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Relationships between Topological Properties of X and Algebraic Properties of Intermediate Rings A(X)

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Setting

Tychonoff spaces

All topological spaces X we consider are Tychonoff spaces: completely regular Hausdorff spaces.

Intermediate rings of continuous functions

All ring we consider are intermediate rings: subrings $A(X)$ of $C(X)$ (ring of all real-valued continuous functions on X) containing $C^*(X)$ (ring of *bounded* real-valued continuous functions on X).

Let

- $\mathcal{C} = \{C(X) \mid X \text{ is a Tychonoff space}\}$
- $\mathcal{C}^* = \{C^*(X) \mid X \text{ is a Tychonoff space}\}$
- $\mathcal{A}$ be the set of all intermediate rings

Example of Non-Trivial Intermediate Ring

Take $X = \mathbb{R}$ and $A(X) = \langle C^*(X), e^x \rangle$.
Note that functions in $A(X)$ have the form

$$f = c_0(x) + c_1(x)e^x + \cdots + c_n(x)e^{nx}, \qquad c_i(x) \in C^*(X).$$

Then

$$C^*(X) \subsetneq A(X) \subsetneq C(X),$$

since all functions in $A(X)$ remain bounded as $x \to -\infty$.

General goals

- A topological property is a class $\mathcal{T}$ of Tychonoff spaces closed under homeomorphism.
- An algebraic property is a class $\mathcal{P}$ of rings closed under ring isomorphism.

Main goal

Relate topological properties of X with algebraic properties of $A(X)$ in various ways.

We begin by examining two topological properties that were originally characterized algebraically: P-spaces and F-spaces.

P-spaces

A zero-set is a set of the form $Z(f) = \{x \in X \mid f(x) = 0\}$ for some $f \in C(X)$.

Definition (P-space)

A Tychonoff space X is a P-space if every zero-set in X is open.

Example

Trivially, any discrete space is a P-space.

("P-space" stands for "pseudo-discrete", though "pseudo-discrete" has taken other meanings.)

P-spaces and algebraic properties

P-spaces have traditionally been defined in terms of algebraic properties of $C(X)$.

Algebraic characterization (defn. in Gillman & Henriksen 1954)

X is a P-space if and only if every prime ideal in $C(X)$ is maximal.

For intermediate rings $A(X)$ of continuous functions, the following are equivalent:

- every prime ideal in $A(X)$ is maximal
- for every $f \in A(X)$, there exists $g \in A(X)$ such that $f = f^2 g$.

A ring with the latter (and hence former) property is called (von Neumann) regular.

This characterization does not extend to intermediate rings

Theorem

If $A(X) \subsetneq C(X)$ is an intermediate ring, then there exists a prime ideal in $A(X)$ that is not maximal.

This property characterizes $C(X)$ among intermediate rings when X is a P-space:

Theorem

If X is a P-space and $A(X)$ is an intermediate ring, then $A(X) = C(X)$ if and only if $A(X)$ is regular.

F-space

Definition (F-space)

A Tychonoff space is an F-space if every two disjoint cozero-sets are completely separated.

- A cozero-set is the complement of a zero-set.
- Two sets A and B are completely separated if there exists a function f, such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$.

Algebraic characterization of F-spaces

An intermediate ring $A(X)$ is Bézout if every finitely generated ideal in $A(X)$ is principal.

Algebraic Characterization (defn. in Gillman & Henriksen 1956)

X is an F-space if and only if $C(X)$ is a Bézout ring.

(Theorem in Gillman & Jerison textbook)

X is an F-space if and only if $C^*(X)$ is a Bézout ring.

Murray, Sack, Watson

Let $A(X)$ be any intermediate ring. X is an F-space if and only if $A(X)$ is a Bézout ring.

Bézout rings "fully correspond" to F-spaces.

Corresponds

Let

- $\mathcal{P}$ be an algebraic property
- $\mathcal{T}$ be a topological property
- $\mathcal{Q}$ be an arbitrary class of intermediate rings.

Definition

Property $\mathcal{P}$ corresponds to $\mathcal{T}$ among $\mathcal{Q}$ iff for any intermediate ring $A(X) \in \mathcal{Q}$,

$$A(X) \in \mathcal{P} \text{ if and only if } X \in \mathcal{T}.$$

Definition

Property $\mathcal{P}$ fully corresponds to $\mathcal{T}$ iff $\mathcal{P}$ corresponds to $\mathcal{T}$ among all intermediate rings.

