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# Uncountable Discrete Sets and Forcing

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*University of South Carolina - Beaufort*, iwasa@uscb.edu

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# 32nd Summer Conference on Topology and its Applications

**University of Dayton**

**June, 2017**

Uncountable discrete sets and forcing

Akira Iwasa
University of South Carolina Beaufort

# Background

**V**: Ground Model

$\mathbf{V}^{\mathbb{P}}$: Extension of **V** by forcing $\mathbb{P}$

$(X, \tau^{\mathbb{P}})$

**V**

$(X, \tau)$

Consider a topological space $(X, \tau)$ in **V**.

We define a topological space $(X, \tau^{\mathbb{P}})$ in $\mathbf{V}^{\mathbb{P}}$ such that

$$\boxed{\tau^{\mathbb{P}} = \text{ the topology generated by } \tau}$$

**Observation.**

- $\tau \subsetneq \tau^{\mathbb{P}}$ New open sets are added by forcing $\mathbb{P}$.
- $\tau$ is a base for $\tau^{\mathbb{P}}$.

I am interested in comparing $(X, \tau)$ and $(X, \tau^{\mathbb{P}})$.

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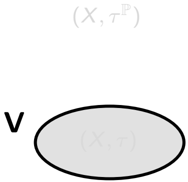


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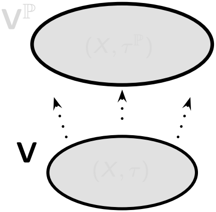


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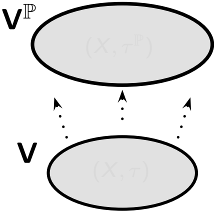


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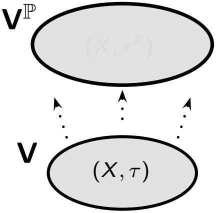


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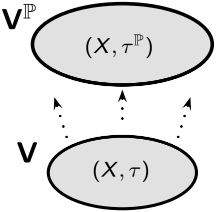


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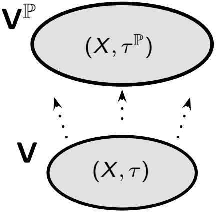


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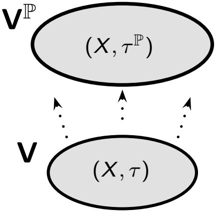


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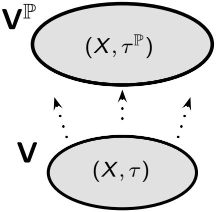


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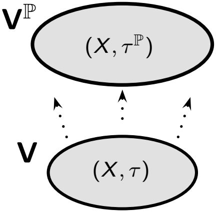


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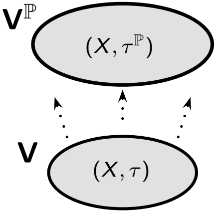


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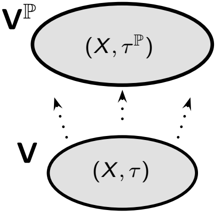


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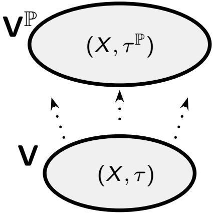


- $(X, \tau)$ is Hausdorff $\implies$ $(X, \tau^{\mathbb{P}})$ is Hausdorff
  "Hausdorffness is preserved by any forcing"
- $(X, \tau)$ is regular $\implies$ $(X, \tau^{\mathbb{P}})$ is regular
- $(X, \tau)$ is completely regular $\implies$ $(X, \tau^{\mathbb{P}})$ is completely regular
- $(X, \tau)$ is normal $\not\Rightarrow$ $(X, \tau^{\mathbb{P}})$ is normal
  "Normality can be destroyed by forcing"
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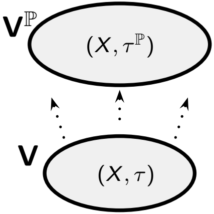


