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A Compact Minimal Space Whose Cartesian Square Is Not Minimal

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A class of compact minimal spaces whose Cartesian squares are not minimal

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Outline

1. Introduction with examples.
2. New classes of spaces that admit minimal noninvertible maps.
3. A class of minimal spaces whose Cartesian powers are not minimal.
4. Questions.

Note: the focus of this talk will be more on topology of minimal spaces, rather than dynamical properties of minimal maps. Both theories are very rich and diverse in examples.

Minimal maps and spaces

G. D. Birkhoff, *Quelques théorémes sur le mouvement des systémes dynamiques*, Bulletin de la Socićé mathématiques de France, 40 (1912), 305-323:

Given a compact metric space X, a map $f : X \to X$ is called **minimal** if for any closed set $A \subset X$ such $f(A) \subseteq A$ we must have $A = X$ or $A = \emptyset$.

Equivalently, f is said to be **minimal** if the forward orbit $\{f^n(x) : n = 1, 2, \ldots\}$ is dense in X.

In such a case X is called a **minimal space**.

What are minimal spaces?

- ***Any dynamical system*** on a compact space ***contains*** minimal subsystems
- Minimal sets/systems are ***building blocks*** for more complicated ones
- ***Periodic orbits*** are among the simplest examples
- Whether a given space admits a minimal map is still ***unknown for large classes of spaces***
- For ***n-manifolds*** the question is fully answered only for $n < 3$
- In higher dimensions mainly **isolated examples** are known
- If X is minimal, and D is its **decomposition into connected components** then the quotient space X/D must be the **Cantor set**, or a **finite set**.

Toy Models

Example 1. ***(Irrational rotations)*** *Let* $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ *be the unit circle and* $\alpha \notin \mathbb{Q}$*. Then the irrational rotation*

$$f(x) = x + \alpha$$

is minimal.

Example 2. ***(Denjoy homeomorphisms)*** *The irrational rotation* f *can be modified by a "“blow-up”" of an orbit to form a minimal Cantor set homeomorphism.*

Denjoy homeomorphisms

The process of inserting the intervals I_n into $S^1(1)$ to obtain a new circle $S^1(1+a)$ is expressed formally by a continuous map $g\colon S^1(1+a) \to S^1(1)$ which collapses each interval $I_n \subset S^1(1+a)$ to the corresponding point $x_n \in S^1(1)$ and is one-to-one outside I_*. We choose $a_n = \text{length}(I_n) > 0$ to satisfy

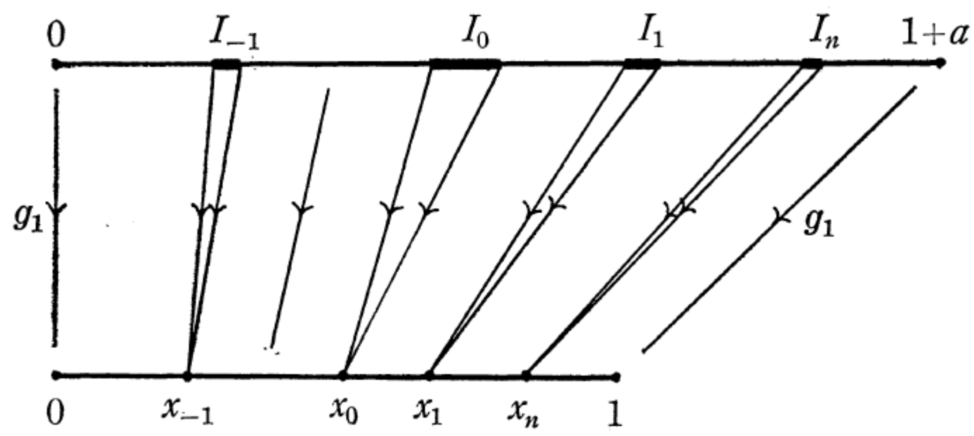
(1) $$a = \sum_{n\in\mathbf{Z}} a_n < \infty$$

so that the disjoint intervals I_n will all fit into $S^1(1+a)$ and

(2) $$\lim_{n\to\pm\infty} a_{n+1}/a_n = 1$$

so that f can be C^1. For example, $a_n = (1+n^2)^{-1}$ suffices. Then the map g is induced by the continuous map $g_1\colon [0, 1+a] \to [0, 1]$ defined (see Figure 4)

