6-2017

Rigidity and Nonrigidity of Corona Algebras

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Joint work with Alessandro Vignati

SUMTOPO 2017, University of Dayton

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Proof.

The isolated points of $\beta\mathbb{N}$ are the singletons $\{n\}$; φ must permute them, and this permutation determines the rest of φ. □

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Theorem (W. Rudin, 1957)

Assume the continuum hypothesis (CH). Then there is a homeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ which is not induced by an almost-permutation of $\mathbb{N}$.

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Theorem (Shelah-Steprans, 1988)

Assume the proper forcing axiom (PFA). Then every homeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ is induced by an almost-permutation of $\mathbb{N}$.

Let H be a (complex) Hilbert space. A linear operator $T : H \to H$ is *bounded* if

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The set $B(H)$ of bounded linear operators has a Banach space structure with the above norm, as well as

1. a multiplication, $ST = S \circ T$, and
2. an involution, $T \mapsto T^*$, defined by $\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle$.

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1. a multiplication, $ST = S \circ T$, and
2. an involution, $T \mapsto T^*$, defined by $\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle$.

This structure (along with some axioms relating $\|\cdot\|$ with the multiplication and involution) makes $B(H)$ a *C*-algebra*.

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Proof.

An orthogonal projection $P_K : H \to K$, where K is a closed subspace of H, is characterized by the algebraic properties $P_K^2 = P_K^* = P_K$. Moreover, $K \subseteq L$ if and only if $P_K P_L = P_K$. The one-dimensional projections are thus characterized algebraically as the smallest projections which are not 0. So any automorphism must permute them. □

Definition

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$K(H)$ forms an ideal in $B(H)$, and the quotient $B(H)/K(H)$ is a C*-algebra called the *Calkin algebra*.

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Assume CH. Then there is an automorphism of $B(H)/K(H)$ which is not induced by an (almost-) change of basis on H.

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Theorem (Farah, 2011)

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Definition

Given a locally compact Hausdorff space X, $C_0(X)$ is the C*-algebra of continuous functions $f : X \to \mathbb{C}$ which "vanish at infinity":

$$\forall \epsilon > 0 \quad \exists K \subseteq X \quad \forall x \in X \setminus K \quad |f(x)| < \epsilon$$

If X is compact, then $C_0(X) = C(X)$.

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Proposition

As C-algebras,* $C(\beta\mathbb{N}) \simeq \ell^\infty$, $C_0(\mathbb{N}) = c_0$, *and* $C(\beta\mathbb{N} \setminus \mathbb{N}) \simeq \ell^\infty / c_0$.

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Every commutative C-algebra is isomorphic to $C_0(X)$ for some locally compact, Hausdorff X.*

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For this reason, *unitizations* in the category of C*-algebras are viewed as analogs of *compactifications* for topological spaces.

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What is the unitization which corresponds to the Čech-Stone compactification?

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$$x \mapsto \|ax\| \qquad \text{and} \qquad x \mapsto \|xa\| \qquad\qquad (a \in A)$$

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- $M(A)$ is a unital C*-algebra containing A as an ideal.
- $M(C_0(X)) = C(\beta X)$ for any locally compact, Hausdorff X.
- $M(K(H)) = B(H)$.
- If A_n $(n \in \mathbb{N})$ is a sequence of unital C*-algebras, then $M\left(\bigoplus A_n\right) = \prod A_n$.

The quotient $M(A)/A$ is called the *corona of A*.

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It will be hard to give any specific structure to the automorphisms of a general corona algebra, but we can isolate the set-theoretic aspects with the following.

Definition

Given a map $\varphi : M(A)/A \to M(B)/B$, the *graph* of φ is

$$\Gamma_\varphi = \{(a, b) \in M(A) \times M(B) \mid \varphi([a]) = [b]\}$$

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Note: if A is separable, then the unit ball of $M(A)$ is Polish in the strict topology. (But $M(A)$ is usually not separable in the norm topology.)

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$\varphi \in \mathrm{Aut}(\ell^\infty/c_0)$ $\xrightarrow{\text{PFA}}$ Γ_φ is Borel $\xleftarrow{\text{PFA}}$ $\varphi \in \mathrm{Aut}(B(H)/K(H))$

Γ_φ is Borel $\xrightarrow{\text{ZFC}}$ φ is a permutation

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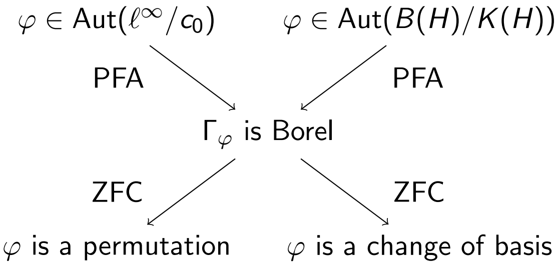

Conjecture (Coskey-Farah)

PFA implies that for every separable C-algebra A and every automorphism φ of $M(A)/A$, Γ_φ is Borel.*

A *UHF algebra* is a direct limit $M_{k_1}(\mathbb{C}) \to M_{k_2}(\mathbb{C}) \to \cdots$ where the connecting maps are of the form

$$A \mapsto \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}$$

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The isomorphism type of a UHF algebra is determined by the limit of the prime factorizations of the k_i's.

E.g. if $k_i = 2^i$ then the corresponding UHF algebra has "type" 2^∞.

Theorem (M.)

Assume PFA. Let A_n and B_n ($n \in \mathbb{N}$) be UHF algebras. Then every isomorphism between $\prod A_n / \bigoplus A_n$ and $\prod B_n / \bigoplus B_n$ has a Borel graph.

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Theorem (M.)

