Spring 4-2015

# Root Cover Pebbling on Graphs

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**Abstract**

Consider a graph, G, with pebbles on its vertices. A pebbling move is defined to be the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The cover pebbling number of a graph, $\gamma(G)$, is the minimum number of pebbles such that, given any configuration of $\gamma(G)$ pebbles on the vertices of G, pebbling moves can be used to place one pebble on each vertex of G. We define the root vertex of a graph and fix an initial configuration of pebbles on G where we place all pebbles on the root vertex of G. We define the root cover pebbling number, R(G), of a graph to be the minimum number of pebbles needed so that, if R(G) pebbles are placed on the root vertex, pebbling moves can be used to place one pebble on each vertex. We obtain formulas for root cover pebbling numbers of two types of graphs. We use these formulas to compare the cover pebbling number with the root cover pebbling number of paths, stars and fuses. We also determine ways to minimize the root cover pebbling number of a graph.

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## Introduction

A *graph* is a mathematical object that consists of a pair of sets (V,E) where V, the vertex set, is non-empty and E, the edge set, contains pairs of elements of V. If a pair of vertices, *uv,* appears in the edge set, we say that *u* and *v* are *adjacent* [3]. Figure 1 is an example of a graph with circles representing vertices and lines representing edges. In the figure, *u* and *v* are adjacent while *u* and *x* are not adjacent.

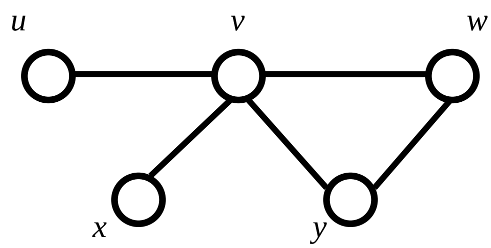


**Figure 1**

Chung [1] defined pebbling on graphs. Pebbles are placed on the vertices of a graph G and moved among the vertices with the goal of moving one pebble to any selected target vertex. Pebbles can only be moved among the vertices of G through the use of pebbling moves. A *pebbling move* consists of removing two pebbles from any vertex, adding one of these pebbles to an adjacent vertex, and removing the other pebble from the graph. This means that for every pebble that is moved, the total number of pebbles on the vertices of G is reduced by one. The *pebbling number* of G, $\pi(G)$, is the minimum number of pebbles needed so that, given any initial configuration with $\pi(G)$ pebbles and any target vertex, pebbling moves can be used to place one pebble on the target vertex.

Cover pebbling is a variation of pebbling defined in [2]. Cover pebbling differs from pebbling in that the goal is to eventually move at least one pebble to each vertex of the graph, instead of trying to place a pebble on just one target vertex. Thus, the *cover pebbling number* of a graph, $\gamma(G)$, is defined to be the minimum number of pebbles needed so that, given any initial configuration of $\gamma(G)$ pebbles on the vertices of G, pebbling moves can be used to place at least one pebble on each vertex of the graph. Crull et al found cover pebbling numbers of certain classes of graphs such as paths, complete graphs and fuses in [2].

In this paper, we define a special case of cover pebbling called *root cover pebbling*. We will fix an initial configuration of pebbles where we can place pebbles on only one designated vertex called the root vertex. We set out the types of graphs we have considered and explain how the root vertex is chosen. We define the *root cover pebbling number,* R(G), to be the minimum number of pebbles needed so that, if all pebbles are initially placed the root vertex, pebbling moves can be used to place at least one pebble on each vertex of the graph. We find root cover pebbling numbers of certain types of graphs. We also determine how to minimize the root cover pebbling number by rearranging the vertices of a graph.

**Types of Graphs**

We consider two types of graphs in this paper.

**Definition**. A *Type 1* graph consists of a *root vertex* and path graphs. The paths, called *pendants*, are attached to the root vertex by adding an edge between the root vertex and exactly one end vertex of each path.

We use $q$ to represent the number of pendants which we label $P_1, P_2, \ldots P_q$. Each pendant contains the same number of vertices as the path that created it. We use $p_1, p_2, \ldots, p_q$ to denote the number of vertices in each pendant with $P_i$ containing $p_i$ vertices.

