2015

Metalogic Notes

Saverio Perugini
University of Dayton, sperugini1@udayton.edu

*Meta*logic Notes

Saverio Perugini

Department of Computer Science
University of Dayton
300 College Park
Dayton, Ohio 45469–2160 USA

Tel: +001 (937) 229–4079
Fax: +001 (937) 229–2193

E-mail: `saverio@udayton.edu`
WWW: `http://academic.udayton.edu/SaverioPerugini`

Contents

1 Reader's Guide: Abbreviations and Notation **3**

2 Foundations from Symbolic Logic **3**

3 Syntactic Notions **3**
3.1 Syntactic *Meta*-theorems of $\mathcal{P}_S$ 6

4 Foundations from Discrete Mathematics **6**
4.1 Reductio proof of the uncountability of $P(\mathbb{N})$ 7
4.2 Cardinality of some common sets 8
4.3 Theorems about infinite sets 8

5 Semantic Notions **8**
5.1 Definition of true for an interpretation of $\mathcal{P}$ 9
5.2 Theorems about truth-functional propositional logic 9
5.3 Some truths about $\vDash_{\mathcal{P}}$. The Interpolation Theorem for $\mathcal{P}$ 10

6 Linkage between Syntax and Semantics **10**

7 Soundness and Completeness of $\mathcal{P}_S$ **10**
7.1 Soundness of $\mathcal{P}_S$: if $\vdash_{\mathcal{P}_S} A$, then $\vDash_{\mathcal{P}_S} A$ 10
7.2 Semantic Completeness of $\mathcal{P}_S$ ('The Big Picture'): if $\vDash_{\mathcal{P}_S} A$, then $\vdash_{\mathcal{P}_S} A$ . . . 11
7.2.1 Corollaries from Semantic Completeness of $\mathcal{P}_S$ 11
7.3 Decidability of $\mathcal{P}_S$ 11
7.4 Informal proof of the incompleteness of a ffs of the full theory of $\mathbb{N}$ 12

8 Quantified (Predicate) Logic: $\mathcal{Q}_S$ **12**
8.1 Semantic Notions 12
8.2 Semantic *Meta*-theorems for $\mathcal{Q}_S$ 14
8.3 Syntactic Notions 15
8.4 Syntactic *Meta*-theorems for $\mathcal{Q}_S$ 15
8.5 Linkage *Meta*-theorems for $\mathcal{Q}_S$ 15
8.6 Con 15

9 First-order Theories **15**
9.1 *Meta*-theorems of K 16

10 Isomorphism of Models, Axioms of Arithmetic, and Non-standard Models **17**
10.1 Example 18
10.2 Isomorphism of Models 19
10.3 Axioms of Arithmetic 19
10.4 Non-standard Models 19

11 Göedel's First Incompleteness Theorem **19**

1 Reader's Guide: Abbreviations and Notation

1. $A \mapsto B$ denotes the relation 'A entails B as an immediate consequence.'
2. $\models_{\mathcal{S}} A$ denotes 'A is a logical truth of formal system $\mathcal{S}$.'
3. $\vdash_{\mathcal{S}} A$ denotes 'A is provable in formal system $\mathcal{S}$.'
4. $\simeq$ denotes 'same cardinality' (e.g., $A \simeq B$)

2 Foundations from Symbolic Logic

1. **Adequacy of connectives**: The set $\{\sim, \supset\}$ is adequate.
2. **DeMorgan's Laws**:
   $\sim (A \wedge B) \equiv \sim A \vee \sim B$
   $\sim (A \vee B) \equiv \sim A \wedge \sim B$
3. **Material Implication**: $p \supset q \equiv \sim p \vee q$
4. **Exportation**: $p \supset q \supset r \equiv\ (p \wedge q) \supset r$
5. **Transposition** (equivalence of a conditional and its contrapositive–mixing of the inverse and the converse): $p \supset q \equiv\ \ \sim q \supset\ \ \sim\ p$
6. : p *iff* $q \equiv\ \ \sim p$ *iff* $\sim q$

3 Syntactic Notions

1. **Formal Language**: A *formal language* $\mathcal{L}$ is specified by

   (a) a set of *symbols* known as the *alphabet* of $\mathcal{L}$

   (b) a set of *formation rules* or *rules of formation* determining which sequences of symbols from the language's alphabet are *wffs* (well-formed formulas) in the language.

   A formal language can be

   (a) *completed defined without reference to any interpretation for it* and

   (b) *identified with the set of its wffs.*

2. **Rule of Formation**: A *rule of formation* determines which sequences of symbols from the alphabet of a formal language are wffs.
3. **Deductive Apparatus**: A *deductive apparatus* can consist of

   (a) *axioms* and/or

(b) *rules of inference* (also know as *transformation rules*).

It must be definable without reference to any intended interpretation of the language: otherwise the system is not a formal system. E.g., Modus Ponens.

4. **Formal System**: A *formal system* $\mathcal{S}$ is
   (a) a formal language $\mathcal{L}$, and
   (b) a deductive apparatus.
5. **Axiom**: An *axiom* is a wff of a formal system determined by its designer.
6. **Rule of Inference**: A *rule of inference* determines which relations between formulas of $\mathcal{L}$ are relations of *immediate (deductive) consequence* in $\mathcal{S}$. Defined with the $\mapsto$ ('entails as an immediate consequence') relation.
7. **Proof in a Formal System**: A *proof in a formal system* of a formula $\mathcal{A}$ is
   (a) a finite, but non-empty, string of wffs of the formal language,
   (b) which satisfies certain purely syntactic requirements and has no meaning, and
   (c) where the last formula in the string is $\mathcal{A}$, and
   (d) each wff in the string is either
      i. an axiom of the formal system, or
      ii. an immediate consequence by the rule of inference of the system of any two wffs preceding it in the string.
8. **Theorem of a Formal System**: A *theorem of a formal system* is a wff (formula) of a formal language that satisfies certain purely syntactic requirements and has no meaning. $\mathcal{A}$ is a theorem of a system $\mathcal{S}$ if $\vdash_S \mathcal{A}$ (i.e., if there is some proof in $\mathcal{S}$ whose last formula is $\mathcal{A}$). Every axiom is a theorem, but not every theorem is an axiom.
9. **Theorem about a Formal System**: A *theorem about a formal system* (also called a *metatheorem*) is a true statement about the system, expressed in the *meta*language. It cannot be proved within the system.
10. **Syntax**: Having to do with formal languages or formal systems without essential regard to their interpretation. Slightly wider than 'proof-theoretic' since it can be applied to properties of formal languages without deductive apparatuses, as well as to formal systems.
11. **Finitary Formal System**: A *finite formal system* (ffs) has the following properties:
    (a) alphabet is countable (finite or denumerable).
    (b) wffs are finite in length.
    (c) rules of inference have finitely many premises (e.g., Modus Ponens has only 2).

