6-2017

Extension Theorems for Large Scale Spaces via Neighbourhood Operators

Thomas Weighill
University of Tennessee, Knoxville, tweighil@vols.utk.edu

Jerzy Dydak

Extension theorems for large scale spaces via neighbourhood operators

Thomas Weighill (joint work with Jerzy Dydak)

University of Tennessee

32nd Summer Conference on Topology and its Applications

Classical extension theorems

Theorem [Tietze Extension Theorem]

Let X be a normal topological space and let A be a closed subset of X. Then any continuous function $f : A \to [0,1]$ extends to a continuous function $g : X \to [0,1]$.

Classical extension theorems

Theorem [Tietze Extension Theorem]

Let X be a normal topological space and let A be a closed subset of X. Then any continuous function $f : A \to [0,1]$ extends to a continuous function $g : X \to [0,1]$.

Theorem [Katetov 1951]

Let X be a uniform space and $A \subseteq X$ a subspace. Then any uniformly continuous function $f : A \to [0,1]$ extends to a uniformly continuous function g on the whole of X.

Slowly oscillating functions

In coarse geometry, the maps of interest with codomain $[0, 1]$ are the **slowly oscillating maps**.

Slowly oscillating functions

In coarse geometry, the maps of interest with codomain $[0, 1]$ are the **slowly oscillating maps**.

Definition

Let X be a metric space and let $f : X \to [0, 1]$ be a map (not necessarily continuous). The map f is called **slowly oscillating** if for every $R > 0$ and $\varepsilon > 0$, there is a bounded set $K \subseteq X$ such that if x and x' are points outside K and $d(x, x') \leq R$, then $|f(x) - f(x')| \leq \varepsilon$.

Slowly oscillating functions

In coarse geometry, the maps of interest with codomain $[0,1]$ are the **slowly oscillating maps**.

Definition

Let X be a metric space and let $f : X \to [0,1]$ be a map (not necessarily continuous). The map f is called **slowly oscillating** if for every $R > 0$ and $\varepsilon > 0$, there is a bounded set $K \subseteq X$ such that if x and x' are points outside K and $d(x,x') \leq R$, then $|f(x) - f(x')| \leq \varepsilon$.

Such functions are also called Higson functions in the literature, or are said to have "vanishing variation at infinity".

Slowly oscillating functions

Slowly oscillating functions appear in many places in coarse geometry. For example,

Slowly oscillating functions

Slowly oscillating functions appear in many places in coarse geometry. For example,

- the Higson corona (originally introduced for the purposes of index theory) is defined using these functions:

$$C(\nu X) = B_h(X)/B_0(X).$$

  (we will see more of the Higson corona later on).

Slowly oscillating functions

Slowly oscillating functions appear in many places in coarse geometry. For example,

- the Higson corona (originally introduced for the purposes of index theory) is defined using these functions:

  $$C(\nu X) = B_h(X)/B_0(X).$$

  (we will see more of the Higson corona later on).
- for a metric space with finite asymptotic dimension, the Roe algebra can be defined as precisely those operators which essentially commute with slowly oscillating functions (this is a very recent result of Tikuisis-Špakula).

A large scale extension theorem

Theorem [Dydak-Mitra 2016]

Given a metric space X, any slowly oscillating function on a subset of X to $[0, 1]$ extends to a slowly oscillating function on the whole of X to $[0, 1]$.

A large scale extension theorem

Theorem [Dydak-Mitra 2016]

Given a metric space X, any slowly oscillating function on a subset of X to $[0, 1]$ extends to a slowly oscillating function on the whole of X to $[0, 1]$.

In this talk we will be interested in generalizing this result to the abstract setting of large scale spaces and providing a unified proof for this result and the two classical ones.

Large scale spaces

To motivate the notion of large scale space (due to Dydak-Hoffland), consider the following analogy:

- a topological space is a set equipped with a collection of covers which are declared to be "open covers",

Large scale spaces

To motivate the notion of large scale space (due to Dydak-Hoffland), consider the following analogy:

- a topological space is a set equipped with a collection of covers which are declared to be "open covers",
- a uniform space is a set equipped with a collection of covers which are declared to be "uniform covers",

Large scale spaces

To motivate the notion of large scale space (due to Dydak-Hoffland), consider the following analogy:

- a topological space is a set equipped with a collection of covers which are declared to be "open covers",
- a uniform space is a set equipped with a collection of covers which are declared to be "uniform covers",
- a **large scale space** is a set equipped with a collection of covers which are declared to be "uniformly bounded covers".