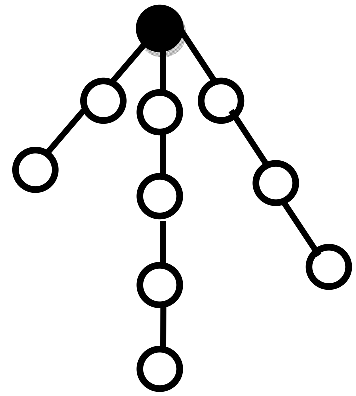

**Figure 2**

The star graph $S_n$ is a Type 1 graph with $p_i = 1$ for all $i$ = 1, 2, ..., $q$. Figure 2 is an example of a Type 1 graph with $q$ = 3, $p_1 = 2, p_2 = 4, p_3 = 3$. In the figure, the root vertex is colored black.

**Definition**. A *Type 2* graph consists of a cycle C, and path graphs. Each path is attached to exactly one vertex on the cycle, called a *root vertex*, by adding an edge from the root to one end vertex of the path. Each path is attached to exactly one root vertex and each root vertex is attached to exactly one path. If a path is attached to a vertex of the cycle it is called a *pendant.*

We use $c$ to denote the length of the cycle and $q$ to denote the number of pendants. Each pendant contains the same number of vertices as the path that created it. We use $p_1, p_2, \ldots, p_q$ to denote the number of vertices in each pendant with $P_i$ containing $p_i$ vertices. Since the number of pendants is equal to the number of root vertices, we use $P_1, P_2, \ldots, P_q$ to denote the pendants and $r_1, r_2 \ldots, r_q$ to denote the root vertices with $P_i$ attached to $r_i$ for all $i$ = 1, 2, ..., $q$.

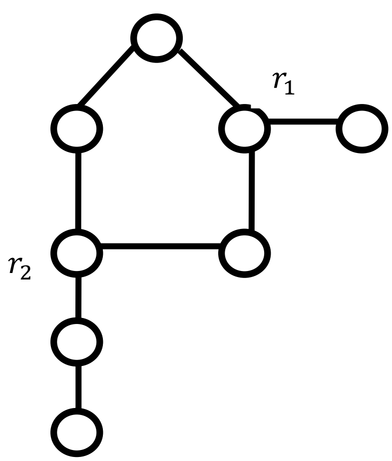


**Figure 3**

A lollipop is a Type 2 graph with $q$ = 1. This special case will be the main Type 2 graph that we discuss. Figure 3 is an example of a Type 2 graph with $q$ = 2, $p_1 = 1$ and $p_2 = 2$. The root vertices are labeled.

## Root Cover Pebbling Number

To find the root cover pebbling number of the types of graphs described above we use the cover pebbling number of path graphs given in [2]. The cover pebbling number for a path of length n is given as follows:

$$\gamma(P_n) = 2^n - 1 = 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0$$

Observe that this can be obtained by adding the numbers of pebbles needed to move one pebble to each vertex independently when all pebbles are initially placed on one end vertex. We can find the root cover pebbling numbers of Type 1 and Type 2 graphs by considering the minimum number of pebbles needed to reach an individual vertex from the root vertex, and then taking the sum over all vertices.

Root Cover Pebbling Number of Type 1 Graphs

Let $G_q$ denote a Type 1 graph with $q$ pendants and $n$ vertices. Recall that $p_1, p_2, \dots, p_q$ denote the number of vertices in each pendant.

**Theorem 1**. $R\left(G_q\right) = 1 + \sum_{i=1}^{q}(2^{p_i+1} - 2)$

*Proof*: The proof is by induction on the number of pendants.

Suppose $q = 1$. Since $G_1 = P_n$, $R(G_1) = 1 + 2^{p_1+1} - 2 = 2^{p_1+1} - 1$. Since $n = p_1 + 1$, we know this is correct from [2]. If $\sum_{i=1}^{q}(2^{p_i+1} - 2)$ pebbles are placed on the root vertex of $G_q$, then each $P_i$ can be covered with $(2^{p_i+1} - 1) - 1$ pebbles since each $P_i$ , along with the root, creates a path of length $p_i + 1$ and the root is not covered. Thus, we have used all of the $\sum_{i=1}^{q}(2^{p_i+1} - 2)$ pebbles and the root vertex cannot be covered so $R(G_q) \geq \sum_{i=1}^{q}(2^{p_i+1} - 2)$.