(3) $$g_1(y) = \limsup\{x_n \mid x_n + \textstyle\sum_{x_k<x_n} a_k < y\}\ .$$



2-adic odometer

Example 3. *Define a Cantor set homeomorphism* $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ *by "add one and carry":*

$$\sigma(0,\ldots,0,1,s_k,s_{k+1},\ldots) = (0,\ldots,0,1,s_{k+1},s_{k+2},\ldots),$$

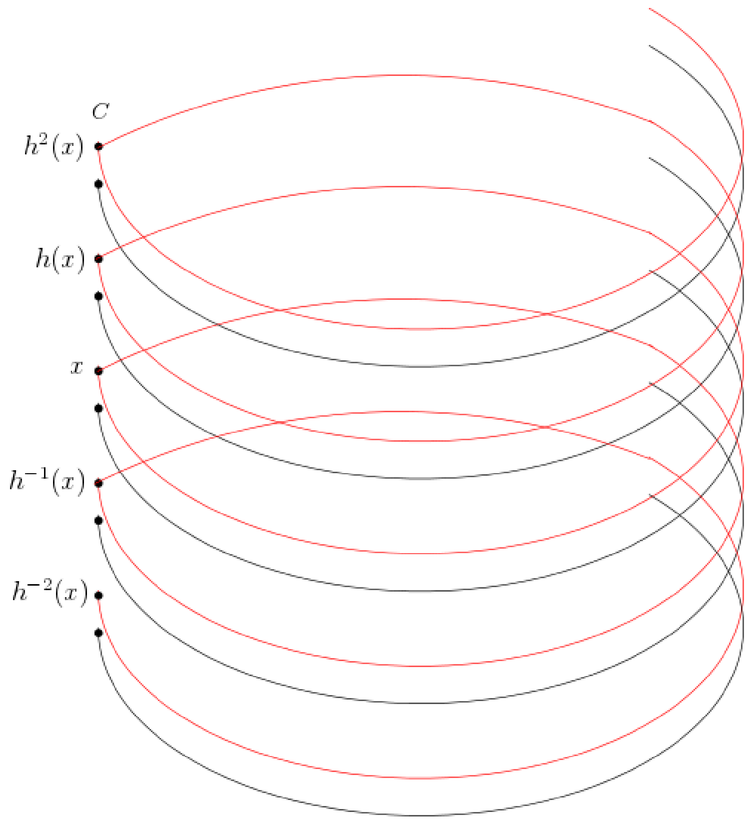
and

$$\sigma(\vec{1}) = \sigma(1,1,1,\ldots) = (0,0,0,\ldots) = \sigma(\vec{0}).$$

The homeomorphism σ *is minimal.*

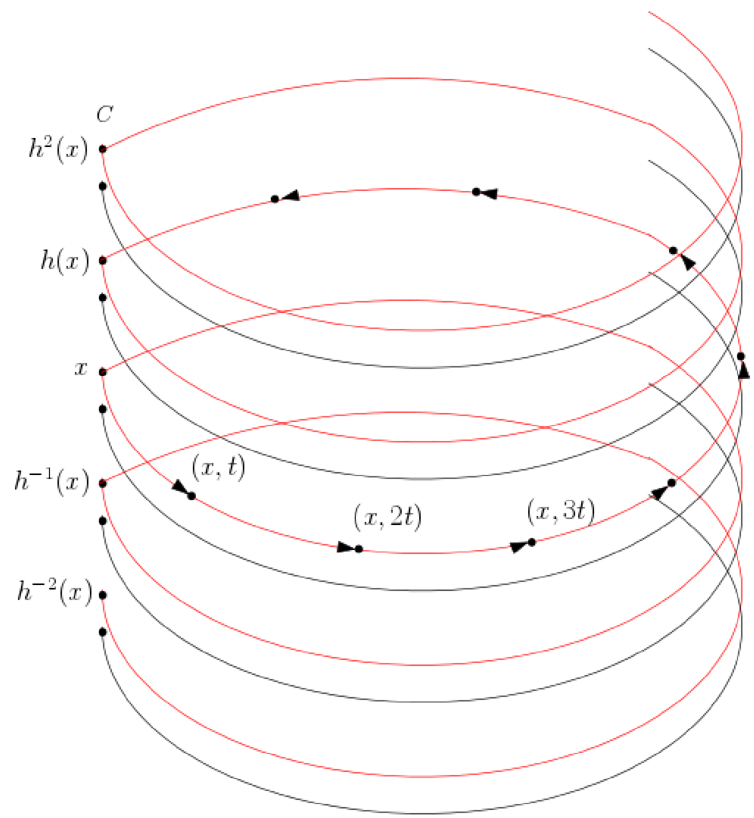
Suspensions

Let $h: C \to C$ be a minimal homeomorphism of a compact metric space C. The **suspension** of (C, h) is the space $X = C \times \mathbb{R}/_{\sim}$, where $\sim$ is the equivalence relation given by: $(x, y) \sim (p, q)$ if $y - q \in \mathbb{Z}$ and $p = h^{-y+q}(x)$.

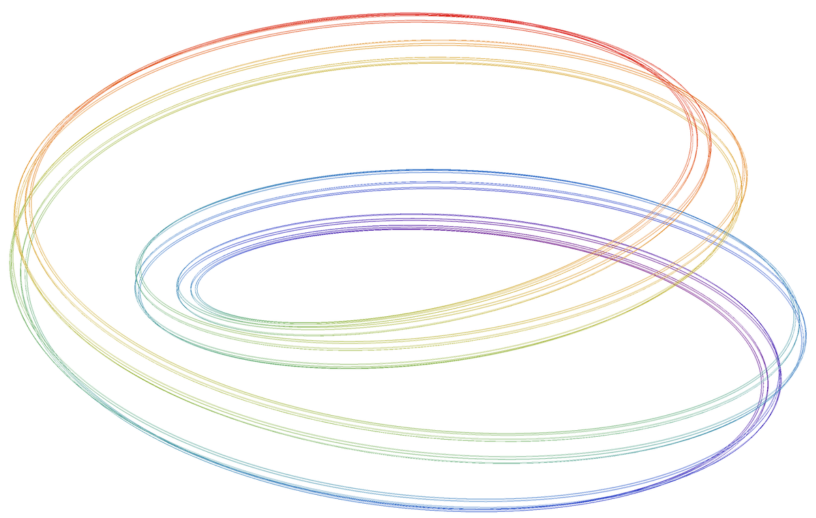


Suspension flows

The **suspension flow** defined by h is the continuous flow induced on X given by $\phi_t(x,s) = (x, s+t)/_\sim$. Since the orbits of h are dense, the flow orbits are dense in X, and so X is a continuum. For a generic choice of parameters t the flow ϕ_t is minimal.



Example 4. *When h is the 2-adic odometer then the suspension is the 2-adic solenoid.*



Questions

