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FORCED OSCILLATIONS OF THE KORTEWEG-DE VRIES EQUATION ON A BOUNDED DOMAIN AND THEIR STABILITY

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Abstract. It has been observed in laboratory experiments that when nonlinear dispersive waves are forced periodically from one end of undisturbed stretch of the medium of propagation, the signal eventually becomes temporally periodic at each spatial point. The observation has been confirmed mathematically in the context of the damped Kortewg-de Vries (KdV) equation and the damped Benjamin-Bona-Mahony (BBM) equation. In this paper we intend to show the same results hold for the pure KdV equation (without the damping terms) posed on a bounded domain. Consideration is given to the initial-boundary-value problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \qquad 0 < x < 1, \ t > 0, \\ \\ u(0,t) = h(t), \qquad u(1,t) = 0, \qquad u_x(1,t) = 0, \quad t > 0. \end{cases} \tag{$*$}$$

It is shown that if the boundary forcing h is periodic with small amplitude, then the small amplitude solution u of $(*)$ becomes eventually time-periodic. Viewing $(*)$ (without the initial condition) as an infinite-dimensional dynamical system in the Hilbert space $L^2(0,1)$, we also demonstrate that for a given periodic boundary forcing with small amplitude, the system $(*)$ admits a (locally) unique *limit cycle*, or *forced oscillation*, which is locally exponentially stable. A list of open problems are included for the interested readers to conduct further investigations.

1. **Introduction.** In this paper we consider an initial-boundary-value problem (IBVP) of the Korteweg-de Vries (KdV) equation posed on the finite domain $(0,1)$, namely,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \qquad 0 < x < 1, \ t > 0, \\ \\ u(0,t) = h(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, \qquad t > 0. \end{cases} \tag{1}$$

Guided by the outcome of laboratory experiments, interest is given to long time effect of the boundary forcing h and large time behavior of solutions of IBVP (1).

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In the experiments of Bona, Pritchard and Scott [1], a channel partly filled with the water mounted with a flap-type wave maker at one end. Each experiment commenced with the water in the channel at rest. The wave-maker was activated and underwent periodic oscillations. The throw of the wave-maker and its frequency of oscillation was such that the surface waves brought into existence were of small amplitude and long wavelength, so it can be modeled by either Benjamin-Bona-Mahoney (BBM) type equation posed in a quarter plane:

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} - \alpha u_{xx} - \gamma u = 0, & x > 0,\ t > 0, \\ u(x,0) = 0, \qquad u(0,t) = h(t), & x \geq 0, \quad t \geq 0, \end{cases} \tag{2}$$

or the Korteweg-de Vries (KdV) type equation posed in a quarter plane:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, & x > 0,\ t > 0, \\ u(x,0) = 0, \qquad u(0,t) = h(t), & x \geq 0, \quad t \geq 0, \end{cases} \tag{3}$$

where α and γ are nonnegative constants that are proportional to the strength of the damping effect. The wave-maker is modeled by the boundary value function $h = h(t)$ which is assumed to be a periodic function of period τ.

It was observed in the experiments that at each fixed station down the channel, for example at a spatial point represented by x_0, that the wave motion $u(x_0, t)$, say, rapidly became periodic of the same period as the boundary forcing. This observation leads to the following conjecture for solutions of IBVPs (2) and (3).

Conjecture *If the boundary forcing h is a periodic function of period τ, then the solution u of IBVP (2) or IBVP (3) becomes eventually time-periodic of period τ, i.e.,*

$$\lim_{t\to\infty} \|u(\cdot, t+\tau) - u(\cdot, t)\|_{L^2(R^+)} = 0.$$

Furthermore, the following mathematical questions arise naturally which are important from the point of view of dynamical systems.

Questions: *Assume the boundary forcing h is a periodic function (of period τ).*

(1) Does the equation in (2) or (3) admit a time periodic solution $u(x,t)$ of periodic τ satisfying the boundary condition? Such a time-periodic solution, if exists, is usually called a forced oscillation.

(2) If such a time periodic solution exists, how many are there?

(3) What are their stability if those forced oscillations exist?