Large scale spaces

To make this precise, we first need the following definition.

Large scale spaces

To make this precise, we first need the following definition.

Definition

Let X be a set. The **star** $\mathrm{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B.

Large scale spaces

To make this precise, we first need the following definition.

Definition

Let X be a set. The **star** $\mathrm{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B.

More generally, for two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathrm{st}(\mathcal{B},\mathcal{U})$ is the family $\{\mathrm{st}(B,\mathcal{U}) \mid B \in \mathcal{B}\}$.

Large scale spaces

Definition [Dydak-Hoffland]

A **large scale structure** $\mathcal{L}$ on a set X is a nonempty set of families $\mathcal{B}$ of subsets of X (which we call the **uniformly bounded families** in X) satisfying the following conditions:

(1) $\mathcal{B}_1 \in \mathcal{L}$ implies $\mathcal{B}_2 \in \mathcal{L}$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ implies $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$.

A set equipped with a large scale structure is a **large scale space**.

Large scale spaces

Definition [Dydak-Hoffland]

A **large scale structure** $\mathcal{L}$ on a set X is a nonempty set of families $\mathcal{B}$ of subsets of X (which we call the **uniformly bounded families** in X) satisfying the following conditions:

(1) $\mathcal{B}_1 \in \mathcal{L}$ implies $\mathcal{B}_2 \in \mathcal{L}$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ implies $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$.

A set equipped with a large scale structure is a **large scale space**.

Note that the notion of large scale space is equivalent to the notion of coarse space in the sense of Roe.

Large scale spaces

Example

A canonical example of a large scale space is as follows. Let (X, d) be a metric space. Define the uniformly bounded families in X to be all those families $\mathcal{U}$ for which there is a $M > 0$ such that every element of $\mathcal{U}$ has diameter at most M.

In a large scale space X, a subset $K \subseteq X$ is declared to be **bounded** if it is an element of some uniformly bounded family (compare: an open set is one that belongs to an open cover).

Slowly oscillating functions

Definition

Let X be a large scale space and let $f : X \to [0,1]$ be a map. The map f is called **slowly oscillating** if for every uniformly bounded family $\mathcal{U}$ and $\varepsilon > 0$, there is a bounded set $K \subseteq X$ such that for every $U \in \mathcal{U}$ outside K, we have $\operatorname{diam}(f(U)) \leq \varepsilon$.

Neighbourhood operators

To unify the proofs of the three extension theorems, we need a notion of neighbourhood operator. For us, a **neighbourhood operator** $\prec$ on a set X is a binary relation on the power set $\mathcal{P}(X)$ such that

$$A \prec B \Rightarrow A \subseteq B.$$

Neighbourhood operators

Here are the three examples we care about in this talk:

- the **topological neighbourhood operator** on a topological space X: define $A \prec B$ if and only if B contains an open set containing $\mathrm{cl}(A)$ (not the usual definition!).

Neighbourhood operators

Here are the three examples we care about in this talk:

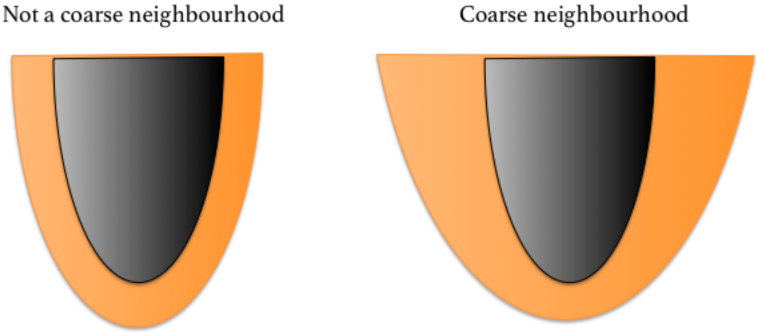
- the **topological neighbourhood operator** on a topological space X: define $A \prec B$ if and only if B contains an open set containing $\mathrm{cl}(A)$ (not the usual definition!).
- the **coarse neighbourhood operator** on a large scale space X: define $A \prec B$ if and only if $A \subseteq B$ and for every uniformly bounded cover $\mathcal{U}$ of X, $\mathrm{st}(A, \mathcal{U})$ is contained in $B \cup K$ for some bounded set K.