(Q1) Is minimality with respect to homeomorphisms preserved under Cartesian product in the class of compact spaces?

(Q2) What spaces admit minimal noninvertible maps?

What spaces admit minimal noninvertible maps?

Question:(Auslander, 1970s) Do there exist minimal noninvertble maps?

- (1979: Auslander, Katznelson) No, on $\mathbb{S}^1$.
- (1980: Auslander, Yorke) Yes, on the Cantor set.
- (2001: Kolyada, Snoha, Trofimchuk) Any minimal skew product homeomorphism of the 2-torus having an asymptotic pair of points has an almost 1-1 factor which is a noninvertible minimal map.
- (2015: Tywoniuk) Yes, on solenoids.

Question:(Bruin, Kolyada, Snoha 2002) Is $\mathbb{S}^1$ the only compact and connected space (continuum) that admit minimal homeomorphisms, but no minimal noninvertble maps?

- (2016: Downarowicz, Snoha, Tywoniuk) No. There are other 1-dimensional continua with this property.

Slovak spaces.
(2016: Downarowicz, Snoha and Tywoniuk)

We say that a compact X is a **Slovak space** if its homeomorphism group

$$H(X) = \{h^n : n \in \mathbb{Z}\},$$

where h is a minimal homeomorphism.

Downarowicz, Snoha and Tywoniuk showed that there exist Slovak spaces that admit minimal homeomorphisms but no minimal noninvertible maps.

