$w = z \in R_N(X)$$



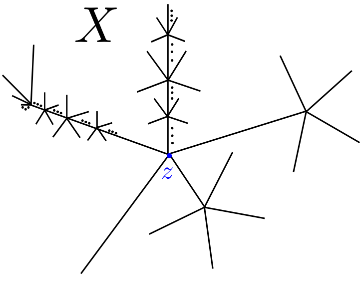


$$ord(z,X)=n \Rightarrow r(\mathcal{F}_2(X)-\{\{z\}\})=\frac{(n-1)(n-2)}{2}$$



$$ord(z, X) = n \Rightarrow r(\mathcal{F}_2(X) - \{\{z\}\}) = \frac{(n-1)(n-2)}{2}$$

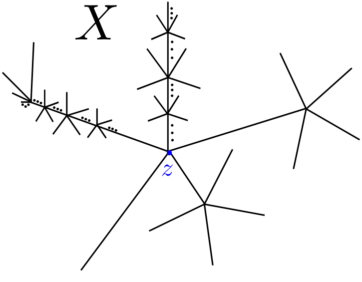
$$r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-1)(m-2)}{2}$$



$$ord(z, X) = n \Rightarrow r(\mathcal{F}_2(X) - \{\{z\}\}) = \frac{(n-1)(n-2)}{2}$$

$$r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-1)(m-2)}{2}$$

$$\Rightarrow \frac{(n-1)(n-2)}{2} = \frac{(m-1)(m-2)}{2} \Rightarrow m = n$$



$$\mathcal{F}_1(R_N(Y)) \subseteq h(\mathcal{F}_1(R_N(X)))$$

Theorem

Let X be a dendrite. If $CR(X) = \cap\{Z \in \mathcal{C}(X) : R_N(X) \subseteq Z\}$, $\Omega_X \subseteq \{5, 6, \ldots\}$ *and either* $|\Omega_X| = 1$ *or*

1. $\Omega_X \cap \left\{\frac{(j-1)(j-2)}{2} + 1 : j \geq 5\right\} = \emptyset$,
2. $\{(n-1)(m-1) : n, m \in \Omega_X\} \cap \left\{\frac{(j-1)(j-2)}{2} : j \geq 5\right\} = \emptyset$,

then X has unique second symmetric product.

Thank you!