Neighbourhood operators

Here are the three examples we care about in this talk:

- the **topological neighbourhood operator** on a topological space X: define $A \prec B$ if and only if B contains an open set containing $\mathrm{cl}(A)$ (not the usual definition!).
- the **coarse neighbourhood operator** on a large scale space X: define $A \prec B$ if and only if $A \subseteq B$ and for every uniformly bounded cover $\mathcal{U}$ of X, $\mathrm{st}(A, \mathcal{U})$ is contained in $B \cup K$ for some bounded set K.
- the **uniform neighbourhood operator** on a uniform space X: define $A \prec B$ if there is a uniform cover $\mathcal{U}$ such that such that $\mathrm{st}(A, \mathcal{U}) \subseteq B$.

Coarse neighbourhoods



Continuity

Definition

Let X and Y be sets equipped with neighbourhood operators $\prec_X$ and $\prec_Y$ respectively. A set map $f : X \to Y$ is called **neighbourhood continuous with respect to** $\prec_X$ **and** $\prec_Y$ if $A \prec_Y B \implies f^{-1}(A) \prec_X f^{-1}(B)$ for any subsets A and B of Y.

Continuity

Definition

Let X and Y be sets equipped with neighbourhood operators $\prec_X$ and $\prec_Y$ respectively. A set map $f : X \to Y$ is called **neighbourhood continuous with respect to** $\prec_X$ **and** $\prec_Y$ if $A \prec_Y B \implies f^{-1}(A) \prec_X f^{-1}(B)$ for any subsets A and B of Y.

This notion of continuity generalizes the three types of continuity we care about.

Continuity

Proposition

Let X and Y be topological spaces with Y a T_1 space. Then a set map $f : X \to Y$ is topologically continuous if and only if it is neighbourhood continuous with respect to the topological neighbourhood operators on X and Y.

Continuity

Proposition

Let X and Y be topological spaces with Y a T_1 space. Then a set map $f : X \to Y$ is topologically continuous if and only if it is neighbourhood continuous with respect to the topological neighbourhood operators on X and Y.

Proposition

Let X be a uniform space and $f : X \to [0, 1]$ a function. Then f is uniformly continuous if and only if it is neighbourhood continuous with respect to the uniform neighbourhood operator on X and the topological neighbourhood operator on $[0, 1]$.

Continuity

Proposition

Let X be an ls-space and $f : X \to [0, 1]$ a set map. Then f is slowly oscillating if and only if f is neighbourhood continuous with respect to the coarse neighbourhood operator on X and the topological neighbourhood operator on $[0, 1]$.

Axioms

We need our neighbourhood operator $\prec$ to satisfy some axioms.

(N0) $A \prec X$ for all $A \subseteq X$.

(N1) if $A \prec B$ then $X \setminus B \prec X \setminus A$.

(N2) if $A \prec B \subseteq C$, then $A \prec C$.

(N3) if $A \prec N$ and $A' \prec N'$ then $A \cup A' \prec N \cup N'$.

(N4) $\forall A \prec C$, $\exists B$ with $A \prec B \prec C$.

Axioms

We need our neighbourhood operator $\prec$ to satisfy some axioms.

(N0) $A \prec X$ for all $A \subseteq X$.

(N1) if $A \prec B$ then $X \setminus B \prec X \setminus A$.

(N2) if $A \prec B \subseteq C$, then $A \prec C$.

(N3) if $A \prec N$ and $A' \prec N'$ then $A \cup A' \prec N \cup N'$.

(N4) $\forall A \prec C$, $\exists B$ with $A \prec B \prec C$.

All three examples above satisfy axioms (N1)–(N3).

Axioms

We need our neighbourhood operator $\prec$ to satisfy some axioms.

(N0) $A \prec X$ for all $A \subseteq X$.

(N1) if $A \prec B$ then $X \setminus B \prec X \setminus A$.

(N2) if $A \prec B \subseteq C$, then $A \prec C$.

(N3) if $A \prec N$ and $A' \prec N'$ then $A \cup A' \prec N \cup N'$.