Now suppose there are $1 + \sum_{i=1}^{q+1}(2^{p_i+1} - 2)$ pebbles on the root vertex of $G_{q+1}$ and assume the hypothesis is true for every $G_q$. If all the pendants are covered, then we are done. Suppose there is some pendant that is not covered. Without loss of generality, call it $P_{q+1}$ Remove $P_{q+1}$ from the graph. This creates a graph with $q$ pendants which, by the induction hypothesis, can be covered with $1 + \sum_{i=1}^{q}(2^{p_i+1} - 2)$ pebbles. Since covering $G_{q+1}$ means that one pebble will remain on the root vertex, we have1 + $2^{p_{q+1}+1} - 2$ pebbles remaining on the root vertex. We know from the base case that $P_{q+1}$ can be covered with $2^{p_{q+1}+1} - 2$ pebbles. Therefore,

$$R(G_{q+1}) = 1 + (2^{p_{q+1}+1} - 2) + \sum_{i=1}^{q}(2^{p_i+1} - 2) = 1 + \sum_{i=1}^{q+1}(2^{p_i+1} - 2).$$

Root Cover Pebbling on Type 2 Lollipop Graphs

**Lemma.** Let *C* be a cycle and *c* denote the number of vertices in the cycle. Let *m* be a positive integer. Then

i. When $c = 2m+1$, $R(C) = 2^{m+2} - 3$
ii. When $c = 2m$, $R(C) = 2^{m+1} + 2^m - 3$

*Proof*: Let *C* be a cycle with *c* vertices, and let *m* be a positive integer. Since all vertices on the cycle are identical, we can choose any vertex to be the root. Fix a root vertex.

i. Let $c = 2m + 1$. Suppose $2^{m+2} - 4$ pebbles are placed on the root vertex. But this is the same as $(2^{m+1} - 2) + (2^{m+1} - 2)$ pebbles. There are two vertices each of shortest distance *m*, *m*-1, ..., 2, 1 away from the root vertex. Thus we need $2(2^m + 2^{m-1} + \cdots + 2^2 + 2^1)$ pebbles on the root vertex. This is equal to $2(2^{m+1} - 2) = 2^{m+2} - 4$ pebbles. But we have used all the pebbles and the root vertex remains uncovered. Thus, adding 1 pebble to the root vertex will cover all vertices of *C* so $R(C) = 2^{m+2} - 3$ when $c = 2m+1$.
ii. The case where $c = 2m$ is similar. Let $c = 2m$ and suppose $2^{m+1} + 2^m - 4$ pebbles are placed on the root vertex. This is the same as $(2^m - 2) + (2^{m+1} - 2)$ pebbles. In this case, there is only one vertex on the cycle with shortest distance *m* from the root. There are two vertices each of shortest distance *m*-1, *m*-2, ..., 2, 1 from the root vertex. Using the formula for cover pebbling paths, we find that we need $(2^m + 2^{m-1} + \cdots + 2^2 + 2^1) + (2^{m-1} + \cdots + 2^2 + 2^1)$ pebbles to cover these vertices. But this is the same as $(2^{m+1} - 2) + (2^m - 2) = 2^m + 2^{m+1} - 4$. Now we have used all the pebbles but not covered the root vertex. Therefore placing one more pebble on the root vertex will cover *C* so $R(C) = 2^{m+1} + 2^m - 3$ pebbles when $c = 2m$.

**Theorem 2**. Let $H_1$ be a type 2 graph with one cycle C and one pendant $P_1$ with $p_1$ vertices in the pendant. Then $R(H_1) = R(C) + (2^{p_1+1} - 2)$.