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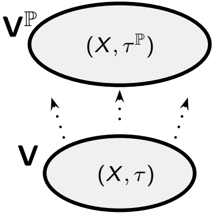


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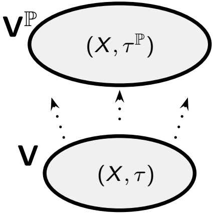


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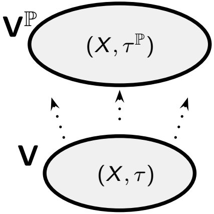


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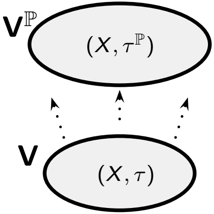


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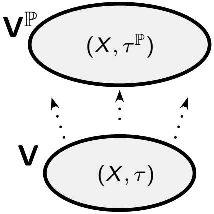


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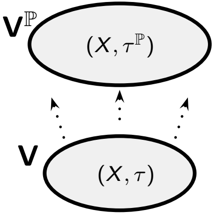


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# Question

**Question:** Suppose that a space $X$ has no uncountable discrete subspace.

**Can forcing create an uncountable discrete subspace of $X$?**

In other words:

Suppose that a space $(X, \tau)$ has no uncountable discrete subspace. For some forcing $\mathbb{P}$, can $(X, \tau^{\mathbb{P}})$ have an uncountable discrete subspace?

No ZFC example so far.

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# Definition

**Definition.** A space $X$ has **countable chain condition** (CCC) if every pairwise disjoint family of open sets is countable.

**Observation:** A space has no uncountable discrete subspace if and only if $X$ is **hereditarily CCC**.

So our question can be rephrased as:

Can forcing destroy hereditarily CCC?

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# HC HL HS

**Notation:**

- Hereditarily CCC (**HC**) $\iff$ No uncountable discrete subspace
- Hereditarily Lindelof (**HL**)
- Hereditarily separable (**HS**)

**HL** $\Longrightarrow$ **HC**

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- **HC**, **HL** and **HS** are similar properties. In fact,

**If forcing destroys HS or HL, then it destroys HC.**
That is, if forcing destroys HS or HL, then it would create an uncountable discrete subspace. So try to destroy **HL** or **HS**.

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# L-space and S-space

- Hereditarily CCC (**HC**)
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**Definition.** **L-space** = HL but not HS

There is an $L$-space in ZFC. (Moore)
But forcing cannot destroy Moore's L-space.
(Tsaban, Zdomskyy)

**Definition.** **S-space** = HS but not HL.

CH implies there is an $S$-space.

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How about spaces which are both HL and HS?

**Example. Filippov Space.** (1969)

• $E \subseteq [0,1] \times [0,1]$ is **Luzin** if every nowhere dense subset is countable.
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$$X_E = (E \times S) \cup ([0,1]^2 \setminus E),$$

where $S$ is the unit circle.

An neighborhood of $(x_1, x_2) \in E \times S$ looks like:

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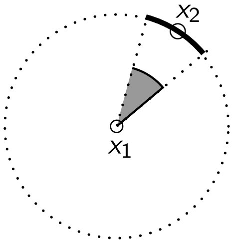

# Weakly Luzin

**Definition.** $E \subseteq [0,1] \times [0,1]$ is **weakly Luzin** if for $E' \subseteq E$, whenever

$$\left\{ \frac{x-y}{\|x-y\|} \in S : x, y \in E' \right\}$$

is not dense in the unit circle $S$, $E'$ is countable. Luzin sets are weakly Luzin.

**Theorem.** (Kunen) The following are equivalent:

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3. Filippov space $X_E$ is HL
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# Destroying Filippov space

**Example.** (CH + $\exists$Super compact cardinal)
There is a HL and HS space $X$ and a proper forcing $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$, $X$ has an uncountable discrete subspace.