Examples of Corresponds

- Being a regular ring corresponds to P-spaces among the class of rings $C(X)$
- Being a regular ring does not fully correspond to P-spaces (among all intermediate rings)
- Being a Bézout ring fully correspond to F-spaces

Refined goals

What properties transfer through certain relations on algebraic and topological structures?

We examine:

- Topological properties invariant under taking A-compact extensions
- Algebraic properties invariant under taking ring localization

Such invariance can help us understand how topological and algebraic properties are related.

z-ultrafilters and Stone-Čech compactification

- A z-filter is a filter on the lattice of zero-sets
- A z-ultrafilter is a maximal z-filter
- The Stone-Čech compactification βX of X is the set of all z-ultrafilters topologized by the hull-kernel topology:
  - (kernel) $k\mathfrak{U} = \bigcap_{\mathcal{U}\in\mathfrak{U}} \mathcal{U}$
    (from set $\mathfrak{U}$ of z-ultrafilters to z-filter)
  - (hull) $h\mathcal{F}$ = set of z-ultrafilters containing $\mathcal{F}$
    (from z-filter $\mathcal{F}$ to set of z-ultrafilters)
  - (closure) $\mathrm{cl}_{\beta X}\,\mathfrak{U} = hk\mathfrak{U}$.

A-stable and A-compact

Given an intermediate ring $A(X)$,

- a z-ultrafilter $\mathcal{U}$ is A-stable if every $f \in A(X)$ is bounded on some member $U \in \mathcal{U}$.
- the A-stable hull of a filter $\mathcal{F}$ is $h^A\mathcal{F}$ = set of all A-stable z-ultrafilters containing $\mathcal{F}$.
- the A-compactification of X is $\upsilon_A X$ consisting of all A-stable z-ultrafilters topologized by the A-stable hull kernel topology:

$$\mathrm{cl}_{\upsilon_A X}\,\mathfrak{U} = h^A k\mathfrak{U}.$$

Special cases

$$\upsilon_C X = \upsilon X \text{ (the Hewitt realcompactification)}$$
$$\upsilon_{C^*} X = \beta X \text{ (the Stone-Čech compactification)}$$

Ring of extensions and C-rings

Each $f \in A(X)$ has a continuous extension $f^{\upsilon_A} : \upsilon_A X \to \mathbb{R}$, where

$$f^{\upsilon_A}(p) = \lim_{\mathcal{U}_p} f, \text{ for } p \in \upsilon_A X$$

Definition (Ring of extensions)

$$A(\upsilon_A X) = \{f^{\upsilon_A} \mid f \in A(X)\}.$$

Then $A(X)$ and $A(\upsilon_A X)$ are isomorphic.

Definition

An intermediate ring $A(X)$ is a C-ring if there exists a Tychonoff space Y, such that $A(X)$ is isomorphic to $C(Y)$.

$A(X)$ is a C-ring if and only if $A(\upsilon_A X) = C(\upsilon_A X)$.

Example of intermediate ring that is not a C-ring

A z-ultrafilter $\mathcal{U}$ is free if $\bigcap_{E \in \mathcal{U}} E = \emptyset$

Example

Let $A(\mathbb{N}) = \langle C^*(\mathbb{N}), x \rangle$.

- $\upsilon_A \mathbb{N} = \mathbb{N}$ (no free z-ultrafilter is A-stable)
- but $A(\mathbb{N}) \neq C(\mathbb{N})$ ($e^x \notin A(\mathbb{N})$)

Cohereditary and P-space example

Topological property $\mathcal{T}$ is cohereditary with respect to $A(X)$ provided X has property $\mathcal{T}$ if and only if $\upsilon_A X$ has property $\mathcal{T}$.
$\mathcal{T}$ is fully cohereditary iff $\mathcal{T}$ is cohereditary with respect to all $A(X) \in \mathcal{A}$.

The property of being a P-space is cohereditary with respect to $C(X)$

$C(X)$ is isomorphic to $C(\upsilon X)$. Hence $C(X)$ is regular iff $C(\upsilon X)$ is.