Noninvertible minimal maps from aperiodic minimal flows

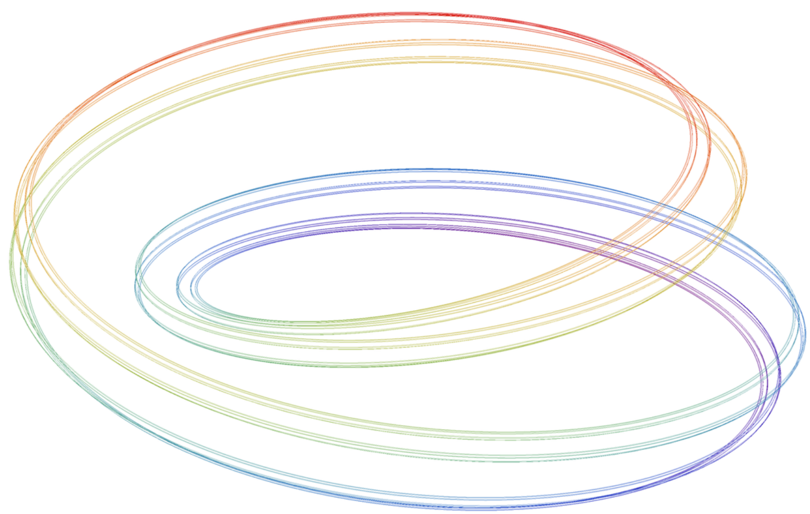
Here we present a new class of spaces that admit minimal nonivertible maps.

Theorem 5. *(B.,Clark,Oprocha) Let* $\phi: M \times \mathbb{R} \to M$ *be a continuous, aperiodic minimal flow on the compact, finite–dimensional metric space* M. *Then* M *admits a noninvertible minimal map.*

Corollary 6. *Suppose that* ϕ *is a suspension flow defined by a minimal homeomorphism* $h: C \to C$ *on an infinite, finite–dimensional compact metric space* C *and let* X *be the phase space of* ϕ. *Then* X *is a continuum that admits a noninvertible minimal map.*

Applications

Example 7. *Let ϕ be the suspension flow of an adding machine and X be its phase space. Then X is a **solenoid**. By Corollary 6 we obtain that X admits a noninvertible dynamical system.*



Example 8. *Consider a Denjoy minimal Cantor set homeomorphism h_α. Let ϕ_α be the suspension flow of h_α and $X(\alpha)$ be its phase space. Then $X(\alpha)$ is a so-called **Denjoy continuum**. By Corollary 6 we obtain that $X(\alpha)$ admits a noninvertible minimal map.*

Example 9. *Let Σ be a **Kuperberg Minimal Set** of a smooth flow ϕ on $\mathbb{S}^3$ without a closed orbit (Smooth Counterexamples to Seifert's Conjecture). Σ is aperiodic, and so it admits a noninvertible minimal map $f\colon \Sigma \to \Sigma$.*

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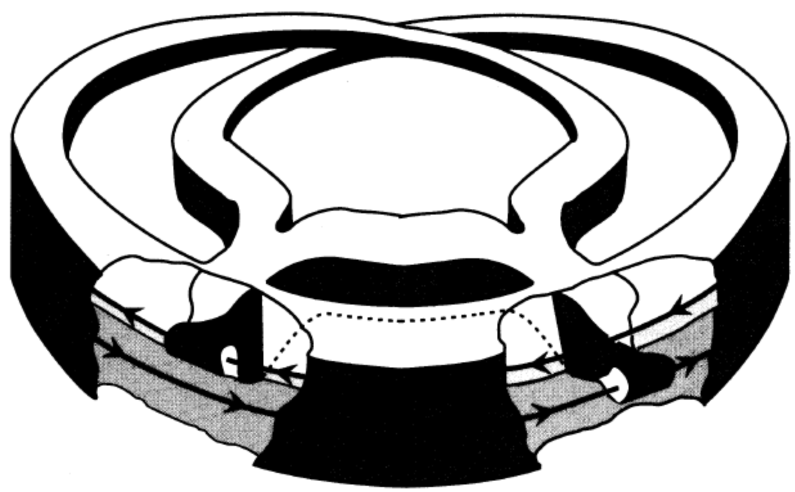


FIGURE 6. A C^∞ aperiodic plug $\widehat{\mathcal{W}}$

Is minimality with respect to homeomorphisms preserved under Cartesian product in the class of compact spaces?

Observation

No product of a homeomorphism $h : X \to X$ with itself

$$(h, h) : X \times X \to X \times X$$

is minimal, by the fact that it keeps the diagonal

$$\Delta = \{(x, x) : x \in X\}$$

invariant.