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What about the structure of φ?

Question

Is φ just composed of a bunch of isomorphisms $A_n \simeq B_n$, after applying an almost-permutation to the indices?

Note that

$$\begin{array}{ccc} \prod A_n / \bigoplus A_n & \xrightarrow{\varphi} & \prod B_n / \bigoplus B_n \\ \uparrow & & \\ \prod M_{k(n)} / \bigoplus M_{k(n)} & & \end{array}$$

where $k(n)$ grows arbitrarily fast.

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In this situation, we have

$$
\begin{array}{ccc}
\prod A_n / \bigoplus A_n & \xrightarrow{\varphi} & \prod B_n / \bigoplus B_n \\
\uparrow & \nearrow{\scriptstyle (\alpha_n)} & \\
\prod M_{k(n)} / \bigoplus M_{k(n)} & &
\end{array}
$$

where $\alpha_n : M_{k(n)} \to B_{f(n)}$ *is a sequence of unital homomorphisms and* f *is an almost-permutation of* $\mathbb{N}$.

Corollary

Assume PFA and suppose A_n and B_n are UHF algebras such that

$$\prod A_n / \bigoplus A_n \simeq \prod B_n / \bigoplus B_n$$

Then there is an almost-permutation f of $\mathbb{N}$ such that for all $n \in \operatorname{dom} f$,

$$A_n \simeq B_{f(n)}$$

In this situation we can also get functions

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such that $\varphi[(x_n)] = [(\beta_n(x_n))]$ for *all* sequences $(x_n) \in \prod A_n$.

But, the maps β_n are not necessarily homomorphisms. They satisfy

$$\begin{aligned} \|\beta_n(x+y) - \beta_n(x) - \beta_n(y)\| &\le \epsilon(\|x\| + \|y\|) \\ \|\beta_n(xy) - \beta_n(x)\beta_n(y)\| &\le \epsilon \|x\| \|y\| \\ \|\beta_n(x^*) - \beta_n(x)^*\| &\le \epsilon \|x\| \\ &\vdots \end{aligned}$$

where $\epsilon \to 0$ as $n \to \infty$. Call such a function an *ϵ-homomorphism*.

Basic problem: given a δ-homomorphism $\beta : A \to B$, can we find a homomorphism $\phi : A \to B$ such that $\|\beta(x) - \phi(x)\| \leq \epsilon$ for all x in the unit ball of A?

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This is a well-studied problem *in the case where β is already linear*. In the nonlinear case it seems much harder.

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This is a well-studied problem *in the case where* β *is already linear*. In the nonlinear case it seems much harder.

Theorem (Farah)

When A and B are finite-dimensional, then yes. In fact δ depends only on ϵ (and not on the dimension of A or B).

Theorem (M.-Vignati)

If A is finite-dimensional and B is any C-algebra, then every δ-homomorphism is within ϵ of a homomorphism, and moreover δ depends only on ϵ.*

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If A is a direct limit of finite-dimensional C-algebras and B is a von Neumann algebra, then every δ-homomorphism is within ϵ of a homomorphism, and moreover δ depends only on ϵ.*

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Theorem (M.-Vignati)

Assume PFA and let A be a separable, nuclear C-algebra with an increasing approximate identity of projections. Then every automorphism of $M(A)/A$ has a Borel graph.*

Theorem (M.-Vignati)

Assume PFA. Let A_n and B_n be separable, unital, nuclear C-algebras, and suppose*

$$\prod A_n / \bigoplus A_n \simeq \prod B_n / \bigoplus B_n$$

Then there is an almost-permutation f and maps

$$\beta_n : A_n \to B_{f(n)}$$

such that β_n is an ϵ-isomorphism where $\epsilon \to 0$ as $n \to \infty$.

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Question (Open so far)

Does this imply that $A_n \simeq B_{f(n)}$?

Theorem (M.-Vignati-BGOS)

Assume PFA and let A be a separable C-algebra with an increasing approximate identity of projections such that A has the metric approximation property as a Banach space. Then every automorphism of $M(A)/A$ has a Borel graph.*

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A Banach space E has the *metric approximation property* if for every finite subset X of E and $\epsilon > 0$, there is a finite-rank operator $T : E \to E$ such that $\|T\| \leq 1$ and $\|Tx - x\| < \epsilon$ for all $x \in X$.

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The class of C*-algebras with the MAP is large; it includes nuclear C*-algebras as well as many non-nuclear C*-algebras (e.g. $C^*(\mathbb{F}_2)$). But there are separable C*-algebras which do not have the MAP.

Theorem (Farah-M.)

Assume PFA. Then if X and Y are a zero-dimensional locally compact Polish spaces, every homeomorphism $\beta X \setminus X \simeq \beta Y \setminus Y$ is induced by a homeomorphism between cocompact subsets of X and Y.

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Removing the zero-dimensional assumption seems very hard. The proof uses the Boolean algebra of clopen sets modulo its ideal of compact-open sets...

Theorem (Coskey-Farah)

Assume CH and let A be a separable, simple, nonunital C-algebra. Then there are $2^{\mathfrak{c}}$-many automorphisms of $M(A)/A$.*

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Theorem (Vignati)

Assume CH and let X be a locally compact, noncompact, metrizable manifold. Then there are $2^{\mathfrak{c}}$-many automorphisms of $C(\beta X \setminus X)$.

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Theorem (Ghasemi)

There exist increasing sequences $k(n)$ and $\ell(n)$ of natural numbers such that $k(n) \neq \ell(n)$ and

$$\prod M_{k(n)} / \bigoplus M_{k(n)} \equiv \prod M_{\ell(n)} / \bigoplus M_{\ell(n)}$$

Thank you!