(N4) $\forall A \prec C$, $\exists B$ with $A \prec B \prec C$.

All three examples above satisfy axioms (N1)–(N3). Axiom (N4) is the "normality" axiom.

Urysohn's Lemma for neighbourhood operators

Theorem [Dydak-W.]

Let X be a set and $\prec$ a neighbourhood operator satisfying (N0)–(N3). Then the following are equivalent:

(1) $\prec$ satisfies (N4),

(2) for any subsets A and B of X such that $A \prec X \setminus B$, there is a neighbourhood continuous function $f : X \to [0, 1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.

Proof

The proof follows the classical proof almost exactly. Since $A \prec X \setminus B$, by normality we have an intermediate neighbourhood

$$A \prec A_{1/2} \prec X \setminus B.$$

Proof

The proof follows the classical proof almost exactly. Since $A \prec X \setminus B$, by normality we have an intermediate neighbourhood

$$A \prec A_{1/2} \prec X \setminus B.$$

By induction we produce a family of neighbourhoods of A, indexed by the dyadic fractions with $A_s \prec A_t$ whenever $s < t$.

Proof

The proof follows the classical proof almost exactly. Since $A \prec X \setminus B$, by normality we have an intermediate neighbourhood

$$A \prec A_{1/2} \prec X \setminus B.$$

By induction we produce a family of neighbourhoods of A, indexed by the dyadic fractions with $A_s \prec A_t$ whenever $s < t$. We then define

$$f(x) = \inf\{t \mid x \in A_t\}.$$

Tiezte Extension Theorem for neighbourhood operators

Theorem [Dydak-W.]

Let X be a set and $\prec$ a neighbourhood operator satisfying $(\mathrm{N0})-(\mathrm{N3})$. Then $\prec$ satisfies (N4) if and only if for any function $f : A \to [0,1]$ from a subset A of X which is neighbourhood continuous with respect to the operator induced by $\prec$ on A and the topological neighbourhood operator on $[0,1]$, there is a neighbourhood continuous function $g : X \to [0,1]$ which extends f.

Tiezte Extension Theorem for neighbourhood operators

Theorem [Dydak-W.]

Let X be a set and $\prec$ a neighbourhood operator satisfying (N0) − (N3). Then $\prec$ satisfies (N4) if and only if for any function $f : A \to [0, 1]$ from a subset A of X which is neighbourhood continuous with respect to the operator induced by $\prec$ on A and the topological neighbourhood operator on $[0, 1]$, there is a neighbourhood continuous function $g : X \to [0, 1]$ which extends f.

Proof.

Easy adaptation of the classical proof using Urysohn's Lemma. □

Extension Theorems

As corollaries, we get

- the classical Tietze Extension Theorem,

Extension Theorems

As corollaries, we get

- the classical Tietze Extension Theorem,
- Katetov's extension theorem for uniform spaces.

Extension Theorems

As corollaries, we get

- the classical Tietze Extension Theorem,
- Katetov's extension theorem for uniform spaces.

Theorem [Dydak-W.]

Let X be a large scale space which is **large scale normal**, that is, such that the coarse neighbourhood operator satisfies (N4). Then any slowly oscillating function on a subset of X to $[0,1]$ extends to a slowly oscillating function on the whole of X to $[0,1]$

Examples

The following are examples of large scale normal spaces.

Examples

The following are examples of large scale normal spaces.

- metric spaces (thus we recover the result of Dydak-Mitra),

Examples

The following are examples of large scale normal spaces.

- metric spaces (thus we recover the result of Dydak-Mitra),
- metric spaces equipped with the C_0 large scale structure (introduced by Wright),

Examples

The following are examples of large scale normal spaces.

- metric spaces (thus we recover the result of Dydak-Mitra),
- metric spaces equipped with the C_0 large scale structure (introduced by Wright),
- sets equipped with the universal bounded geometry large scale structure (in the sense of Roe).

Examples

The following are examples of large scale normal spaces.

- metric spaces (thus we recover the result of Dydak-Mitra),
- metric spaces equipped with the C_0 large scale structure (introduced by Wright),
- sets equipped with the universal bounded geometry large scale structure (in the sense of Roe).