Toy Models

1. If X is a **Cantor set** then $X \times X$ is a **Cantor set** as well, so minimality is preserved.

2. If $X = \mathbb{S}^1$ then $X \times X = \mathbb{T}^2$ and

$$H(x, y) = (x + \alpha, y + \beta)$$

is minimal if and only if $1, \alpha, \beta$ are $\mathbb{Q}$-independent.

Here we shall show that a **Cartesian power of a minimal spaces need not be minimal**.

Theorem 10. *(B.,Clark,Oprocha)There exists a compact connected metric minimal space* Y *such that* $Y \times Y$ *is not minimal.*

Almost Slovak space

A compact space X is an **almost Slovak space** if its homeomorphism group

$$\mathsf{H}(X) = \mathsf{H}_+(X) \cup \mathsf{H}_-(X),$$

with

$$\mathsf{H}_+(X) \cap \mathsf{H}_-(X) = \{\mathsf{id}_X\},$$

where $\mathsf{H}_+(X)$ is cyclic and generated by a minimal homeomorphism, and for every $g \in \mathsf{H}_-(X)$ we have $g^2 \in \mathsf{H}_+(X)$.

The Pseudo-circle

The *pseudo-arc* P is defined as the unique ***arc-like*** continuum that is ***hereditarily indecomposable***.

- P is ***arc-like*** means that for each $\epsilon > 0$ there exists a map $f_\epsilon : P \to [0, 1]$ with $diam(f^{-1}(t)) < \epsilon$ for every t; alternatively, $P = \lim_{\leftarrow}\{[0, 1], f_i\}$

- P is ***indecomposable*** means that it ***does not decompose*** into the union of ***two proper subcontinua***

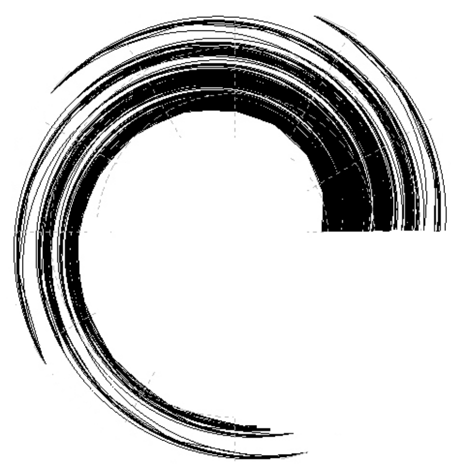
- P is ***hereditarily indecomposable*** means that all subcontinua of P are indecomposable

The *pseudo-circle* is defined as the unique ***circle-like***, ***hereditarily indecomposable*** and **plane separating** continuum.

The Pseudo-arc

The pseudo-arc P may be considered as a very ***bad fractal***, as it is hereditarily equivalent, and so it has a self-similarity feature.

Hereditary equivalence means that every subcontinuum of P is homeomorphic to P.



Note: No indecomposable continuum is a continuous image of $[0, 1]$!

Pseudo-arcs and Pseudo-circles in Dynamics (select results)

- (1982) Handel: pseudo-circle as an attracting minimal set of a C^∞-smooth diffeomorphism of the plane
- (1996) Kennedy&Yorke: constructed a C^∞ diffeomorphism on a 7-manifold which has an invariant set with an uncontable number of pseudocircle components and is stable to C^1 perturbations
- (2010) Chéritat: pseudo-circle as a boundary of a Siegel disk of a holomorphic map
- (2014) B.&Oprocha: pseudo-circle as a Birkhoff-type attractor on the 2-torus
- (2016) Rempe-Gillen: pseudo-arc as a compactification Julia set component of entire functions of disjoint type

Theorem 11. *(B.,Clark,Oprocha) Let* (C, h) *be a minimal homeomorphism of a Cantor set* C*. There exists a minimal suspension* (X, F) *of* (C, h)*, a continuum* Y*, a minimal homeomorphism* (Y, H) *and a factor map* $\pi: (Y, H) \to (X, F)$ *such that:*

(i) π *is almost 1-1,*

(ii) all non-singleton fibers $\pi^{-1}(q)$ *are pseudo-arcs,*

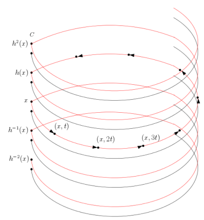
(iii) there exists a composant $W \subset Y$ *such that if* $|\pi^{-1}(x)| > 1$ *then* $\pi^{-1}(x) \subset W$*.*