*Proof*: Suppose $H_1$ is a Type 2 graph with one cycle *C* and one pendant $P_1$ containing $p_1$ vertices. Suppose there are $R(C) + (2^{p_1+1} - 3)$ pebbles on the root vertex. We know from the lemma that $R(C)$ pebbles will cover the cycle including the root vertex. This leaves $(2^{p_1+1} - 3)$ pebbles plus one more from covering the cycle. We therefore have $1 + (2^{p_1+1} - 3) = (2^{p_1+1} - 2)$ pebbles on the root vertex. We know from Theorem 1 that we can use $(2^{p_1+1} - 2)$ pebbles to cover the pendant with $p_1$ vertices. But this leaves the root vertex uncovered. Therefore adding one pebble to the root vertex will cover the graph. So $R(H_1) = R(C) + (2^{p_1+1} - 2)$.

**Root Cover Pebbling vs.**
**Cover Pebbling**

We use the formula for root cover pebbling number of Type 1 graphs above to compare the cover pebbling number and root cover pebbling number of paths, stars and fuses. The formulas for cover pebbling numbers of these graphs come from [2].

Path Graphs

We know from [2] that the cover pebbling number of $P_n$, $\gamma(P_n) = 2^n - 1$, and we know from the base case of Theorem 1 that this is equal to the root cover pebbling number of a Type 1 graph with $n$ vertices and one pendant.

Another way to represent a path graph $P_n$ is as a Type 1 graph with two pendants. This gives a path that is "bent" somewhere in the middle as shown in Figure 4 where $n = 5$.

Thus, $R(P_n) = 1 + (2^{p_1+1} - 2) + (2^{n-p_1} - 2)$ where $0 < p_1 < n$. Therefore, $(p_1 + 1) \leq n$ and $(n - p_1) < n$ giving us

$$R(P_n) = 1 + 2^{p_1+1} + 2^{n-p_1} - 4$$

$$< 2(2^{n-1}) - 3 = 2^n - 3$$

$$< 2^n - 1 = \gamma(P_n)$$

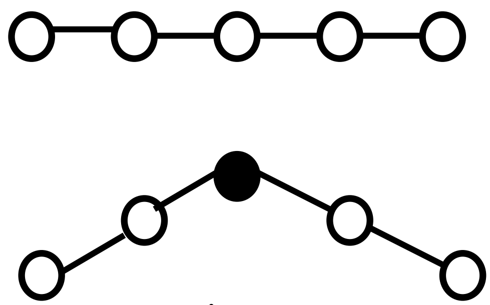

**Figure 4**

So when we choose a root such that $P_n$ is a Type 1 graph with 2 pendants, $R(P_n) < \gamma(P_n)$ for any choice of the root.

Star Graphs

If $G_q$ is a Type 1 graph with $n$+1 vertices and $q$ pendants, then the star graph, $S_{n+1}$, is a special case of this graph where $q = n$. Thus $p_i = 1$ for $i$ = 1, 2, ..., $q$. Figure 5 shows the star graph as a Type 1 graph with one root vertex and 8 pendants each containing one vertex. From the formula for the root cover pebbling number of Type 1 graphs, we get given as

$$R(S_{n+1}) = 1 + \sum_{i=1}^{q}(2^{p_i+1} - 2) = 1 + \sum_{i=1}^{q}(2^{1+1} - 2)$$

$$= 1 + q(2^2 - 2) = 2q + 1 = 2n + 1$$

$$< 4(n + 1) - 5 = 4n - 1 = \gamma(S_{n+1})$$

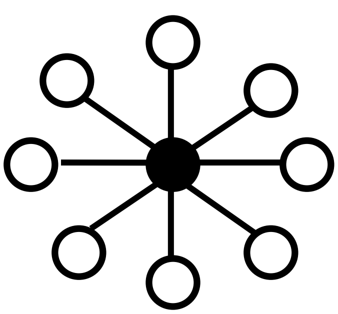

**Figure 5**

This inequality is true when $n \geq 2$. When $n = 1$, $R(S_{n+1}) = \gamma(S_{n+1})$.

Fuse Graphs

The fuse graph $F_l(n+1)$ is a Type 1 graph on $n$+1 vertices with $q$ = (n+1)-(l-1) pendants. One of these pendants contains $l$-1 vertices and the remaining $n$-$l$+1 pendants each contain one vertex. From the formula for root cover pebbling number of Type 1 graphs, we have that

$$R(F_l(n+1)) = 1 + (2^l - 2) + (n - l + 1)(2^2 - 2)$$

$$= (2^l - 1) + 2(n - l + 1)$$

$$= 2^l + 2n - 2l$$

$$< (n - l + 2)2^l - 1 = \gamma(F_l(n+1))$$

whenever $n > 2\ , 1 < l < n$. So the root cover pebbling number of fuse graphs is less than the cover pebbling number of fuse graphs for these values.

