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Shift Maps and Their Variants on Inverse Limits with Set-Valued Functions

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A scheme for constructing examples of inverse limits with set-valued functions

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Generalized Inverse Limits

Suppose X is a compact metric space and $f : X \to 2^X$ is an upper semicontinuous map. Let

$$M = \varprojlim f = \{(x_0, x_1, ...) : x_{i-1} \in f(x_i) \text{ for } i > 0\}.$$

Then M is a *generalized inverse limit*, or equivalently an *inverse limit with set-valued functions*. It is, of course, possible to define these generalized inverse limits more generally (i.e., for X a compact space, or for a sequence $\{f_i\}_{i=1}^{\infty}$ of functions, rather than just one function repeated over and over).

These objects were introduced in 2003 by William Mahavier. They behave very differently than do standard inverse limits.

Bonding maps

Suppose $f : X \to 2^X$ is an upper semicontinuous map.

- The *graph* $G(f) = \{(x, y) \in X \times X : x \in f(y)\}$ [Note that this notation is not the ones most authors currently use, but it is convenient for our purposes.]
- f is upper semicontinuous if and only if the graph $G(f)$ is a closed subset of $X \times X$.
- f is *surjective* if for each $y \in X$, there is some $x \in X$ such that $y \in f(x)$.
- f is called the *bonding map*.

Differences from standard inverse limits

- The (generalized) inverse limit need not be connected, even if the bonding maps have connected graphs. (Ingram example)
- Generalized inverse limits can raise dimension (Hilbert cube example)
- Generalized inverse limits can be a harmonic sequence, or a Cantor sets (Mahavier example) or something "in between" but an important factor here was that the map was not surjective.
- "Finite" generalized inverse limits can be interesting - a "finite" standard inverse limit (on intervals) is JUST topologically an arc.
- Standard inverse limits over intervals are always connected, one-dimensional, and chainable.

Shift on Generalized Inverse Limits

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Suppose $f : X \to 2^X$ is a surjective, upper semicontinuous bonding map. Let $M = \varprojlim f$.

[The *graph* of f is the set of all points (x, y) such that $x \in f(y)$. Saying that f is upper semicontinuous is equivalent to saying that the graph of f is closed in $X \times X$.]

Even though f is not even a function in the usual sense, it induces a continuous function σ from M onto M.

The function σ is called the *shift map* on M, since for $\mathbf{x} = (x_0, x_1, ...) \in M$, $\sigma(\mathbf{x}) = \sigma(x_0, x_1, ...) = (x_1, x_2, ...)$.

Dynamical implications

Def. Suppose X, Y, Z are compact metric spaces, $A \subset X \times Y$, $B \subset Y \times Z$. Define the *Mahavier product* $A \star B$ to be $\{(x, y, z) \in X \times Y \times Z : (x, y) \in A, (y, z) \in B\}$.

Mahavier products make it easy to talk about closed subset of generalized inverse limits, whether they are sub-generalized inverse limits or not. They also make it easy to talk about "finite" generalized inverse limits.

Notation Suppose for each i, X_i is a compact metric space $(i \geq 0)$, and $A_i \subset X_{i-1} \times X_i$ $(i \geq 1)$. Then $\star_{i=1}^{\infty} A_i$ denotes $A_1 \star A_2 \star \cdots$, and if n is a positive integer, then $\star_{i=1}^{n} A_i$ denotes $A_1 \star A_2 \star \cdots \star A_n$.

Note that if $f : X \to 2^X$ is upper semicontinuous and Γ denotes the graph of f, then

$$M = \varprojlim f = \{(x_0, x_1, ...) : x_{i-1} \in f(x_i) \text{ for } i > 0\} = \star_{i=1}^{\infty} \Gamma.$$

Natural ways a set-valued function might arise

Suppose X and Y are compact metric spaces. We consider two natural constructions.