(iv) $\lim_{|i|\to\infty} H^i(\pi^{-1}(x)) = 0$ *for all* x

Outline of the proof:

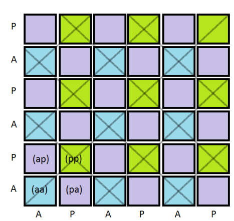
Theorem. *(B.,Clark,Oprocha)There exists a compact connected metric minimal space Y such that $Y \times Y$ is not minimal.*

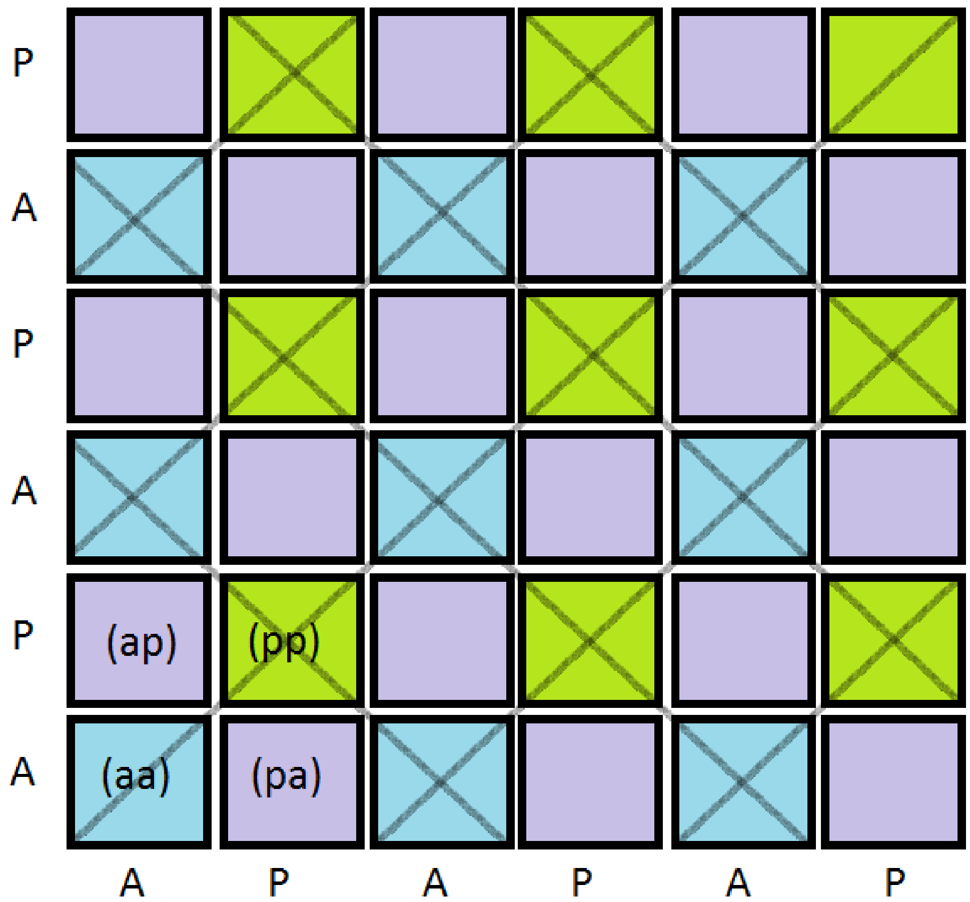
1. Start with a **minimal suspension flow homeomorphism**.



2. Perform a **"surgery"** inserting obstacles in place of one of the orbits.

3. The resulting space will have an **almost cyclic homeomorphism group** and will be **factorwise rigid**.





The square $W \times W$ of the special composant W

- **Question 1:** Does there exist a minimal space Y, such that Y^m is minimal for some $m > 1$, but Y^k is not minimal for some $k > m$?

- **Question 2:** Do the almost Slovak spaces presented here embed into the 2-torus $\mathbb{T}^2$? Can their minimal homeomorphisms be extended to homeomorphisms of $\mathbb{T}^2$?

- **Question 3:** Does the pseudo-circle admit a minimal noninvertible map?

- **Question 4:** Does the pseudo-circle admit a (minimal) homeomorphism semi-conjugate to an irrational rotation of $\mathbb{S}^1$?

- **Question 5:** Does the pseudo-arc with a point removed admit a minimal homeomorphism?

Thank You for Your Attention