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Aperiodic Colorings and Dynamics

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Finite case

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$$\deg G = \max\{\deg(v) | v \in G\},$$

is finite, then $D(G) \leq \deg(G) + 1$, where equality is attained only for complete graphs K_n, complete bipartite graphs $K_{n,n}$ and the cyclic graph with 5 vertices C_5.

Limits of graphs

A pointed colored graph (G', z, ϕ') is a *limit* of (G, ϕ) if there is a sequence of balls $B_G(x_i, R_i)$, with $x_i \in G$ and $R_i \to \infty$ such that $(B_G(x_i, R_i), x_i, \phi)$ and $(B_G(z, R_i), z, \phi')$ are isomorphic as pointed colored graphs.

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Theorem

If an infinite connected graph G has bounded degree $\deg G < \infty$, then it admits a limit distinguishing coloring by $\deg G$ colors.

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We have a reserve color that we can use to create many different distinguishing colorings.

Idea of the proof 2

We want to prove for a coloring χ a estimate of the following type: there are $R, S > 0$ such that if $B_G(x, R) \to B_G(y, R)$ is an isomorphism of graphs preserving χ, then either $x = y$ or $d(x, y) > S$.

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In fact for χ to be limit distinguishing is equivalent to countably many such estimates for pairs R_n, S_n with $R_n, S_n \to \infty$.

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If the number of different distinguishing colorings for each cluster is greater that the cardinality of the balls $B(C_n, S')$ in G_1 for a suitable S', we obtain the desired estimate.
Proceeding inductively (we divide G_1 into clusters, etc.), we obtain a strongly distinguishing coloring by $\deg G$ colors.

A word on repetitiviness

A graph is *repetitive* if every ball $B(x, r) \subset G$ appears uniformly on G, that is, there is some $K_{x,r}$ such that for every $y \in G$ there is some z with $d(y, z) \leq K_{x,r}$ and $B(x, r) \simeq B(z, r)$.

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Theorem

If the graph G is of bounded degree and repetitive, then there is a limit distinguishing repetitive coloring by $\deg G$ colors.

Foliated spaces

A *foliated space* is a topological space X with an equivalence class of atlas consisting of charts of the form $\phi_i \colon U_i \subset X \to \mathbb{R}^n \times Z_i$, and such that the change of coordinates $\phi_i \circ \phi_j^{-1}$ sends the plaques $\mathbb{R}^n \times \{z\}$ to plaques smoothly .

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In this context, we have a differentiable structure along the leaves, but only topological structure in the transverse direction.

This structure gives rise to a special type of dynamical system, a *pseudogroup*. If this dynamical system is in some sense trivial, we say that the foliation is *without holonomy*.

Universal space

We will consider triples (M, x, f), where M is a Riemannian n-manifold, $x \in M$ a distinguished point, and $f\colon M \to \mathbb{H}$ a smooth map into a separable Hilbert space.

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Two such triples (M, x, f) and (N, y, g) are equivalent if there is an isometry $\phi \colon M \to N$ sending x to y and such that $g\phi = f$.

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Let $\widehat{\mathcal{M}}_*^\infty(n)$ be the space of equivalence classes of triples, then choosing a manifold M and a map $f\colon M \to \mathbb{H}$ determines an inclusion $M \hookrightarrow \widehat{\mathcal{M}}_*^\infty(n)$ sending $x \in M$ to $[M, x, f]$.

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If f is an isometric embedding with some minor additional conditions, we get an embedding of M into a compact lamination $X \subset \widehat{\mathcal{M}}_*^\infty(n)$.

A sort of translator

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In this spirit, we have that "no holonomy" on the foliated space is equivalent to "limit distinguishability" of the map, and "minimality" of the space is equivalent to "repetitivity" of the map.

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The existence of maps with these conditions is equivalent to the existence of the aforementioned colorings.

Theorem

Every manifold of bounded geometry can be realized as a leaf in a compact foliated space without holonomy. Moreover, if the manifold is repetitive, then the space can be taken to be minimal.

Tilings

Consider a finite set of prototiles $\mathcal{T} = \{T_1, \dots, T_n\}$ which are polygons, polyhedrons, etc. which tile some space X meeting face to face.

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Given a set of colors K, we can constructed the associated set $\mathcal{T}_K$ of colored prototiles, i.e. with a color associated to each face.

Tilings II

From the theorem of existence of limit distinguising colorings, we can deduce a similar theorem for edge colorings.

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Theorem

Given a tiling as before, we can color it by finitely many colors so that the associated colored tiling is limit aperiodic. Moreover, if the tiling was repetitive, then the colored tiling can be chosen to be repetitive.

Thank you!