1. (**Bottom**) Suppose $T : Y \to Y$ is a continuous surjection, and $h : Y \to X$ is a continuous surjection. Then $h^{-1} : X \to 2^Y$ is an upper semicontinuous surjective mapping. Let $f = h \circ T \circ h^{-1}$, so $f : X \to 2^X$ is also a surjective upper semicontinuous mapping.
2. (**Top**) Suppose $T : Y \to Y$ is a continuous surjection, and $h : X \to Y$ is a continuous surjection. Then $h^{-1} : Y \to 2^X$ is an upper semicontinuous surjective mapping. Let $f = h^{-1} \circ T \circ h$, so $f : X \to 2^X$ is also a surjective upper semicontinuous mapping.

Situation Bottom

Proposition 1. Suppose situation **Bottom** holds. Then for each $y \in Y$, $h \circ T(y) \in f(h(y))$.

Proposition 2. Suppose property situation **Bottom** holds, and

$M = \varprojlim(X, f) = \star_{i=1}^{\infty} \Gamma(f)$, and $N = \varprojlim(Y, T)$.

Then $h_\infty : N \to M$ defined by $h_\infty(y_0, y_1, \ldots) = (h(y_0), h(y_1), \ldots)$ is a continuous function. If σ_T denotes the shift function on N and σ_f denotes the shift function on M, then $h_\infty \circ \sigma_T = \sigma_f \circ h_\infty$.

$\Gamma(f)$ must be connected!

Proposition 3. Suppose X and Y are continua.

1. Suppose $g : Y \to X$ is a continuous surjection and K is a subcontinuum of $X \times Y$ such that $\varphi(K) = X$. Then $\Gamma(g) \star K$ is a continuum.
2. Suppose $T : Y \to Y$ is a continuous surjection, and $k : Y \to X$ is continuous surjection. Suppose that K is a continuum in $Y \times X$ such that $\varphi(K) = Y$. Then $\Gamma(kT) \star K$ is a continuum in $X \times Y \times X$.
3. Suppose $T : Y \to Y$ is a continuous surjection, and $k : Y \to X$ is continuous surjection. Then $\Gamma(k^{-1})$ is a continuum in $Y \times X$ such that $\varphi(\Gamma(k^{-1})) = Y$, and $\Gamma(kT) \star \Gamma(k^{-1})$ is a continuum in $X \times Y \times X$.
4. Suppose $T : Y \to Y$ is a continuous surjection, and $k : Y \to X$ is continuous surjection. Then $\Gamma(kTk^{-1})$ is a continuum in $X \times X$.

Let $g = kTk^{-1}$, so that $g(\cos(t)) = \{\cos(t+\lambda), \cos(-t+\lambda)\}$. Now if $x = \cos(t)$, $y = \cos(t+\lambda)$, we get a parametric curve in $[-1,1] \times [-1,1]$, which is an ellipse, but is at any rate a simple closed curve which sits in $[-1,1] \times [-1,1]$ very symmetrically. Also, $x = \cos(t)$, $y = \cos(-t+\lambda)$ is likewise a simple closed curve sitting in $[-1,1] \times [-1,1]$ very symmetrically. In fact, both of these represent the same curve, since $\cos(t) = \cos(-t)$ and $\cos(-t+\lambda) \in g(\cos(-t))$, so $(\cos(-t+\lambda), \cos(-t)) = (\cos(-t+\lambda), \cos(t)) \in \Gamma(g)$. Hence the graph of g is one simple closed curve in $[-1,1] \times [-1,1]$. However, different parts of the curve are traversed at different times under the two possibilities. Let $g_1(\cos(t)) = \cos(t+\lambda)$ and $g_2(\cos(t)) = \cos(-t+\lambda)$.