**Proof.**

- CH implies there is a Luzin set $E$.
- Filippov space $X_E$ is HL and HS.
- Supercompact cardinal implies that there is a proper forcing which forces PFA.
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# Spaces where it is impossible to shoot an uncountable discrete subspace

**Spaces where it is impossible to shoot an uncountable discrete set by forcing.**

- Metrizable.
- Developable.
- Stratifiable.

All the above spaces have a **countable network** if they have no uncountable discrete subspace, and forcing preserves countable network.

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# Monotone normality

**Theorem.** (Borges) $\langle X, \tau \rangle$ is **monotonically normal** if and only if for a base $\mathcal{B}$ for $X$, there is an operator $H(x, B)$, where $B \in \mathcal{B}$ and $x \in B$ such that

$$H(x, B) \cap H(x', B') \neq \emptyset \implies x \in B' \text{ or } x' \in B.$$

Forcing preserves a base of a topology, and so the operator $H$ will have the same property in any forcing extension. Therefore,

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# Monotone normality II

Here are usefull theorems.

**Theorem.** (Williams, Zhou) The following are equivalent:

1. There is no Souslin tree.
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**Theorem.** (Gartside) For a monotonically nomral space $X$, $d(X) = hd(X)$. In particular, every separable monotonically normal space is hereditarily separable (HS).

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# Impossible to shoot an uncountable discrete set to Monotonically normal space

**Theorem.** (No Souslin tree) Let $X$ be a monotonically normal space with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing.

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- By Williams and Zhou's theorem, $X$ is separable.
- Forcing preserves separability and monotone normality.
- In the forcing extension, $X$ remains separable and monotonically normal.
- By Gartside's theorem, separable monotonically normal spaces are hereditarily separable.
- Therefore, in the forcing extension, $X$ is hereditarily separable, and so it has no uncountable discrete subspace.

# Impossible to shoot an uncountable discrete set to LOTS

**Corollary.** (No Souslin tree) Suppose that $X$ is a **linearly ordered topological space** (LOTS) with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing.

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# Scattered spaces

**Definition.** A space $X$ is **scattered** if every subspace contains an isolated point in the relative topology.

**Lemma.** Assume that there is no $S$-space. Then every scattered space with no uncountable discrete subspace is countable.

**Theorem.** Assume that there is no $S$-space. Let $X$ be a scattered space with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing (because $X$ is countable).

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# Kunen line

**Definition.** A space $\langle X, \tau \rangle$ is **cometrizable** if there is a weaker metric topology $\mu \subseteq \tau$ such that each point of $X$ has a neighborhood base consisting of sets which are closed with respect to $\mu$. (Sorgenfrey line is cometrizable.)

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# Destroying Kunen line

**Example.** (CH & $2^{\aleph_1} = \aleph_2$) There are a scattered $S$-space $X$ with no uncountable discrete subspace and a ccc forcing such that in the forcing extension, $X$ contains an uncountable discrete subspace.

**Proof.**

- Let $X$ be the Kunen line, which is a scattered cometrizable $S$-space.
- Let $\mathbb{P}$ be a ccc forcing which forces $MA$ and $2^{\aleph_0} = \aleph_2$. So $MA(\aleph_1)$ holds.
- Forcing preserves cometrizability so by the Todorcevic's theorem, $X$ is not an $S$-space in the forcing extension.
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# Upshot

**Can forcing shoot an uncountable discrete set?**

**(1). NO.**

- Metrizable; developable; stratifiable

**(2) Consistently, NO.**

- Monotonically normal; LOTS (if there is no Souslin tree)
- Scattered (if there is no $S$-space)

**(3) Consistently, YES.**

- Monotonically normal; LOTS; scattered
- Compact; quasi-metrizable; non-archimedean
  (Souslin line can have these properties.)
- Submetrizable (cometrizable)

**(4)** So far no ZFC example of a space where forcing create an uncountable discrete subspace.

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# Thank you

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