12. **Derivation**: A *derivation* of a formula $\mathcal{A}$ in $\mathcal{P}_S$ from a set Γ of wffs of $\mathcal{P}$ is
    (a) a finite, but non-empty, string of wffs of $\mathcal{P}$,
    (b) where the last formula in the string is $\mathcal{A}$, and
    (c) each wff in the string is either
        i. an axiom of $\mathcal{P}_S$,
        ii. the result of applying MP to any two wffs preceding it in the string, or
        iii. a member of the set Γ

    The concept of a derivation addresses the desire to prove things from a set of assumptions.

13. **Difference between a derivation and a proof**:
    (a) In a *proof in* $\mathcal{P}_S$ *every* formula is a *theorem* of $\mathcal{P}_S$.
    (b) In a *derivation in* $\mathcal{P}_S$ formulas may occur in the string that are *not* theorems of $\mathcal{P}_S$ (e.g., formulas from Γ, if Γ is a set of formulas that are not theorems of $\mathcal{P}_S$).

14. **Syntactic consequence**: A formula A is a *syntactic consequence* in $\mathcal{P}_S$ of a set of Γ of formulas of $\mathcal{P}[\Gamma \vdash_{\mathcal{P}_S} A]$ *iff* there is a derivation in $\mathcal{P}_S$ of A from the set Γ.

15. **Proof-theoretic consistent (or p-consistent or $\mathsf{p\text{-}Con}$)**: A set Γ of formulas is $\mathsf{p\text{-}Con}$ (denoted by $\mathsf{Con}\ \Gamma$) *iff* for no wff A, $\Gamma \vdash A$ and $\Gamma \vdash \sim A$.

16. **Proof-theoretic inconsistent (or p-inconsistent)**: A set Γ of formulas is p-inconsistent *iff* for some wff A of $\mathcal{P}$ both $\Gamma \vdash A$ and $\Gamma \vdash \sim A$.

17. **Deduction Theorem for $\mathcal{P}_S$**: if $\Gamma, A \vdash_{\mathcal{P}_S} B$, then $\Gamma \vdash_{\mathcal{P}_S} A \supset B$

18. **Simple consistency**: A formal system $\mathcal{S}$ is *simply consistent iff* for no formula A of $\mathcal{S}$ are both A and $\sim A$ theorems of $\mathcal{S}$ (i.e., the deductive apparatus of $\mathcal{S}$ will not lead to contradictions).

19. **Absolute consistency**: A formal system $\mathcal{S}$ is *absolutely consistent* if at least one formula of $\mathcal{S}$ is not a theorem of $\mathcal{S}$ (i.e., $\exists A$ s.t. $\nvDash A$).

20. **Maximally p-consistent**: Γ is a maximally p-consistent set of $\mathcal{P}_S$ (denoted by $\mathsf{MaxCon}\ \Gamma$) *iff*
    (a) $\mathsf{Con}\ \Gamma$, and
    (b) any for any arbitrary formula A of $\mathcal{P}$, if $A \notin \Gamma$, then $\not\mathsf{Con}\ \Gamma \cup \{A\}$. I.e., if A is any arbitrary formula of $\mathcal{P}$, then either
        i. $A \in \Gamma$ or
        ii. $\not\mathsf{Con}\ \Gamma \cup \{A\}$ (i.e., $\Gamma \cup \{A\} \vdash_{\mathcal{P}_S} B$ and $\Gamma \cup \{A\} \vdash_{\mathcal{P}_S} \sim B$ for some formula B of $\mathcal{P}$).

    A MaxCon set is the biggest possible p-consistent set of wffs. The concept of a maximally p-consistent set was introduced by Henkin to help prove the completeness of $\mathcal{P}_S$.

3.1 Syntactic *Meta*-theorems of $\mathcal{P}_S$

1. $A \vdash A$
2. If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$ (Augmentation)
3. If $\Gamma \vdash A$ and $A \vdash B$, then $\Gamma \vdash B$
4. If $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$, then $\Gamma \vdash B$ (*Meta*-MP)
5. If $\vdash A$, then $\Gamma \vdash A$
6. $\vdash A$ *iff* $\emptyset \vdash A$
7. $\Gamma \vdash A$ *iff* $\Delta \subseteq \Gamma$ and $\Delta \vdash A$ (if $A \in \Gamma$, then $\Gamma \vdash A$; A is the derivation)
8. $\cancel{\mathsf{Con}}\ \Gamma \cup \{\sim A\}$ *iff* $\Gamma \vdash A \equiv \mathsf{Con}\ \Gamma \cup \{\sim A\}$ *iff* $\Gamma \nvdash A$
9. $\cancel{\mathsf{Con}}\ \Gamma \cup \{A\}$ *iff* $\Gamma \vdash \sim A \equiv \mathsf{Con}\ \Gamma \cup \{A\}$ *iff* $\Gamma \nvdash \sim A$
10. **Fullness of MaxCon sets**: For any MaxCon set Γ, exactly one of A or $\sim A$ is in Γ.
11. **Maximally p-consistent** (alternate way): Γ is a maximally p-consistent set of $\mathcal{P}_S$ (denoted by MaxCon Γ) *iff*
    (a) Con Γ, and
    (b) Γ is full (i.e., for any arbitrary wff A of $\mathcal{P}_S$, either $A \in \Gamma$ or $\sim A \in \Gamma$, but not both).
12. **Deductive closure of MaxCon sets**: For any MaxCon set Γ and any formula A, if $\Gamma \vdash_{\mathcal{P}_S} A$, then $A \in \Gamma$ (implies every theorem is in a MaxCon set; if $\vdash A$ and MaxCon Γ, then $A \in \Gamma$).
13. **Enumeration Theorem for $\mathcal{P}$**: The wffs of $\mathcal{P}$ are effectively enumerable (i.e., given an arbitrary wff, $\exists$ an effective method for determining its position in the enumeration, and given a position in the enumeration, an effective method exists for determining to which wff it refers).
14. (32.12) **Lindenbaum's Lemma**: if Con Γ, then $\exists$ a set of wffs Δ s.t. $\Gamma \subseteq \Delta$ and MaxCon Δ (i.e., informally, every Con set is a subset of a MaxCon set).

4 Foundations from Discrete Mathematics

1. **Effective method**: An *effective method* for solving a problem is a method for computing the answer that, if followed correctly and as far as may be necessary, is logically bound to give the right answer (and no wrong answers) in a finite number of steps.
2. **Decidable set**: A set is *decidable iff* there is an effective method for telling, for each item that might be a member of the set, whether or not it really is a member.