Let $a = \cos(\lambda)$. This curve intersects the boundaries of $[-1,1] \times [-1,1]$ at precisely the points $(1,a), (-a,-1), (-1,-a), (a,1)$. The line $y = x/a$ contains the points $(1,a), (-1,-a)$ and divides the curve into a left curve and a right curve with the curves being reflections of each other about this line. Call the left curve L_1 and the right curve L_2. Also $L_1 \cap L_2 = \{(1,a), (-1,-a)\}$, the set of endpoints of L_1 and L_2. We can regard each of these curves as the graph of a continuous function from $[-1,1]$ into $[-1,1]$. Applying the two previous propositions, we can conclude that for each n, $\star_{i=1}^{n}\Gamma(g)$ is connected, and then that $[-1,1]_g$ is connected.

The inverse limit $[-1,1]_g$

The inverse limit develops as follows: Consider the graph $\Gamma(g)$ to be step one. For step two, $\Gamma(g)\star\Gamma(g)=\Gamma(g)\star(L_1\cup L_2)=(\Gamma(g)\star L_1)\cup(\Gamma(g)\star L_2)$. Since $\{(a,1,a),(-a,-1,-a)\}=(\Gamma(g)\star L_1)\cap(\Gamma(g)\star L_2)$, it is precisely at these two points at which the new "connection" happens. This continues. For n even, $\star_{i=1}^{n}\Gamma(g)=(\star_{i=1}^{n-1}\Gamma(g)\star L_1)\cup(\star_{i=1}^{n-1}\Gamma(g)\star L_2)$ with $(\star_{i=1}^{n-1}\Gamma(g)\star L_1)\cap(\star_{i=1}^{n-1}\Gamma(g)\star L_2)=$ $\{(a,1,a,\ldots,1,a),(-a,-1,\ldots,-1,-a)\}$. For n odd, $\star_{i=1}^{n}\Gamma(g)=(\star_{i=1}^{n-1}\Gamma(g)\star L_1)\cup(\star_{i=1}^{n-1}\Gamma(g)\star L_2)$ with $(\star_{i=1}^{n-1}\Gamma(g)\star L_1)\cap(\star_{i=1}^{n-1}\Gamma(g)\star L_2)=$ $\{(1,a,1,a,\ldots,1,a),(-1,-a,-1,\ldots,-1,-a)\}$. At each step, the new curve $\Gamma(g)$ is attached to the already formed "bundle" of curves at precisely two points.

This example is rather similar to the inverse limit Example 2.11 and Figure 2.10 in Tom Ingram's book. Tom says that his example leads to a Hurewicz continuum.

Whether these two inverse limits (our ellipse graph example and Tom Ingrams's Hurewicz continuum example) are homeomorphic or not, we do not know at this point. But the dynamics are quite different.

The map k_∞ takes a point $(e^{it}, e^{i(t-\lambda)}, e^{i(t-2\lambda)}, \cdots) =$ $((\cos(t), \sin(t)), (\cos(t-\lambda), \sin(t-\lambda)), \cdots)$ to $(k(e^{it}), k(e^{i(t-\lambda)}), \cdots) = (\cos(t), \cos(t-\lambda), \cos(t-2\lambda), \cdots)$. The map k_∞ is one-to-one.

Thus, since S^1_T is just topologically a circle, so is $k_\infty(S^1_T)$. Since $[-1,1]_g$ is not even locally connected, and contains infinitely many simple closed curves, k_∞ cannot be surjective. Points in $k_\infty(S^1_T)$ have orbits dense in $k_\infty(S^1_T)$.

Indeed, there is a dense set of eventually period two points (under the action of the shift) in $[-1,1]_g$ and none of these periodic points is in $k_\infty(S^1_T)$. Consider for example the point $(\cos(t), \cos(-t+\lambda), \cos(t), \cos(-t+\lambda), \cdots)$. Since $g_2(\cos(t)) = \cos(-t+\lambda)$ and $g_2(\cos(-t+\lambda)) = \cos(-(-t+\lambda)+\lambda) = \cos(t)$, this point is in $[-1,1]_g$

Now on to situation Top

Proposition 7. Suppose situation Top holds, and $M = \varprojlim(X, f)$ and $N = \varprojlim(Y, T)$.
Then if $x \in f(z)$, $T(h(z)) = h(x)$. Then $h_\infty : M \to N$ defined by $h_\infty(x_0, x_1, \ldots) = (h(x_0), h(x_1), \ldots)$ is a continuous surjection. If σ_f denotes the shift on M and σ_T denotes the shift on N, then $h_\infty \circ \sigma_T = \sigma_f \circ h_\infty$.