**Minimizing Root Cover Pebbling Number**

we consider how best to distribute the vertices among the pendants in order to obtain the Given a Type 1 graph with a fixed number of vertices and a fixed number of pendants, lowest possible root cover pebbling number. The next theorem is an answer to this problem.

**Lemma.** Let $G_q$be a graph with $n$+1 vertices, $q$ pendants and let $R(G_q)$ denote the root cover pebbling number. Consider any two pendants $P_i$ and $P_j$ with $p_i$ and $p_j$vertices, respectively. If the pendants are made more even (that is if for, $p_i < p_j$, one vertex is added to $P_i$ and one vertex is removed from $P_j$.) then the rroot cover pebbling number of the graph decreases.

*Proof*: Let $G_q$be a Type 1 graph with $q$ pendants and $n$+1 vertices. Consider $P_i$ and $P_j$ with $p_i$ and $p_j$ vertices and $p_i \leq p_j$. Then we need $2^{p_i+1} - 2$ pebbles on the root vertex to cover $p_i$ and $2^{p_j+1} - 2$ pebbles on the root vertex to cover $P_j$. Consider the graph $G_q{}'$ which is obtained by adding one vertex to $P_i$ and removing one vertex from $P_j$. This is equivalent to removing pendants of lengths $p_i$ and $p_j$, and adding pendants of lengths $(p_i + 1)$ and $(p_j - 1)$. Then $G_q{}'$ still has $n$+1vertices and $q$ pendants. Note that if $p_i = p_j - 1$, then adding one vertex to $p_i$ and removing one vertex from $p_j$creates a graph that is isomorphic to the original. So suppose $p_i$<$p_j - 1$. Then we have

$$R(G_q') = R(G_q) - (2^{p_i+1} - 2) - (2^{p_j+1} - 2) + (2^{p_i+2} - 2)\ + (2^{p_j} - 2)$$

This gives us

$$R(G_q') = R(G_q) + 2^{p_i}(-2^1 - 2^{p_j-p_i+1} + 2^2 + 2^{p_j-p_i})$$
$$= R(G_q) + 2^{p_i}(2^{p_j-p_i} - 2^{p_j-p_i+1} + 2)$$
$$= R(G_q) + 2^{p_i}(2^{p_j-p_i} - 2(2^{p_j-p_i}) + 2)$$
$$R(G_q') = R(G_q) + 2^{p_i}(2 - 2^{p_j-p_i})$$

Since $0 < p_i < p_j - 1$, $2^{p_j-p_i} > 2$. Therefore $2^{p_i}(2 - 2^{p_j-p_i}) < 0$ and $R(G_q') < R(G_q)$.

**Theorem 3.** Let $G_q$ be a Type 1 graph with *n*+1 vertices and *q* pendants. The root cover pebbling number of $G_q$ is minimized when all pendants are of equal length.

*Proof*: Let $G_q$ be a Type 1 graph with *n*+1 vertices and *q* pendants. Assume for sake of contradiction that $R(G_q)$ is minimized but all pendants are not equal. Then there exists a pair of pendants $P_i$ and $P_j$ such that $p_i < p_j$. We create $G_q'$ by removing one vertex from $P_j$ and adding it to $P_i$. $G_q'$ is a graph with *n*+1 vertices and *q* pendants. If $p_i = p_j - 1$, then $G_q$ is isomorphic to $G_q'$ so the root cover pebbling number does not change. If $p_i < p_j - 1$ we know from the lemma that $R(G_q') < R(G_q)$. But this is a contradiction because $R(G_q)$ is the minimum root cover pebbling number for Type 1 graphs with *n*+1 vertices and *q* pendants. We have a contradiction and our assumption that the pendants do not all contain an equal number of vertices is incorrect. Therefore the root cover pebbling number of a Type 1 graph with *n*+1 vertices and *q* pendants is minimized when the pendants contain equal an equal number of vertices.

