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Topologically Homogeneous Continua, Isometrically Homogeneous Continua, and the Pseudo-Arc

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Topologically Homogeneous Continua, Isometrically Homogeneous continua, and the Pseudo-arc

Janusz R. Prajs

Department of Mathematics and Statistics
California State University, Sacramento
University of Opole, Poland

32nd Summer Conference on Topology and Its Applications
University of Dayton, OH

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We use accumulated knowledge on topologically homogeneous continua, and in particular on the pseudo-arc, to investigate the properties of isometrically homogeneous continua.

About This Talk

This talk is a review of six result. There is an entire story behind each of these results. For each result, telling the story and only sketching the proof would fill out entire talk. I make no effort to do that.

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This talk is addressed to non-specialists. I hope they will have a chance to see something about the nature of the presented investigation.

Major concepts are reduced to definitions.

Major results, which took effort of numerous authors over decades, are reduced to single statements.

I do not sketch the proofs. But I can answer some questions regarding the proofs.

Topologically Homogeneous Spaces

(Sierpiński, 1920)

A space X is called **topologically homogeneous** provided for every $x, y \in X$ there is a homeomorphism $h : X \to X$ with $h(x) = y$.

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Homogeneous spaces are within the focus of the study in topology and mathematics.

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In the talk all spaces are metric.

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Studying **homogeneous continua** has been my greatest adventure in research in topology.

Motivation for Studying Homogeneous Continua I

Homogeneous continua is an excellent field for further exploration.

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If focusing on manifolds, homogeneous continua may appear a vast and perhaps too large area. To a general topologist, homogeneous continua may be a rather narrow class. The study of these continua is somewhere halfway between these two viewpoints.

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- Variety of methods have been used to study homogeneous continua.
- Rich in creativity: new ideas, new methods, new important examples have been found.
- Association with space-time modeling.
- Esthetical motivation.
- Still mostly unexplored. There is a lot to discover!

K-connected Continua and Three Results

Let **K** be a continuum.

Definition

A continuum X is **K**-*connected* provided for each pair $x, y \in X$ there is a map $f : \mathbf{K} \to X$ with $x, y \in f(\mathbf{K})$.

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There is no non-degenerate continuum **K** such that for every continuum X, the continuum X is **K**-connected if and only if X is a continuous image of **K**.

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The proof follows the pattern of constructing uncountable collections of path connected continua without common model. (Modified constructions of Waraszkiwicz and Maćkowiak).

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Classic Theorem for Special ‘Elite’ Homogeneous Continua

If a continuum *X* admits a topological group stricture, then *X* is locally connected if and only if *X* is path connected.

Result I: The Case **K** Is an Arc

In this case, our question takes the form of the following question by K. Kuperberg.

Question (K. Kuperberg, 1970s)

Is every homogeneous path connected continuum locally connected? (Or equivalently, is it a continuous image of an arc?)

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Thus our question, for **K** being an arc, has been answered in the negative.

Other **K**'s

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- The union of two continuous images of **K** with non-empty intersection is a continuous image of **K**.
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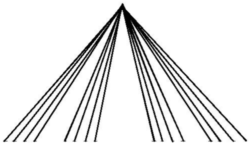
All three conditions are satisfied if **K** is an arc with the class of locally connected continua as continuous images of **K**.
(*Discussion*)

The Case **K** Is the Cantor Fan



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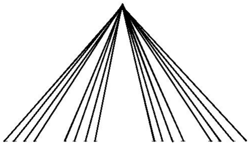

Cantor Fan

If **K** the Cantor fan, **K**-connectedness is equivalent to path connectedness.
Continuous images of the Cantor fan have been studied, and they are known as **uniformly path connected** continua (W. Kupeberg, 1970s), which are the continua admitting a compact collection of paths such that each two points are in the image of at least one path in the collection.

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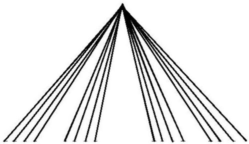

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All three postulated conditions are satisfied if **K** is the Cantor fan.

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Our Question in the Case **K** Is the Cantor Fan

Our general question

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in the case **K** Is the Cantor fan, takes the form of the following question by D. P. Bellamy.

Question (D. P. Bellamy, 1980s)

Is every homogeneous path connected continuum uniformly path connected?