3. **Finite set**: A set is *finite iff* it has only a finite number of members.
Every finite set is decidable.

4. **Denumerable set**: A set is *denumerable iff* there is a 1–1 correspondence between it and the set of natural numbers (so a denumerable set is an infinite set).

5. **Countable set**: A set is *countable iff* it is either finite or denumerable (so an uncountable set is an infinite set).

6. **Sequence**: A *sequence* is an ordering of objects, called *terms* in the sequence.

7. **Enumeration**: An *enumeration* of a set A is a finite or denumerable sequence of which every member of A is a term and every term is a member of A (e.g., the sequences $\prec 3, 1, 2 \succ$ and $\prec 1, 2, 3, 1 \succ$ are both enumerations of the set $\{1, 2, 3\}$.

8. **Effective enumeration**: An *effective enumeration* is an enumeration which is finite or for which there is an effective method for telling what the nth term is, for positive integer n (Every finite enumeration is effective, because there is an effective method—remember it need not be known to anyone—for enumerating the members of any finite set).

9. **The Power Set Axiom**: For every set there exists its power set. Used to establish the relationship between the cardinality of a set and its power set (i.e., $\bar{\bar{S}} < \overline{\overline{P(S)}}$). Also used to establish that there are sets with cardinal numbers greater than $\aleph_1 = 2^{\aleph_0} = c$.

10. **The Continuum Hypothesis**: There is no cardinal number greater than $\aleph_0$ and smaller than $\aleph_1 = 2^{\aleph_0} = c$ (has yet to be proved).

4.1 Reductio proof of the uncountability of $P(\mathbb{N})$

Any alleged 1–1 correspondence between $P(\mathbb{N})$ and $\mathbb{N}$ entails stating that $\exists$ a list or enumeration $L = \{s_0, s_1, s_2, \ldots, s_m, \ldots\}$ of the subsets of $\mathbb{N}$.

Claim: There is a subset of $\mathbb{N}$, call it $\bar{D}$, which cannot be included in L.

Definition: Define $\bar{D}$ as, for each $n \in \mathbb{N}$, $n \in \bar{D}$ *iff* $n \notin s_n$.

Spoze: $\bar{D} \in L$. Say $\bar{D} = s_m$ for some natural number m.

Case 1: $m \in \bar{D}$. By definition of $\bar{D}$, $m \notin s_m$, and then $m \notin \bar{D}$ since we have spozed $\bar{D} = s_m$.

Case 2: $m \notin \bar{D}$. So then $m \notin s_m$ because we have spozed that $\bar{D} = s_m$, and then $m \in \bar{D}$ by definition of $\bar{D}$.

Case 1 and 2 together yield $m \in \bar{D}$ *iff* $m \notin \bar{D}$. This is a contradiction.

$\therefore$ **$P(\mathbb{N})$ is not denumerable. It also is not finite since it contains denumerably many unit (singleton) sets.**

Since $P(\mathbb{N})$ is neither finite, nor denumerable, it is uncountable.

4.2 Cardinality of some common sets

1. The power set of the set of natural numbers, i.e., $P(\mathbb{N})$, is uncountable ($\aleph_1$).
2. The set of truths of the full theory of $\mathbb{N}$ is uncountable ($\aleph_1$).
3. The set of theorems of a ffs of the full theory of $\mathbb{N}$ is denumerable ($\aleph_0$).
4. The set of all *finite* strings of 1's and 0's (or +'s and -'s) is denumerable ($\aleph_0$).
5. The set of all *denumerable* strings of 1's and 0's (or +'s and -'s) that from some point on are all 0's is denumerable ($\aleph_0$).
6. The set of all *denumerable* strings of 1's and 0's (or +'s and -'s) is uncountable ($\aleph_1$).
7. The set of all finite subsets of $\mathbb{N}$ is denumerable ($\aleph_0$).
8. The set of truths of the full theory of arithmetic is uncountable ($\aleph_1$).
9. The set of irrational numbers (i.e., the real numbers ($\aleph_1$) minus the rational numbers ($\aleph_0$); those numbers which cannot be expressed as a ratio of two integers, e.g., π or $\sqrt{2}$) is uncountable ($\aleph_1$).

4.3 Theorems about infinite sets

1. Any subset of a countable set is countable.
2. The union of a denumerable set and a finite set is denumerable.
3. The union of a denumerable set and a denumerable is a denumerable set.
4. The union of a countable set and a finite set is countable.
5. The union of a countable set and a countable set is countable.
6. The removal from an uncountable set of countably many members leaves an uncountable set remaining.

5 Semantic Notions

1. **Semantics**: Having to do with the interpretation of formal languages or simply 'model-theoretic.'

2. **Interpretation**: An *interpretation* of a formal language is an assignment of meanings to its symbols and/or wffs.

   An *interpretation* of P is an assignment to each propositional symbol of P of one or the other (but not both) of the truth values truth and falsity, and an assignment to the connectives of P of their usual truth-functional meanings.

3. **Model Theory**: *Model theory* is the theory of interpretation of formal languages (i.e., $\models_I A$).

4. **Model**: A *model* of a wff (formula) of a language is an interpretation of the language for which the formula comes out true.

5. **Semantic consequence**: A formula B of $\mathcal{P}$ is a *semantic consequence* of a formula A (or a set Γ of formulas) of $\mathcal{P}$ $[A \models_{\mathcal{P}_S} B]$ $([\Gamma \models_{\mathcal{P}_S} B])$ *iff* there is no interpretation of $\mathcal{P}$ for which A (every formula in Γ) is true and B is false.

6. **Model-theoretic consistency**: A formula or set Γ of formulas is *model-theoretically consistent* (or m-consistent or $\mathsf{m\text{-}Con}$ or satisfiable; denoted by $\mathsf{Sat}\ \Gamma$) *iff* it has a model $(\forall A \in \Gamma, \models_I A)$.

7. **Model-theoretic inconsistency (or m-inconsistent)**: A formula or set Γ of formulas is *model-theoretically inconsistent* (or m-inconsistent or unsatisfiable; denoted by $\mathsf{Sat}\ \Gamma$) *iff* it has no model (i.e., $\exists A$, s.t. $\not\models_I A$).

5.1 Definition of true for an interpretation of $\mathcal{P}$

Let I be an interpretation of $\mathcal{P}$, and A and B any formulas of $\mathcal{P}$. Then:

1. If A is a propositional symbol, then A is true for I *iff* I assigns the truth value truth to A.

2. $\sim A$ is true for I *iff* A is not true for I.