The property of being a P-space is not cohereditary with respect to any intermediate C-ring $A(X) \subsetneq C(X)$.

$$\begin{aligned} X \text{ is a } P\text{-space} &\Rightarrow A(X) \text{ is not regular} \\ &\Rightarrow C(\upsilon_A X) \text{ is not regular} \quad (A(X) \cong C(\upsilon_A X)) \\ &\Rightarrow \upsilon_A X \text{ is not a } P\text{-space} \end{aligned}$$

Coheredtiary and F-space example

Proposition

The property of being an F-space is cohereditary (with respect to any intermediate ring $A(X)$).

$$\begin{aligned} X \text{ is an } F\text{-space} &\Leftrightarrow A(X) \text{ is Bézout} \\ &\Leftrightarrow A(\upsilon_A X) \text{ is Bézout} \qquad (A(X) \cong A(\upsilon_A X)) \\ &\Leftrightarrow \upsilon_A X \text{ is an } F\text{-space} \end{aligned}$$

Relationship between cohereditary and corresponds

Let $\mathcal{P}$ be an algebraic property and $\mathcal{T}$ be a topological property.

Theorem

If $\mathcal{P}$ fully corresponds to $\mathcal{T}$ then $\mathcal{T}$ is fully cohereditary.

Proof.

Suppose $\mathcal{P}$ fully corresponds to $\mathcal{T}$ and let $A(X) \in \mathcal{A}$. Then the following are equivalent:

- $X \in \mathcal{T}$
- $A(X) \in \mathcal{P}$
- $A(\upsilon_A X) \in \mathcal{P}$
- $\upsilon_A X \in \mathcal{T}$,

($\mathcal{T}$ is cohereditary with respect to the arbitrary $A(X)$). □

Relationship between cohereditary and corresponds (cont'd)

Theorem

If $\mathcal{T}$ is fully cohereditary, then the following are equivalent

- *$\mathcal{P}$ corresponds to $\mathcal{T}$ among all rings $C(X) \in \mathcal{C}$.*
- *$\mathcal{P}$ corresponds to $\mathcal{T}$ among all intermediate C-rings.*

Suppose

(1) $\mathcal{T}$ is cohereditary (among all intermediate rings) and

(2) $\mathcal{P}$ corresponds to $\mathcal{T}$ among all rings $C(X) \in \mathcal{C}$.

Then for any C-ring $A(X)$, the following are equivalent

- $A(X) \in \mathcal{P}$
- $C(\upsilon_A X) \in \mathcal{P}$ (as $A(X) \cong C(\upsilon_A X)$)
- $\upsilon_A X \in \mathcal{T}$ by (2)
- $X \in \mathcal{T}$ by (1)

Localization

Given an intermediate ring $A(X)$ and a multiplicatively closed subset $S \subseteq A(X)$, the localization of $A(X)$ with respect to S is

$$S^{-1}A(X) = \{f/s \mid f \in A(X), s \in S\},$$

identifying f/s with g/t when $ft = gs$.

Theorem (Domínguez et al. 1997)

Let $A(X) \in \mathcal{A}$ and let S be the set of bounded units of $A(X)$. Then $A(X) \cong S^{-1}C^*(X)$.

Note that the set of bounded units is multiplicatively closed.

Cohereditary algebraic properties

A set of functions S is saturated if $fg \in S$ implies $f, g \in S$.

Definition

An algebraic property $\mathcal{P}$ is cohereditary if for any saturated multiplicatively closed subset $S \subseteq C^*(X)$. $C^*(X) \in \mathcal{P}$ if and only if $S^{-1}C^*(X) \in \mathcal{P}$ ($S^{-1}C^*(X)$ is isomorphic to a ring in $\mathcal{P}$).

Example

The property of being Bézout is cohereditary.

Relationship between localization and corresponds

Theorem

If $\mathcal{P}$ is cohereditary, then the following are equivalent:

- *$\mathcal{P}$ corresponds to $\mathcal{T}$ among $\mathcal{C}^*$*
- *$\mathcal{P}$ corresponds to $\mathcal{T}$ among $\mathcal{A}$.*

Suppose

(1) $\mathcal{P}$ is cohereditary, and

(2) $\mathcal{P}$ corresponds to $\mathcal{T}$ among $\mathcal{C}^*$.

Then for any intermediate ring $A(X)$, there exists a saturated multiplicatively closed $S \subseteq C^*(X)$ such that $A(X) \cong S^{-1}C^*(X)$. Then, the following are equivalent:

- $A(X) \in \mathcal{P}$
- $S^{-1}C^*(X) \in \mathcal{P}$
- $C^*(X) \in \mathcal{P}$ by (1)
- $X \in \mathcal{T}$ by (2)

Conclusion

- Use notion of corresponds to relate topological properties of X and algebraic properties of $A(X)$.
- Illustrate relationships using topological properties of P-spaces and F-spaces and algebraic properties of regular and Bézout.
- Examined what property preserving topological or algebraic transformations tell us about the relationships among topological and algebraic properties

THANK YOU!