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Balanced and Functionally Balanced P-Groups

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The left uniformity $\mathcal{L}_G$ of a topological group G is formed by the sets $U_L := \{(x, y) \in G^2 : x^{-1}y \in U\}$, where U is a neighborhood of the identity element of G. The right uniformity $\mathcal{R}_G$ is defined analogously.

Definition

A topological group G is:

1. **balanced** *if $\mathcal{L}_G = \mathcal{R}_G$;*
2. **strongly functionally balanced** *if every left-uniformly continuous real-valued function on G is also right-uniformly continuous;*
3. **functionally balanced** *if every bounded left-uniformly continuous real-valued function on G is also right-uniformly continuous.*

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Itzkowitz's Problem

Is every (strongly) functionally balanced group balanced?

In the survey paper from 2007 "**Problems about the uniform structures of topological groups**", A. Bouziad and J. -P. Troallic ask:

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Functionally balanced= Balanced

1. Locally compact groups (Itzkowitz, Milnes and Protasov indep.)
2. Metrizable groups (Protasov)
3. Locally connected groups (Megrelishvili-Nickolas-Pestov)

Strongly functionally balanced= Balanced

1. Non-archimedean $\aleph_0$-bounded groups (Hernández)
2. A non-archimedean group that is strongly functionally generated by the set of all its subspaces of countable o-tightness (Troallic)

Definition

1. *A topological group in which the intersection of countably many open sets is still open is a* **P-group**.
2. *A* **uniform P-space** $(X, \mathcal{U})$ *is a uniform space in which the intersection of countably many elements of* $\mathcal{U}$ *is again in* $\mathcal{U}$.

It is easy to prove that if G is a P-group, then it has a local base at the identity consisting of open subgroups. That is, G is **non-archimedean**. Also, if $(X, \mathcal{U})$ is a uniform P-space, then it has a base of equivalence relations. Such a uniform space is called **non-archimedean**.

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Definition

Let τ be an infinite cardinal.

1. *A topological group G is called* **τ-bounded** *if for every neighborhood U of the identity, there exists a set F of cardinality not greater than τ such that $FU = G$.*
2. *A uniform space $(X, \mathcal{U})$ is* **τ-narrow** *if for every $\varepsilon \in \mathcal{U}$, there exists a set $\{x_\alpha : \alpha < \tau\}$ such that $X = \bigcup_{\alpha<\tau} \varepsilon[x_\alpha]$.*

The following is a reformulation of a Lemma from **"Topological groups and related structures"**:

Lemma (Arhangel'skii-Tkachenko)

An $\aleph_0$-bounded P-group is balanced.

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Extension of the main definitions

Definition

We say that a symmetric subset B of a topological group G is:

1. **balanced** *if the left and right uniformities of G coincide on B;*
2. **strongly functionally balanced** *if every $\mathcal{L}_G^B$-uniformly continuous function $f : B \to \mathbb{R}$ is $\mathcal{R}_G^B$-uniformly continuous;*
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Theorem

Let G be a P-group with a symmetric subset B. Then the following assertions are equivalent:

1. *B is strongly functionally balanced.*
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3. *If $\varepsilon \in \mathcal{L}_G^B$ is an equivalence relation with at most c equivalence classes, then $\varepsilon \in \mathcal{R}_G^B$.*

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Corollary

Let G be a P-group. Then, G is functionally balanced iff it is strongly functionally balanced.

Corollary

Let G be a $\mathfrak{c}$-bounded P-group. Then G is balanced iff it is functionally balanced.

This should be compared to the following result:

Theorem (Hernández)

Let G be a non-archimedean $\aleph_0$-bounded topological group. Then G is balanced iff it is strongly functionally balanced.

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This should be compared to the following result:

Theorem (Hernández)

Let G be a non-archimedean $\aleph_0$-bounded topological group. Then G is balanced iff it is strongly functionally balanced.