3. $(A \supset B)$ is true for I *iff* either A is not true for I or B is true for I (i.e., $A \supset B \models_{\mathcal{P}} \sim A \vee B$).

5.2 Theorems about truth-functional propositional logic

1. $\emptyset \models_{\mathcal{P}} A$ *iff* $\models_{\mathcal{P}} A$ (A is a tautology).

2. For any given interpretation a given formula is either true or false.

3. No formula is both true or false for the same interpretation.

4. A is false for a given interpretation *iff* $\sim A$ is true for that interpretation; and A is true for a given interpretation *iff* $\sim A$ is false for that interpretation.

5. If A and $A \supset B$ are both true for a given interpretation, then B is true for that interpretation (Modus Ponens).
6. If $\models_{\mathcal{P}} A$ and $\models_{\mathcal{P}} A \supset B$, then $\models_{\mathcal{P}} B$.
7. B is a semantic consequence of A *iff* $A \supset B$ is logically valid: i.e., $A \models_{\mathcal{P}} B$ *iff* $\models_{\mathcal{P}} A \supset B$.

5.3 Some truths about $\models_{\mathcal{P}}$. The Interpolation Theorem for $\mathcal{P}$

1. $A \models_{\mathcal{P}} A$
2. If $\Gamma \models_{\mathcal{P}} A$, then $\Gamma \cup \Delta \models_{\mathcal{P}} A$
3. If $\Gamma \models_{\mathcal{P}} A$ and $A \models_{\mathcal{P}} B$, then $\Gamma \models_{\mathcal{P}} B$
4. If $\Gamma \models_{\mathcal{P}} A$ and $\Gamma \models_{\mathcal{P}} A \supset B$, then $\Gamma \models_{\mathcal{P}} B$ (*Meta*-MP)
5. If $\models_{\mathcal{P}} A$, then $\Gamma \models_{\mathcal{P}} A$
6. ~~SAT~~ $\Gamma \cup \{\sim A\}$ *iff* $\Gamma \models A \equiv$ SAT $\Gamma \cup \{\sim A\}$ *iff* $\Gamma \not\models A$
7. ~~SAT~~ $\Gamma \cup \{A\}$ *iff* $\Gamma \models \sim A \equiv$ SAT $\Gamma \cup \{A\}$ *iff* $\Gamma \not\models \sim A$

6 Linkage between Syntax and Semantics

1. **Completeness**: A formal system is *complete* if you could prove all of its tautologies (truths) within the system (i.e., if $\models_{\mathbb{N}} A$ then $\vdash_{\mathbb{N}} A$).
2. **Completeness of a ffs of** $\mathbb{N}$: A ffs of $\mathbb{N}$ is *complete iff* all truths of $\mathbb{N}$ are provable in the ffs (i.e., $\vdash_{\mathbb{N}} A$ *iff* $\models_{\mathbb{N}} A$).
   (a) if $\vdash_{\mathbb{N}} A$ then $\models_{\mathbb{N}} A$ (soundness)
   (b) if $\models_{\mathbb{N}} A$ then $\vdash_{\mathbb{N}} A$ (completeness)
3. **Incompleteness of a ffs of** $\mathbb{N}$: A ffs of $\mathbb{N}$ is *incomplete* if there are truths of $\mathbb{N}$, i.e., $\models_{\mathbb{N}} A$, and yet not $\vdash_{\mathbb{N}} A$.
4. **A ffs of** $\mathbb{N}$: A *ffs of* $\mathbb{N}$ is a formal system some or all of whose theorems can be interpreted as expressing truths of $\mathbb{N}$.
5. **Full theory of** $\mathbb{N}$: A *full theory* of $\mathbb{N}$ is all truths of $\mathbb{N}$ (e.g., full theory of propositional logic $\equiv$ all tautologies).

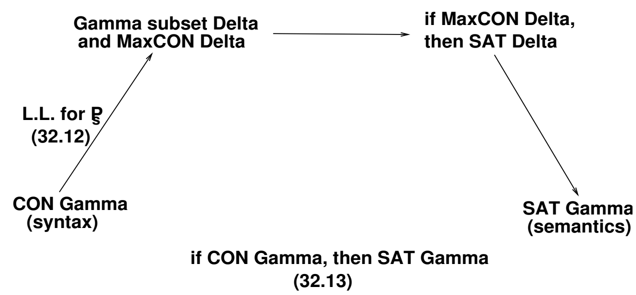
7 Soundness and Completeness of $\mathcal{P}_S$

7.1 Soundness of $\mathcal{P}_S$: if $\vdash_{\mathcal{P}_S} A$, then $\models_{\mathcal{P}_S} A$

(Every theorem is a tautology; if a wff is provable (there is a proof for it in $\mathcal{P}_S$), then A is a valid (true) wff under all interpretations I of $\mathcal{P}_S$ ($\forall I$ of $\mathcal{P}_S$, $\models_I A$)).

7.2 Semantic Completeness of $\mathcal{P}_S$ ('The Big Picture'): if $\models_{\mathcal{P}_S} A$, then $\vdash_{\mathcal{P}_S} A$

(Every tautology is a theorem; if a wff A is a valid (true) wff under all interpretations I of $\mathcal{P}_S$ ($\forall I$ of $\mathcal{P}_S$, $\models_I A$), then A is provable (there is a proof for it in $\mathcal{P}_S$))



Goal: if $\models_{\mathcal{P}_S} A$, then $\vdash_{\mathcal{P}_S} A$

1. Says, 'if a wff A is valid (true) under all interpretations I of $\mathcal{P}_S$, then A is a theorem (i.e., it can be proved or there exists a proof in $\mathcal{P}_S$ for A).
2. We get 1) by proving 'strong completeness': if $\Gamma \models_{\mathcal{P}_S} A$, then $\Gamma \vdash_{\mathcal{P}_S} A$ (which means $\Gamma = \emptyset$).
3. We get 2) by proving if $\mathsf{Con}\ \Gamma$, then $\mathsf{SAT}\ \Gamma$ (difficult) and with 32.8.
4. We get 3) by Lindenbaum's Lemma and by showing that $\mathsf{MaxCon}\ \Delta$ has a model.

7.2.1 Corollaries from Semantic Completeness of $\mathcal{P}_S$

1. **Syntactic finiteness**: an infinite $\Gamma \vdash A$ *iff* $\exists$ a finite Δ, where $\Delta \subseteq \Gamma$, s.t. $\Delta \vdash A$
2. **Semantic finiteness**: an infinite $\Gamma \models A$ *iff* $\exists$ a finite Δ, where $\Delta \subseteq \Gamma$, s.t. $\Delta \models A$
3. **Syntactic compactness**: If for every finite subset Δ of Γ, $\mathsf{Con}\ \Delta$, then $\mathsf{Con}\ \Gamma$.
4. **Semantic compactness**: If for every finite subset Δ of Γ, $\mathsf{SAT}\ \Delta$, then $\mathsf{SAT}\ \Gamma$.