Non-examples

If G is a locally compact group, then it carries a natural large scale structure wherein the uniformly bounded covers are precisely those that refine a cover of the form

$$\{g \cdot K \mid g \in G\}$$

for a compact set K.

Non-examples

If G is a locally compact group, then it carries a natural large scale structure wherein the uniformly bounded covers are precisely those that refine a cover of the form

$$\{g \cdot K \mid g \in G\}$$

for a compact set K.

Theorem [Dydak-W.]

Let G be a locally compact abelian group. Then the following conditions are equivalent:

(1) G is large scale normal,

(2) G is σ-compact,

(3) the large scale structure on G is induced by a metric.

Non-examples

In particular, $\mathbb{R}$ equipped with the discrete topology and the corresponding large scale structure (*not* the usual metric large scale structure!) is not large scale normal.

Non-examples

In particular, $\mathbb{R}$ equipped with the discrete topology and the corresponding large scale structure (*not* the usual metric large scale structure!) is not large scale normal.

Other examples can be constructed but they are little tedious to describe.

The Higson compactification

Let X be a large scale space and endow X with the discrete topology. Then we can define a unique compactification hX of X by requiring that any map $f : X \to [0, 1]$ extends over hX if and only if it is slowly oscillating.

The Higson compactification

Let X be a large scale space and endow X with the discrete topology. Then we can define a unique compactification hX of X by requiring that any map $f : X \to [0, 1]$ extends over hX if and only if it is slowly oscillating. The compactification so obtained is the **Higson compactification**. The complement $hX \setminus X$ is the **Higson corona**.

The Higson compactification

Let X be a large scale space and endow X with the discrete topology. Then we can define a unique compactification hX of X by requiring that any map $f : X \to [0,1]$ extends over hX if and only if it is slowly oscillating. The compactification so obtained is the **Higson compactification**. The complement $hX \setminus X$ is the **Higson corona**.

Definition

Let X be a large scale space and A, B be subsets of X. We say that A and B are **coarsely separated** if A has coarse neighbourhood $X \setminus B$.

The Higson compactification

Proposition

Let X be an large scale space with the discrete topology. Then the following are equivalent, where for $Y \subseteq X$, $\overline{Y}$ denotes the closure of Y in hX:

(1) X is large scale normal,

(2) two disjoint closed subsets A and B of X are coarsely separated if and only if $\overline{A} \cap \overline{B} = \varnothing$.

The Higson compactification

Proposition

Let X be an large scale space with the discrete topology. Then the following are equivalent, where for $Y \subseteq X$, $\overline{Y}$ denotes the closure of Y in hX:

(1) X is large scale normal,

(2) two disjoint closed subsets A and B of X are coarsely separated if and only if $\overline{A} \cap \overline{B} = \varnothing$.

Property (2) (which by the above is true for metric spaces) is often used in the literature to prove things about the Higson corona of a space.

Hybrid large scale spaces

Definition [Austin-Dydak-Holloway]

A **hybrid large scale space** is a set X equipped with both a large scale structure and a topology such that there is a uniformly bounded cover of X which consists of open sets.

Hybrid large scale spaces

Definition [Austin-Dydak-Holloway]

A **hybrid large scale space** is a set X equipped with both a large scale structure and a topology such that there is a uniformly bounded cover of X which consists of open sets.

Definition

If A and N are subsets of a hybrid large scale space X then we say that A has **hybrid neighbourhood** N if N is a topological and a coarse neighbourhood of A (as defined in this talk). If the neighbourhood operator so obtained satisfies (N4) we call X **hybrid large scale normal**.

Hybrid large scale spaces

Theorem [Dydak-W.]

If X is a hybrid large scale space, then the following conditions are equivalent:

(1) X is hybrid large scale normal,

(2) X is large scale normal as a large scale space and the topology of X is normal.

Hybrid large scale spaces

Theorem [Dydak-W.]

If X is a hybrid large scale space, then the following conditions are equivalent:

(1) X is hybrid large scale normal,

(2) X is large scale normal as a large scale space and the topology of X is normal.

Corollary (of Tietze Theorem for neighbourhood operators)

Let X be a hybrid large scale space. Then X is hybrid large scale normal if and only if for any closed subset A of X, any continuous slowly oscillating function $f : A \to [0, 1]$ extends to a continuous slowly oscillating function $g : X \to [0, 1]$.

Thank you.