Another circle rotation example.

Suppose $X = [0, 1]$, and $Y = S^1$ (so Y is the unit circle). We denote the points in S^1 as $(\cos(2\pi x), \sin(2\pi x))$, for $x \in [0, 1]$. Now $(\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi x + 2\pi n), \sin(2\pi x + 2\pi n))$, for n an integer, and if $x \in \mathbb{R}$, there is $z \in X = [0, 1]$ such that $(\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi z), \sin(2\pi z))$. Define $h : X \to Y$ by $h(x) = (\cos(2\pi x), \sin(2\pi x))$. Then h is a continuous surjection, and $h|(0, 1)$ is one-to-one, while $h(0) = h(1) = (1, 0)$. Hence, $h^{-1}(\cos(2\pi x), \sin(2\pi x)) = x$ for $0 < x < 1$, and $h^{-1}((1, 0)) = \{0, 1\}$.

Let $\lambda \in (0, 1)$ denote an irrational number. Define $T((\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi(x + \lambda)), \sin(2\pi(x + \lambda)))$. Hence, T is an irrational rotation on Y, the unit circle.

Then define $f(x) = h^{-1} \circ T \circ h(x)$, so that $f : X \to 2^X$ is upper semicontinuous and surjective. Then for $x \neq 1 - \lambda$, $f(x) = h^{-1}(T((\cos(2\pi x), \sin(2\pi x))) = h^{-1}((\cos(2\pi(x + \lambda)), \sin(2\pi(x + \lambda)))$, and, if $x > 1 - \lambda$, $(\cos(2\pi(x + \lambda)), \sin(2\pi(x + \lambda))) = (\cos(2\pi(x - 1 + \lambda)), \sin(2\pi(x - 1 + \lambda))$. Then, if $0 \leq x < 1 - \lambda$, $f(x) = \{x + \lambda\}$, and if $1 - \lambda < x \leq 1$, $f(x) = \{x - 1 + \lambda\}$. If $x = 1 - \lambda$, then $f(x) = f(1 - \lambda) = h^{-1}(T((\cos(2\pi(1 - \lambda)), \sin(2\pi(1 - \lambda)))) = h^{-1}((\cos(2\pi), \sin(2\pi))) = \{0, 1\}$. It is easy to check that if $x \in f(z)$, then $h(x) = T(h(z))$.

Properties of this example

Proposition 8. In the circle rotation example, recall that $f = h^{-1}Th$. Then $f^{-1} = h^{-1}T^{-1}h$. Moreover, for each $n \neq 0$, $y \in f^n(z)$ implies $h(y) = T^n(h(z))$.

Let M denote the inverse limit space generated by f.

Proposition 9. M is a Cantor set.
Proposition 10. The shift map $\sigma : M \to M$ is minimal.

Some notation and remarks. Let $L_0 = \{(x, x-\lambda) \in I_0 \times I_1 : \lambda \leq x \leq 1\}$ and $L_1 = \{(x, x-\lambda+1) \in I_0 \times I_1 : 0 \leq x \leq \lambda\}$. Then $L_0 \cup L_1 = \Gamma(f)$ and $L_0 \cap L_1 = \emptyset$. If $s = (s_1, s_2, \ldots) \in \Sigma_2$, let $L_s = \star_{i=1}^{\infty} L_{s_i}$.

Now add a vertical line to the graph to make it connected

Still studying this inverse limit, but it has the following properties so far:

- It is $\frac{1}{2}$-indecomposable.
- It is treelike (in fact, I think, arclike) and one-dimensional.
- It has a natural decomposition into points, arcs, and dense collections of arcwise connected sets.
- If we call this new inverse limit M', then $M \subset M'$.

Thanks for listening!