The Case **K** Is the Cantor Fan: Solution

Theorem (Result II, 2016)

Every path connected homogeneous continuum is uniformly path connected, that is, it is a continuous image of the Cantor fan.

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(*Pay attention to dates ...*)

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In fact, a somewhat stronger condition has been proven.

Theorem (2016)

If X is a homogeneous path connected continuum and $x_0 \in X$, then there is a topologically complete separable collection $\mathcal{P}$ of paths $p_\alpha : [0,1] \to X$ with $p_\alpha(0) = x_0$ for each α such that the destination map $p_\alpha \mapsto p_\alpha(1)$ from $\mathcal{P}$ to X is surjective and open.

The Case **K** Is the Cantor Fan: Main Conclusion

Definition

Two continua X and Y are called **continuously equivalent** provided there exist continuous surjections $f : X \to Y$ and $g : Y \to X$.

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Theorem (2016)

If X is a homogeneous path connected continuum, then X is either continuously equivalent to an arc (the case X is locally connected), or continuously equivalent to the Cantor fan (the case X is non-locally connected).

The Pseudo-arc

Definitions

Let X be a continuum:

- X is **decomposable** if there are two proper subcontinua A and B such that $X = A \cup B$.
- X is **indecomposable** if X is not decomposable.
- X is **hereditarily indecomposable** if every subcontinuum of X is indecomposable.
- X is **arc-like** if for every $\varepsilon > 0$ there is an ε-map $f_\varepsilon : X \to [0, 1]$.

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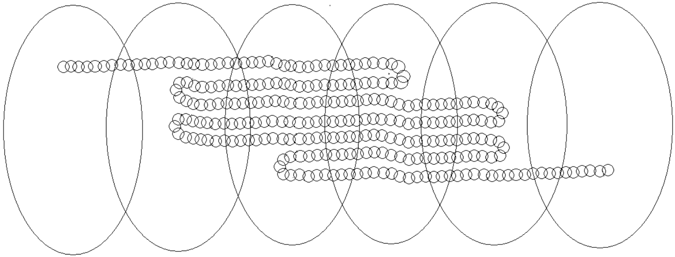
Theorem (R. H. Bing,1950)

All non-degenerate hereditarily indecomposable arc-like continua are mutually homeomorphic.

A hereditarily indecomposable arc-like continuum is called the **pseudo-arc**.

The Pseudo-arc

First example by B. Knaster (1922). In 1948 E. Moise presented another construction, introduced the name *pseudo-arc*, and proved the property that all its subcontinua are homeomorphic to the whole continuum. Shortly afterwards Bing showed that Moise' example is homogeneous.



The two initial steps of the construction of the pseudo-arc

Continuous Images of the Pseudo-arc

Continuous images of the pseudo-arc have been characterized independently by L. Fearnley and A. Lelek in early 1960s as **weakly chainable continua**. The significance of the class of weakly chainable continua is similar to that of locally connected continua. However weak chainability is not a local property, and it is less intuitive.

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Continuous maps from the pseudo-arc to a space are called **pseudo-paths**. They introduce the structure of **pseudo-path components** of a space.

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Theorem (Result III, 2015)

Every pseudo-path connected homogeneous continuum is weakly chainable.

Isometrically Homogeneous Continua: Definitions

Let X be a space and G a topological group with the identity e.

A map $F : G \times X \to X$ is an **action** of G on X provided $F(e, x) = x$ and F(gh,x)=F(g,F(h,x)) for all $x \in X$ and $g, h \in G$.

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Compact metric groups, as topological spaces, are a sub-category of isometrically homogeneous compact spaces.

The two following propositions are well-known and classic.

Proposition

A continuum X is homogeneous if and only if X admits a transitive action by a Polish group (that is, by a separable, topologically complete metric group).