7.3 Decidability of $\mathcal{P}_S$

1. **Decidability** of a system: A system $\mathcal{S}$ is *decidable* if, for any wff A of $\mathcal{S}$, there is an effective (algorithmic) method for determining if A is a theorem of $\mathcal{S}$ ($\vdash_{\mathcal{S}} A$).

$\mathcal{P}_S$ is decidable.
We only have effective methods (e.g., truth tables and trees) on the semantic side for determining whether a wff A is valid or not. There are no effective methods for constructing proofs (syntactic side). It is only by semantic completeness that we can determine whether A is a theorem.

7.4 Informal proof of the incompleteness of a ffs of the full theory of $\mathbb{N}$

1. There are uncountably many truths of a full theory of $\mathbb{N}$.

   (a) For each member S of $P(\mathbb{N})$, which is uncountable, there is a truth (e.g., for each $n \in \mathbb{N}$, either $n \in S$ or $n \notin S$).

2. Any ffs has only denumerably many wffs and thus only denumerably many theorems because all wffs are theorems.

   (a) A $\aleph_0$ alphabet has no greater expressive capacity than a finite alphabet or even a 2-symbol alphabet.

   (b) Since each distinct finitely long wff can be represented as sequence containing no more than 2 distinct symbols, each wff can be mapped to a natural number, and thus the set of wffs is denumerable.

   $\therefore$ Any ffs of a full theory of $\mathbb{N}$ is incomplete. There are uncountably many truths ($\aleph_1$) of $\mathbb{N}$ that do not match up with the denumerably many theorems ($\aleph_0$) of a ffs of the full theory of $\mathbb{N}$.

For n distinct propositional symbols there are 2^n distinct possible interpretations.

Since P has $\aleph_0$ (i.e., denumerably many) propositional symbols, there are $2^{\aleph_0} = c$ (and so uncountably many) distinct possible interpretations of P.

There are 2^{2^n} truth functions of n arguments.

8 Quantified (Predicate) Logic: $\mathcal{Q}_S$

8.1 Semantic Notions

1. An *interpretation* of $\mathcal{Q}$ consists in the specification of some non-empty set (called the *domain* of the interpretation) and the following assignments:

   (a) To each propositional symbol is assigned one or the other (but not both) of truth values truth and falsity.

   (b) To each individual constant is assigned some member of the domain of the interpretation.

   (c) To each function symbol is assigned a function with arguments and values in the domain.

   (d) To each predicate symbol is assigned some property or relation defined for objects in the domain.

   The connectives are given their usual truth-functional meanings.

2. *Satisfaction*:

   (a) If A is a propositional symbol, then s satisfies A *iff* I assigns the truth value truth to A.

   (b) If A is an atomic wff of the form $F\ t_1, \ldots, t_n$, where F is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then s satisfies A *iff* $\prec t_1 \star s, \ldots, t_n \star s \succ$ is a member of the set of ordered n-tuples assigned by I to F.

   (c) If A is of the form $\sim\ B$, then s satisfies A *iff* s does not satisfy B.

   (d) If A is of the form $(B \supset C)$, then s satisfies A *iff* either s does not satisfy B or s does satisfy C.

   (e) If A is of the form $\wedge v_k B$, where v_k is the kth variable in our enumeration, then s satisfies A *iff* every denumerable sequence of members of $\mathbb{D}$ that differs from s in at most the kth term satisfies B.

3. A wff A of $\mathcal{Q}$ is *true for a given interpretation* I of $\mathcal{Q}$ *iff every* denumerable sequence of members of the domain of I satisfies A.

4. A wff A of $\mathcal{Q}$ is *false for a given interpretation* I of $\mathcal{Q}$ *iff no* denumerable sequence of members of the domain of I satisfies A.

5. A *closed* wff is a sentence.

6. 'To satisfy' does not mean 'to make true,' unless the wff is closed.

7. An interpretation I of $\mathcal{Q}$ is a *model of a wff* A of $\mathcal{Q}$ *iff* A is true for I (i.e., $\vDash_I\ A$).

8. An interpretation I of $\mathcal{Q}$ is a *model of a set* Γ *of wffs of* $\mathcal{Q}$ *iff* every wff in Γ is true of I *iff* A is true for I (i.e., if $A \in \Gamma$, then $\vDash_I\ A$).

9. A *formal system has a model iff* the set of all its theorems has a model.

10. A wff A of $\mathcal{Q}$ is a *logically valid wff of* $\mathcal{Q}$ $[\vDash_{\mathcal{Q}} A]$ *iff* A is true for *every* interpretation of $\mathcal{Q}$ (i.e., for any I in $\mathcal{Q}$, $\vDash_I\ A$). [Remember that every interpretation of $\mathcal{Q}$ must, by definition, have a non-empty domain.]

11. A wff A of $\mathcal{Q}$ is a *semantic consequence of a set* Γ *of wffs of* $\mathcal{Q}$ $[\Gamma \vDash_{\mathcal{Q}}\ A]$ *iff* for every interpretation of $\mathcal{Q}$ every sequence that satisfies every member of Γ also satisfies A (i.e., there is no sequence that satisfies every member of Γ and does not satisfy A).

12. Progressive nature of $\mathcal{Q}$

    (a) s satisfies A

    (b) A is true for I ($\vDash_I\ A$)

    (c) A is logically valid ($\vDash\ A$; i.e., $\forall s, \forall I, s\mathsf{SAT}I$)

8.2 Semantic *Meta*-theorems for $\mathcal{Q}_S$

1. (40.1) If $\vDash A$ then $\sim A$ is unsatisfiable.

2. (40.2) MP preserves satisfaction-by-s (i.e., if a sequence s satisfies A and also $A \supset B$, then it also satisfies B.

3. (40.3) MP preserves truth-for-I (i.e., if A and $A \supset B$ are both true for an interpretation I, then B also is true for I.

4. (40.4) MP preserves logical validity (i.e., if A and $A \supset B$ are both logically valid, then so is B. Or: If $\vDash_Q A$ and $\vDash_Q A \supset B$, then $\vDash_Q B$.

5. (40.5) A is false (all sequences fail to satisfy A) for a given interpretation I *iff* $\sim A$ is true for I; and A is true for I *iff* $\sim A$ is false for I.