**Corollary**. Let $H_1$ be a Type 2 graph with *n* vertices and one pendant (a lollipop graph). Let $p_1$ be the number of vertices in the pendant. Then $R(H_1)$ is minimized when $c = 2p_1 + 1$.

*Proof*: Let $H_1$ be a Type 2 graph with *n* vertices and 1 pendant. Let c = $2p_1 + 1$. Consider altering $H_1$ by removing the edge on the cycle that is farthest from the root vertex, breaking the cycle into two equal pendants of length $\frac{c-1}{2} = \frac{2p_1}{2} = p_1$ vertices each since the root vertex is not included in either pendant. We now have a Type 1 graph with three pendants with an equal number of vertices and, by Theorem 3, this minimizes the root cover pebbling number of $H_1$.

**Number of Pendants**

We now consider a Type 1 graph and fix the number of vertices to be *n*+1 (one root vertex and *n* vertices distributed among the pendants). Observe that $R(S_{n+1}) = 2n + 1$

while $R(P_{n+1}) = 2^{n+1} - 1$. Thus $R(S_{n+1})$ is linear and $R(P_{n+1})$ is exponential. $R(S_{n+1}) < R(P_{n+1})$ for all positive integers *n*. It appears that increasing the number of pendants could minimize the root cover pebbling number of the graph. We will show that for certain choices of *n* and *q*, this is not the case.

**Conjecture 1**. Let $G_q$ be a Type 1 graph with *n*+1 vertices and *q* equal (or nearly equal) pendants. Let $G_{q+1}$ be a Type 1 graph with *n*+1 vertices and *q*+1 pendants. In $G_{q+1}$, let *q* pendants contain one vertex and one pendant contain *n*-*q* vertices. We want to show that $R(G_q) \leq R(G_{q+1})$ for certain choices of *n* and *q*.

Let $G_q$ be a Type 1 graph with *n*+1 vertices and *q* pendants of equal (or near equal) length. Then each pendant contains either $\left\lfloor \frac{n}{q} \right\rfloor$ or $\left\lfloor \frac{n}{q} \right\rfloor + 1$ vertices, say, *i* pendants with $\left\lfloor \frac{n}{q} \right\rfloor$ vertices and *j* pendants with $\left\lfloor \frac{n}{q} \right\rfloor + 1$ vertices. From the formula for root cover pebbling number of Type 1 graphs we have

$$R(G_q) = 1 + \sum_{k=1}^{i} \left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 1} - 2\right) + \sum_{k=1}^{j} \left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 2} - 2\right)$$

$$= 1 + i\left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 1} - 2\right) + j\left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 2} - 2\right)$$

$$< 1 + q\left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 2} - 2\right)$$

For $G_{q+1}$, the formula from root cover pebbling of Type 1 graphs gives us

$$R(G_{q+1}) = 1 + (2^{n-q+1} - 2) + q(2^2 - 2)$$

$$= 1 + (2^{n-q+1} - 2) + 2q$$

We want to find integers *n* and *q* for which

$$q\left(2^{\left\lfloor \frac{n}{q} \right\rfloor + 2} - 2\right) < (2^{n-q+1} - 2) + 2q$$

For example, we know that when *n* = 20 and *q* = 2 this inequality holds. We therefore know that, in a Type 1 graph, it is not always the case hat increasing the number of pendants decreases the root cover pebbling number for a fixed number of vertices. Finding details about the relationship between number of pendants and root cover pebbling number is a topic for further exploration.

**Open Questions**

- For a Type 1 graph with a fixed number of vertices, which distributions of the vertices among $q$ pendants gives a lower root cover pebbling number than a distribution among $q+1$ pendants?
- For a Type 2 graph with $q > 1$, how can the pendants be arranged on the cycle so that the root cover pebbling number of the graph is minimized?
- For a Type 2 graph with $q > 1$, a fixed number of vertices and a fixed configuration of pendants, how can we choose the root vertex to minimize root cover pebbling number for the graph?
- For a Type 2 graph with $q > 1$, how can pebbles be distributed among multiple root vertices to minimize the root cover pebbling number of the graph?

**Acknowledgements**

Thank you to the University of Dayton Honors Program and to the University of Dayton Department of Mathematics.

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