Definition

Let Ω be a subclass of **TGr** *and $(X, \mathfrak{U})$ be a uniform space. By an* **Ω-free topological group of $(X, \mathfrak{U})$** *we mean a pair $(F_\Omega(X, \mathfrak{U}), i)$, where $F_\Omega(X, \mathfrak{U}) \in \Omega$ and $i : X \to F_\Omega(X, \mathfrak{U})$ is a uniform map satisfying the following universal property. For every uniformly continuous map $\varphi : (X, \mathfrak{U}) \to G$ where $G \in \Omega$ there exists a unique continuous homomorphism $\Phi : F_\Omega(X, \mathfrak{U}) \to G$ for which the following diagram commutes:*

$$\begin{array}{ccc} (X, \mathfrak{U}) & \xrightarrow{\ i\ } & F_\Omega(X, \mathfrak{U}) \\ & {\scriptstyle \varphi} \searrow & \downarrow {\scriptstyle \Phi} \\ & & G \end{array}$$

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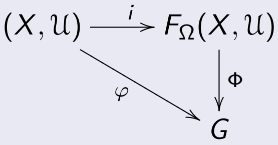

Some Ω-free groups

For $\Omega = \mathbf{TGr}$ the universal object $F_\Omega(X, \mathcal{U})$ is the *uniform free topological group* of $(X, \mathcal{U})$. This group was invented by Nakayama and studied by Numella and Pestov. In particular, Pestov described its topology.

Let $(X, \mathcal{U})$ be a non-archimedean uniform space.

1. $\Omega = \mathbf{NA}$. The *free non-archimedean group* $F_{\mathcal{NA}}$.
2. $\Omega = \mathbf{NA_b}$. The *free non-archimedean balanced group* $F^b_{\mathcal{NA}}$.

These groups were defined and studied by Megrelishvili and Shlossberg.

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Remark

Let $(X, \mathcal{U})$ be a non-archimedean uniform space. By the universal properties of the universal objects one can show that:

1. *if $F(X, \mathcal{U})$ is non-archimedean, then $F(X, \mathcal{U})$ coincides with $F_{\mathcal{NA}}(X, \mathcal{U})$;*
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Theorem

Let $(X,\mathfrak{U})$ be a uniform space. Suppose that there exists an infinite cardinal τ such that $\bigcap_{i\in I}\varepsilon_i \in \mathfrak{U}$ for any family of entourages $\{\varepsilon_i : \ i \in I\} \subseteq \mathfrak{U}$ with $|I| \le \tau$. Then,

1. *the intersection of any family of cardinality at most τ of open subsets of $F(X,\mathfrak{U})$ is open. In particular, $F(X,\mathfrak{U})$ is a P-group and we have $F(X,\mathfrak{U}) = F_{\mathcal{NA}}(X,\mathfrak{U})$.*
2. *if the uniform space $(X,\mathfrak{U})$ is also τ-narrow then*

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Omitting the τ-narrowness assumption from the last theorem we obtain the following counterexample.

Example

By a result of Nickolas and Tkachenko, for every cardinal $\tau > \aleph_1$, there exists a Hausdorff uniform P-space such that $w(X, \mathcal{U}) = \aleph_1 < \tau < \chi(F(X, \mathcal{U}))$. By a theorem of Megrelishvili and Shlossberg, $\chi(F^b_{\mathcal{NA}}(X, \mathcal{U})) = w(X, \mathcal{U})$. We conclude that $F(X, \mathcal{U}) = F_{\mathcal{NA}}(X, \mathcal{U}) \neq F^b_{\mathcal{NA}}(X, \mathcal{U})$.

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For $n \in \mathbb{N}$ denote by B_n the subset of $F(X)$ consisting of all words of length at most n.

Theorem

Let $(X, \mathfrak{U})$ be a uniform P-space. Then, $F(X, \mathfrak{U})$ is balanced if and only if B_n is balanced for every $n \in \mathbb{N}$.

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