6. (40.6) $\vDash_I A$ *iff* $\vDash_I \wedge x A$

7. (40.7) $\vDash_I A$ *iff* $\vDash_I A^c$

8. (40.8) $\vDash A$ *iff* $\vDash A^c$

9. (40.9) $\vee x A$ is satisfiable for an I *iff* A is not satisfiable by I.

10. (40.10) Every instance of a tautological schema of $\mathcal{Q}$ ($\mathcal{Q}^+$) is logically valid.

11. (40.11) $\wedge x_k(A \supset B) \supset (\wedge x_k A \supset \wedge x_k B)$ is logically valid for arbitrary wffs A and B and an arbitrary variable x_k.

12. (40.13) If x_k does not occur free in A, then $A \supset \wedge x_k A$ logically valid (A an arbitrary wff).

13. (40.16) $\wedge x_k A \supset At/x_k$ is logically valid if t is free for x_k in A.

14. (40.17) If A is a closed wff, then exactly one of A or $\sim A$ is true for I and exactly one false for I.

15. (40.18) If A and B are closed wffs, then $\vDash_I A \supset B$ *iff* $\nvDash_I A$ or $\vDash_I B$.

16. (40.19) If A and B are closed wffs, then $\nvDash_I A \supset B$ *iff* $\vDash_I A$ or $\nvDash_I B$.

17. (40.20) If a wff A with exactly one free variable x_k is true for I, then each wff that results from substituting a closed term for the free occurrences of the variable is true for I. In other words, if A is true for all, it is true for each instance.

18. (40.21) Let I be an interpretation with domain $\mathbb{D}$. Let A be a wff with exactly one free variable, x_k. If each member of $\mathbb{D}$ is assigned by I to some closed term or another, and At/x_k is true for I for each closed term t, then $\wedge x_k A$ is true for I. I.e., if $\vDash_I Bt/x_k$ for all members of $\mathtt{D}$, then $\vDash_I \wedge x_k A$.

8.3 Syntactic Notions

1. **Proof-theoretically consistent**: Definitions, proof, Con, $\vdash$ – same as in $\mathcal{P}_S$.

8.4 Syntactic *Meta*-theorems for $\mathcal{Q}_S$

1. (43.1) The Deduction Theorem (if $\Gamma, A \vdash B$, then $\Gamma \vdash (A \supset B)$) holds for $\mathcal{Q}_S$. as in $\mathcal{P}_S$.
2. (43.2) if $\Gamma \vdash (A \supset B)$, then $\Gamma, A \vdash B$ (the converse of the Deduction Theorem)
3. (43.3) if Δ is a closed set of wffs, then if $\Delta \vdash A$ then $\Delta \vdash \wedge x A$.

8.5 Linkage *Meta*-theorems for $\mathcal{Q}_S$

1. (43.4) If A is an instance of a tautological of $\mathcal{Q}$, then $\vdash_Q A$.
2. (43.5) **Soundness, consistency of a system**: if $\vdash A$, then $\models A$ (i.e., every theorem of $\mathcal{Q}_S$ is logically valid).

   Soundness proof has same pattern in all systems:

   (a) All axioms are valid.

   (b) RoI (MP) preserves validity.

   Shows Kon of $\mathcal{Q}_S$ (of a system).

   (a) **Simple konsistency**: for no wff A, $\vdash_{\mathcal{Q}_S} A$ and $\vdash_{\mathcal{Q}_S} \sim A$.

   (b) **Absolute konsistency**: there is one A s.t. $\nvdash_{\mathcal{Q}_S} A$.
   E.g., p' is not valid in $\mathcal{Q}_S$ ($\not\models p'$ so $\nvdash p'$).

8.6 Con

1. (43.6) if $\Gamma \vdash A$ then there is a finite subset Δ of Γ s.t. $\Delta \vdash A$ (definition of a derivation).
2. (43.7) if $\Gamma \vdash A$, then $\Gamma \models A$.
3. (43.9) if $\not\text{Con}$ $\Gamma \cup \{\sim A\}$, then $\Gamma \vdash A$.

9 First-order Theories

1. A *first-order theory* (FOT) K is the same as a logical formal system.
2. Let K be a first-order theory equal to all of $\mathcal{Q}_S$ plus possibly additional proper axioms (closed).
3. Let $K + A$ be the system resulting from adding wff A as an axiom to K.

9.1 *Meta*-theorems of K

1. (**45.6a**) (If A is closed)

   If $\nvdash_K A$, then $K + \{\sim A\}$ is a konsistent FOT $\equiv$ If $\not\text{Kon}\ K + \{\sim A\}$, then $\vdash_K A$ (i.e., if $\nvdash_K A$, then $\text{Kon}\ K + \{\sim A\} \equiv$ if $\not\text{Kon}\ K + \{A\}$, then $\vdash_K\ A$).

2. (**45.6b**) (If A is closed and) If $\nvdash_K \sim A$, then $\text{Kon} K + \{A\} \equiv$ if $\not\text{Kon}\ K + \{A\}$, then $\vdash_K\ A$.

3. A *model* of system K makes *all theorems* of K true.

4. (**45.8**) If a FOT has a model, then it is konsistent (analog in $\mathcal{P}_S$: if $\text{SAT}\ \Gamma$, then $\text{Con}\ \Gamma$).

   Proof: does not require completeness

   (a) Spoze a FOT K has a model, yet $\not\text{Kon}\ K$
   (b) $\vdash\sim A$ and $\vdash A$ (by a and def. of Kon).
   (c) $\vDash_M\sim A$ and $\vDash_M A$ (by a and def. of a model).
   (d) c contradicts truth.
   (e) $\therefore$ the claim is true.

5. A system S' is an *extension* of a system S *iff* every theorem of S is a theorem of S'.

6. If S' is an extension of S, then any model of S' is a model of S.

7. A system is *negation complete iff* for every sentence (closed wff) A of S either A or the negation of A ($\sim A$) is a theorem of S (i.e., $\vdash_S A$ or $\vdash_S\sim A$; like fullness).

8. A system is *not negation complete iff* $\exists$ a sentence (closed wff) A of S s.t. neither A or its negation ($\sim A$) are theorems of S (i.e., $\nvdash_S A$ and $\nvdash_S\sim A$).

9. $\mathcal{Q}_S$ is not negation complete because $\nvdash_{\mathcal{Q}_S} p', \nvdash_{\mathcal{Q}_S}\sim p'$.

10. (45.10) **Lindenbaum's Lemma for fot**: if Kon K, then $\exists$ a FOT K' that is a konsistent, negation-complete extension of K with the same wffs of K.

11. (**45.13**) if Kon K, then $\exists$ a FOT K' that is a konsistent, closed, and negation-complete extension of K.