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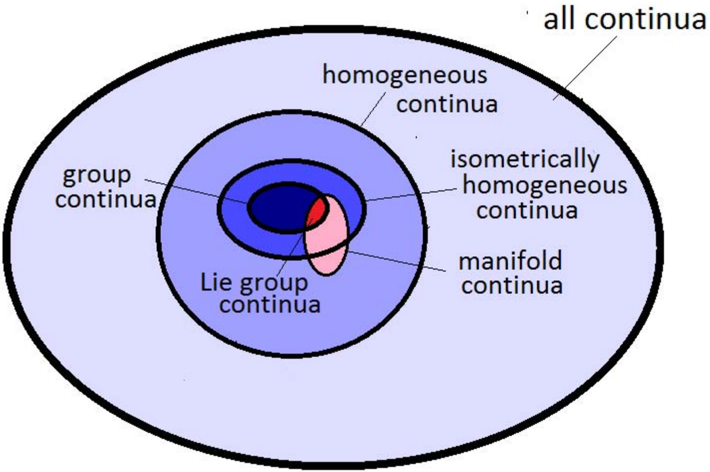
Thus isometrically homogenous continua are a narrow, 'elite' class among all (topologically) homogeneous continua.

The Stage: Schematic Picture



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Isometrically Homogeneous Continua: the Question

For isometrically homogeneous continua we ask the question discussed before. Let **K** be a continuum.

Question

Suppose X is a **K**-connected isometrically homogeneous continuum. Is X necessarily a continuous image of **K** ?

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The next result shows that our question, for isometrically homogeneous continua, has positive answer in the three considered cases for **K**. Not only that. What is more surprising is that for isometrically homogeneous continua **K**-connectedness, in the three cases for **K**, are equivalent conditions.

Isometrically Homogeneous Continua: Result IV

The following is a generalization and extension of the classic result for compact connected topological groups mentioned earlier in this talk.

Theorem (Result IV, 2015)

Suppose X is an isometrically homogeneous continuum. Then the following conditions are equivalent:

1. **X is pseudo-path connected.**
2. **X is weakly chainable.**
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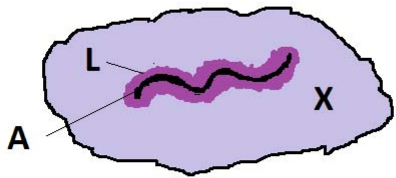
(1) $\Leftrightarrow$ (2) the only equivalence holding for all homogeneous continua.

Ample Continua: Definition



Definition

A continuum A in a space X is called **ample** if for every $\varepsilon > 0$ the ε-neighborhood of A contains a continuum L such that $A \subset \mathrm{Int} L$.

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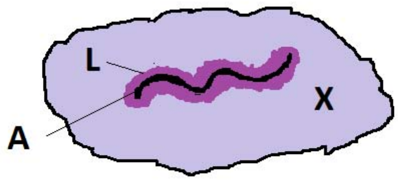

Definition

A continuum *A* in a space *X* is called **ample** if for every $\varepsilon > 0$ the ε-neighborhood of *A* contains a continuum *L* such that $A \subset \mathrm{Int} L$.

The continuum *A* may be 'thin' or not, but it can be slightly 'thickened' to a continuum having *A* in its interior.

Ample Diagonal of $X \times X$

The following two results are in a modified version. The authors did not use the term *ample continuum*.

Theorem (D. P. Bellamy and J. Łysko, 2014)

The product $P \times P$ of two (or more) pseudo-arcs has ample diagonal $\Delta = \{(x, x) \mid x \in P\}$.

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Theorem (D. P. Bellamy and J. Łysko, 2014)

If a continuum X is a topological group, then X is locally connected if and only if the diagonal $\Delta = \{(x, x) \,|\, x \in X\}$ is ample in $X \times X$.

Ample Diagonal of $X \times X$ and Isometric Homogeneity

Homogenous continua X that are very far from being locally connected, such as the pseudo-arc, can have ample diagonal in $X \times X$ by the first result of Bellamy and Łysko. If we additionally assume isometric homogeneity, it is not the case any more.

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The following is an addition to the result IV and a generalization of the second result by Bellamy and Łysko.

Theorem (Theorem, an addition to Result IV)

If X is an isometrically homogeneous continuum, then X is locally connected if and only if the diagonal in the product $X \times X$ is ample.

Ample Diagonal of $X \times X$ and Topological Homogeneity

Bellamy and Łysko have shown that the product of two pseudo-arcs has ample diagonal. It turns out that a much more general statement holds, which is the next result.

Theorem (Theorem, Result V)

Let X be a homogeneous, weakly chainable continuum. Then the diagonal in the product $X \times X$ is ample.

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Bellamy and Łysko have shown that the product of two pseudo-arcs has ample diagonal. It turns out that a much more general statement holds, which is the next result.