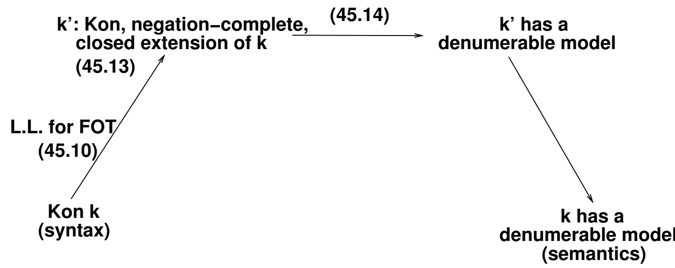
12. (**45.14**) Any konsistent, closed, and negation-complete FOT has a denumerable model.

13. (**45.13 + 45.14 = 45.15**) Any konsistent FOT has a denumerable model.

    Proof:

    (a) 45.13 + 45.14
    (b) if S' is an extension of S, then any model of S' is a model of S.

14. (45.20) **The Compactness Theorem**: If every finite subset of the set of proper axioms of a FOT K has a model, then K has a model.



if Kon k, then k has a denumerable model
(45.15)

15. (**45.8 + 45.15 = 45.18**) **The Löwenheim-Skolem Theorem**: If a FOT has a model, then it has a denumerable model.

16. (**46.1**) **Semantic Completeness of $\mathcal{Q}_S$**: if $\vDash_{\mathcal{Q}} A$, then $\vdash_{\mathcal{Q}_S} A$.

    Proof of the contrapositive: if $\nvdash_{\mathcal{Q}_S} A$, then $\nvDash_{\mathcal{Q}} A$.

    (a) Spoze given 45.15.
    (b) A is closed.
    (c) Spoze $\nvdash_{\mathcal{Q}_S} A$ (show $\nvDash_{\mathcal{Q}} A$).
    (d) Kon $\mathcal{Q}_S + \{\sim A\}$ (by 45.6a).
    (e) $\mathcal{Q}_S + \{\sim A\}$ has a denumerable model M (by d and 45.15).
    (f) $\vDash_M \sim A$ (by e and the def. of a model of a sytem).
    (g) $\nvDash_M A$ (by f and the def. of truth).
    (h) $\nvDash_{\mathcal{Q}_S} A$ (by g).
    (i) $\therefore$ the claim is true.

17. (46.2) **'Strong' Semantic Completeness of $\mathcal{Q}_S$**: if $\Gamma \vDash_{\mathcal{Q}} A$, then $\Gamma \vdash_{\mathcal{Q}_S} A$.

18. (43.7 + 46.2 = 46.3) $\Gamma \vDash_{\mathcal{Q}} A$ *iff* $\Gamma \vdash_{\mathcal{Q}_S} A$.

10 Isomorphism of Models, Axioms of Arithmetic, and Non-standard Models

1. $\mathcal{Q}_S^{=}$ is $\mathcal{Q}_S$ + new axioms.

2. An interpretation I is a *normal interpretation* of $\mathcal{Q}$ *iff* I assigns identity to $F^{\star\star\prime}$.

3. If K has $\mathcal{Q}_S^{=}1$ and $\mathcal{Q}_S^{=}2$ as axioms, then $K^{=}$.

4. An interpretation I is a *normal model* of a FOT $K^=$ *iff* I is a normal interpretation and I is a model of $K^=$.

5. **Adequacy of $\mathcal{Q}_S^=$**

   (a) if $\models_{\mathcal{Q}} A$, then $\vdash_{\mathcal{Q}^=} A$.

   (b) if $\underbrace{\models_{I^N} A}_{A \text{ is true for any normal interpretation}}$, then $\vdash_{\mathcal{Q}_S^=} A$, where I^N is a normal model of $\mathcal{Q}$ (e.g., $A = F^{\star\star\prime}xx$, $x = x$).

6. (47.2) if $\mathsf{Kon}\ K^=$, then $K^=$ has a countable normal model $\equiv$ if $K^=$ has no countable normal model, then $\not\mathsf{Kon}\ K^=$.

7. (**47.3**) if $\models_{I^N} A$, then $\vdash_{\mathcal{Q}_S^=} A$ (i.e., $\mathcal{Q}_S^=$ is semantically complete).

   Proof:

   (a) Spoze C is a closed wff of $\mathcal{Q}$ and $\models_{I^N} C$, for all normal interpretations.

   (b) $\mathcal{Q}_S^= + \{\sim C\}$.

   (c) If M is any normal model for $\mathcal{Q}_S^= + \{\sim C\}$, then $\models_M C$ and $\models_M \sim C$.

   (d) c contradicts the def. of truth.

   (e) $\mathcal{Q}_S^= + \{\sim C\}$ has no normal model (by d).

   (f) $\not\mathsf{Kon}\ \mathcal{Q}_S^=$ (by the contrapositive of 47.2).

   (g) $\vdash_{\mathcal{Q}} C$ (by the contrapositive of 45.6a)

10.1 Example

Prove if (if some closed wff A is true under some interpretation I (of a FOT K), then A is true for all interpretations of K), then K is NC.
Proof of the contrapositive: if K is not NC, then (if some closed wff A is true under some interpretation I (of a FOT K), then A is not true for all interpretations of K).

1. Spoze K is not NC.

2. $\vdash_K A$ and $\vdash_K \sim A$, for some closed wff A (by 1).

3. $\mathsf{Kon}\ K + \{\sim A\}$ and $\mathsf{Kon}\ K + \{A\}$ (by 2 and 45.6a and 45.6b, resp).

4. $\mathsf{Kon}\ K$ (by 3).

5. K has a denumerable model M (by 4 and 45.15).

6. $\models_M \sim A$ and $\models_{M'} A$ (by 2 and 5).

7. $\therefore A$ is true under some model, but not all.

10.2 Isomorphism of Models

isomorphic: 'same form, but different content' (has all the same truths)

Let K be a FOT and M, $M^{'}$ are 2 models of K with domains $\mathbb{D}$, $\mathbb{D}^{'}$, resp.

$M\ (c) = d \in \mathbb{D}$, $M^{'}\ (c) = d^{'} \in \mathbb{D}^{'}$,

$M\ (F^n) = R$, $M^{'}\ (F^n) = F^{'}$

M is *isomorphic* to $M^{'}$ (i.e., $M \simeq M^{'}$) *iff* there is a 1-1 correspondence between $\mathbb{D}$ and $\mathbb{D}^{'}$ s.t. $M\ (c) = d$ *iff* $M^{'}\ (c) = d^{'}$, i.e., $\prec d_1,\ \ldots,\ d_n \succ\ \in M\ (F^n)$ *iff* $\prec d_1^{'}, \ldots d_n^{'} \succ \in M^{'}\ (F^n)$. To be isomorphic, M and $M^{'}$ must have domains with the same cardinality.