Theorem (Theorem, Result V)

Let X be a homogeneous, weakly chainable continuum. Then the diagonal in the product $\mathbf{X \times X}$ is ample.

The question of the converse to this result is intriguing.

Question

Suppose X is a homogeneous continuum having the diagonal in the product $X \times X$ ample. Is X necessarily a weakly chainable continuum ? In other words, is X a continuous image of the pseudo-arc ?

Aposyndesis: Definitions

A continuum X is called (a) **aposyndetic** ((b) **mutually aposyndetic**) provided for all $x, y \in X$ with $x \neq y$, respectively:

- (a) there exists a continuum $K \subset X$ such that $x \in \mathrm{Int} K$ and $y \notin K$.
- (b) there exist disjoint continua K, L with $x \in \mathrm{Int} K$ and $y \in \mathrm{Int} L$.

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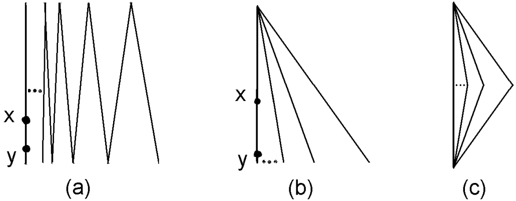

(a) Neither aposyndetic at x with respect to y, nor at y with respect to x. (b) Aposyndetic at x with respect to y but not at y with respect to x. (c) Aposyndetic.

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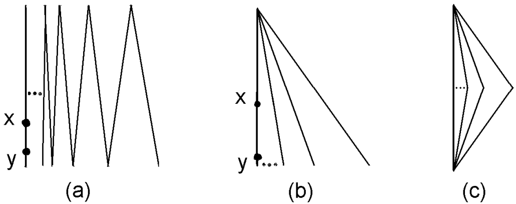

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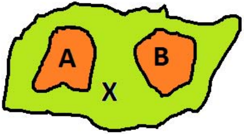
Only the continuum (c) is aposyndetic. (c) is also mutually aposyndetic.

Semi-indecomposable Continua

Definition. A continuum X is **semi-indecomposable** provided for every two disjoint subcontinua of X at least one has empty interior.



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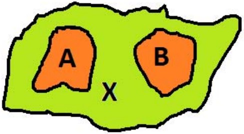

Semi-indecomposable continua have been introduced by Charls L. Hagopian under the name *strictly non-mutually aposyndetic continua*.

Example(C.L. Hagopian)

The product $P \times P$ of two pseudo-arcs is an example of a homogeneous, semi-indecomposable, aposyndetic (and thus decomposable) homogeneous continuum.

Two Pairs of Opposites

Aposyndesis and **indecomposability** are a pair of opposite properties in the sense of the statement below. (They are not negations of each other!)

Let X be a continuum.

- X is aposyndetic if and only if X is aposyndetic at each point with respect to any other point.
- X is indecomposable if and only if X is **not** aposyndetic at each point with respect to any other point.

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Similarly, **mutual aposyndesis** and **semi-indecomposability** are a pair of opposite properties. (They are not negations of each other!)

Let X be a continuum.

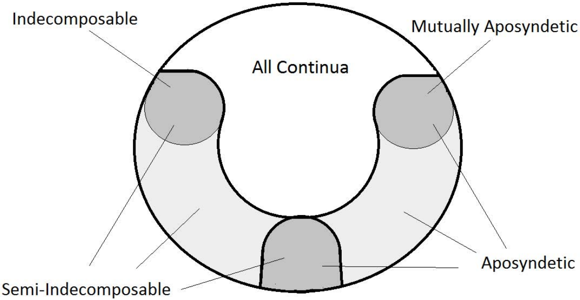
- X mutually aposyndetic if and only if X is mutually aposyndetic between every two different points in X.
- X semi-indecomposable if and only if X is **not** mutually aposyndetic between every two different points in X.

Result VI

Theorem (2016, Result VI)

If X is an isometrically homogeneous continuum, then exactly one of the following conditions holds:

1. **X is indecomposable.**
2. **X is semi-indecomposable and aposyndetic.**
3. **X is mutually aposyndetic.**



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Classify all semi-indecomposable, aposyndetic, compact connected topological groups.

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Question

Does there exist a non-degenerate homogeneous hereditarily non-weakly chainable continuum ?