1. (48.2) if M and $M^{'}$ are isomorphic models of a FOT K, then $\vDash_M A$ *iff* $\vDash_{M^{'}} A$, for a wff A of K.

2. (**48.3**) If all normal models of a $K^=$ are isomorphic (have all the same truths), then $K^=$ is NC (i.e., if $K^=$ is *categorical*, then $K^=$ is NC).

   Proof of the contrapositive: if $K^=$ is not NC, then all of its normal models are not isomorphic.

   (a) Spoze $K^=$ is not NC, for some closed wff A (i.e., $\nvdash_{K^=} A$ and $\nvdash_{K^=} \sim A$).
   (b) Kon $K^= + \{\sim A\}$ and Kon $K^= + \{A\}$ (by a and 45.6a and 45.6b, resp).
   (c) These two FOTs have countable models, call them M, $M^{'}$ (by b and 47.2)
   (d) $\nvDash_M A$ and $\vDash_{M^{'}} A$ (by a and def. of a model).
   (e) M and $M^{'}$ are not isomorphic.

10.3 Axioms of Arithmetic

1. Arithmetic can be modeled using standard first-order theory with identity.

2. function S_x is 'the successor of x' (e.g., $S_5 = 5 + 1 = 6$).

10.4 Non-standard Models

$\mathcal{Q}_S^=$ + axioms of arithmetic = Theory $\mathcal{R}$.
$M^{'}$ is a model of $\mathcal{R}^{'}$ and $\therefore$ a model of $\mathcal{R}$. $\therefore \mathcal{R}$ has a *non-standard* model.

11 Göedel's First Incompleteness Theorem

Recall that $\mathbb{N}$ is a formal theory (FOT) of arithmetic.

Abbreviation: $A_n\ (x)$ = the property with one free variable x that has Göedel # ($g\#$) = n.

There is a proof in $\mathbb{N}$ that 2 has the even property, i.e., $\vdash_{\mathbb{N}} E\ (2)$.

Proofs also get g#'s

A *number-theoretic* property $W\ (\underbrace{n}_{\text{g\# of a wff } A\ (x)},\ \underbrace{k}_{\text{g\# of a proof in } \mathbb{N} \text{ of } A\ (\overline{n})})$ holds between natural numbers n and k *iff*

1. n is the g# of a wff $A\ (x)$ with one free variable x, and
2. k is the g# of a proof in $\mathbb{N}$ of $A\ (\overline{n})$.

W is *representable* in $\mathbb{N}$, meaning there is a wff $B\ (x,\ y)$ in $\mathbb{N}$ with free variables x and y s.t. for all natural numbers n and k:

1. if $W\ (n,\ k)$ holds, then $\vdash_{\mathbb{N}} B\ (\overline{n},\ \overline{k})$.
2. if $W\ (n,\ k)$ does not hold, then $\vdash_{\mathbb{N}} \sim B\ (\overline{n},\ \overline{k})$.

E.g., if 7100 were the g# associated with $P\ (x)$, being prime, then we would have $\vdash_{\mathbb{N}}\ (y)\ \sim\ B\ (\overline{7100},\ y)$, where (y) is the universal quantifier y.

Now consider the common property $C\ (x)\ =\ (y)\ \sim\ B\ (x,\ y)$, and $C\ (x)$ has g# $= m$.

g#	**Property with one free variable**
i	$A_i\ (x) =$ the property with one free variable
$\vdots$	$\vdots$
54	$A_{54}\ (x) = O\ (x)$, x is odd
$\vdots$	$\vdots$
540	$A_{540}\ (x) = E\ (x)$, x is even
$\vdots$	$\vdots$
600	$A_{600}\ (x) = O\ (x)$, x is odd
$\vdots$	$\vdots$
7100	$A_{7100}\ (x) = P\ (x)$, x is prime
$\vdots$	$\vdots$
m	$A_m\ (x) = C\ (x)$, there is no proof in $\mathbb{N}$ that x has the property with g# $= x$, i.e., $(y) \sim\ B\ (x,\ y)$; $C\ (\overline{m}) = (y)\ \sim\ B\ (\overline{m},\ y)$

$C\ (\overline{m})$ is the Göedel formula G.

Let us marry W and C.

$W\ (n,\ k)$ holds *iff* k is the g# of a proof (in $\mathbb{N}$) of $A_n\ (\overline{n})$.

Thus, $W\ (m,\ k)$ holds *iff* k is the g# of a proof (in $\mathbb{N}$) of $A_m\ (\overline{m})$.

So $W\ (m,\ k)$ holds *iff* k is the g# of a proof (in $\mathbb{N}$) of G (which says there is no proof).
So $W\ (m,\ k)$ holds for some k *iff* G is provable in $\mathbb{N}$.

We are now ready for the first part of Göedel's incompleteness result.

Consider the question: Is there a proof in $\mathbb{N}$ that $A_m\ (\overline{m})$?

But again, $A_m\ (\overline{m}) = C\ (\overline{m}) = G$.

So our question is equivalent to: Is G provable in $\mathbb{N}$, i.e., is $\vdash_{\mathbb{N}} G$?

Proof that if G is provable in $\mathbb{N}$, then $\mathbb{N}$ inconsistent.

1. Spoze $\vdash_{\mathbb{N}}$ G, i.e., $\vdash_{\mathbb{N}} C\ (\overline{m})$.
2. Then $\vdash_{\mathbb{N}} (y) \sim B\ (m,\ y)$, by the def. of $C\ (\overline{m})$.
3. But we know that G is provable in $\mathbb{N}$ *iff* $W\ (m,\ k)$ holds for some k.
4. So, from (1), we have $W\ (m,\ k)$ holds for some number k, call it k'; so, $W\ (m,\ k')$.
5. Since B *represents* W, it follows from (4) that $\vdash_{\mathbb{N}} B\ (\overline{m},\ \overline{k'})$.
6. But it follows from (2), by quantifier instantiation or by the axioms of $\mathbb{N}$, that $\vdash_{\mathbb{N}}\ \sim B\ (\overline{m},\ \overline{k'})$.
7. Then, from (5) and (6), $\mathbb{N}$ is an inkonsistent FOT.

$\therefore$ if $\mathbb{N}$ is konsistent, then G is not provable in $\mathbb{N}$ is true. ______________________________ □
G says of itself, it is not provable. So, if G is provable, then G is not provable. So, if we assume $\mathbb{N}$ has its intended model, then G is not provable.