In [3], Bona, Sun and Zhang studied the KdV type equation (3). Assuming the damping coefficient $\gamma > 0$, they showed that if the the amplitude of the boundary forcing h is small, then the solution u of (3) is asymptotically time-periodic satisfying

$$\|u(\cdot, t+\tau) - u(\cdot, t)\|_{L^2(R^+)} \leq Ce^{-\beta t} \qquad \text{for any } t \geq 0,$$

where C and β are two positive constants. In addition, they demonstrated that the equation in (3) admits a unique time-periodic solution $u^*(x,t)$ of period τ satisfying the boundary condition, which is shown to be globally exponentially stable in the space $H^s(R^+)$ with $s \geq 1$, i.e, for a given initial data $\phi \in H^s(R^+)$, the unique solution $u(x,t)$ of (3) with zero initial data replaced by $\phi(x)$ has the property

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{H^s(R^+)} \leq Ce^{-\beta t}$$

for any $t \geq 0$ where $C > 0$ is a constant depending only on $\|\phi\|_{H^s(R^+)}$. Later, the BBM type equation (3) was studied by Yang and Zhang [20]. The similar results have been established while assuming that the damping coefficients α and γ are both positive.

Earlier, the same problems had been studied by Zhang for the BBM equation [21] and the KdV equation [22] posed on the finite interval $(0,1)$:

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} - \alpha u_{xx} - \gamma u = 0, \qquad 0 < x < 1,\ t > 0, \\ u(x,0) = 0, \qquad 0 \leq x \leq 1, \\ u(0,t) = h(t), \qquad u(1,t) = 0 \quad t \geq 0, \end{cases} \tag{4}$$

and

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, \qquad x > 0,\ t > 0, \\ u(x,0) = 0, \qquad 0 \leq x \leq 1, \\ u(0,t) = h(t), \qquad u(1,t) = 0, \qquad u_x(1,t) = 0, \quad t \geq 0. \end{cases} \tag{5}$$

Assuming either $\alpha > 0$ or $\gamma > 0$, Zhang [21, 22] showed that if the boundary forcing h is a periodic function of period τ with small amplitude, then both solutions of the IBVPs (4) and (5) are asymptotically time periodic (of period τ). Moreover, both equations admit a unique time periodic solution satisfying the boundary conditions which is globally exponentially stable.

In the above cited works, it is required that the damping coefficients α and γ are greater than zero, or at least one of them is greater than zero. It would be interesting to see whether the results reported in those works still hold when both damping coefficients α and γ are zeros. This leads us to consider the IBVP (1) of the KdV equation posed on the finite interval $(0,1)$. We will show that, though there is no explicit damping term in the equation, the solution u of the IBVP (1) is asymptotically time-periodic if the boundary forcing h is periodic with small amplitude and the initial date ϕ is small in certain space. We will also demonstrate that for a given small amplitude, time-periodic boundary forcing h, the equation in (1) admits a (locally) unique time-periodic solution u^* satisfying the boundary conditions. This time-periodic solution will be demonstrated to be locally exponential stable in the space $L^2(0,1)$.

The paper is organized as follows. Section 2 is devoted to the associated linear systems. Both homogenous and non-homogeneous boundary values problems are studied. Long time behavior of their solutions is investigated. The results presented in this section are essential for the study of the nonlinear systems. The study of long-time behavior of solutions of the nonlinear system (1) is conducted in Section 3. In particular, time-independent bounds on solutions are presented, which are essential for the main analysis that is developed in Section 4 in which the existence, the local uniqueness and the local exponential stability of time periodic solutions(forced oscillation) are established. The paper is concluded with Section 5 where a list of open problems are provided for the interested readers to conduct further investigations.

We end our introduction with a review of terminology and notation. For an arbitrary Banach space X, the associated norm is denoted by $\|\cdot\|_X$. If (a,b) is a bounded interval in $(0,\infty)$ and k is a nonnegative integer, we denote by $C^k(a,b)$

the collection of functions that along with their first k derivatives, are continuous on $[a,b]$ with the norm

$$\|f\|_{C^k(a,b)} = \sup_{x\in(a,b),0\le j\le k} |f^{(j)}(x)|.$$

For $1 \le p \le \infty$, $L^p(a,b)$ connotes those functions f which are pth-power absolutely integrable on (a,b) with the usual modification in case $p=\infty$. If $k \ge 0$ is an integer let $H^k(a,b)$ be the Sobolev space consisting of those $L^2(a,b)$-functions whose first k generalized derivatives lie in $L^2(a,b)$, equipped with the usual norm

$$\|f\|^2_{H^k(a,b)} = \sum_{j=0}^{k} \|f^{(j)}\|^2_{L^2(a,b)}.$$

If $s \ge 0$ is a real number and $k < s < k+1$ for some integer k, then $H^s(a,b)$ is the standard interpolation space of $H^k(a,b)$ and $H^{k+1}(a,b)$. In this paper, the space $H^s(a,b)$ will occur often with $(a,b)=(0,1)$. Because of their frequent occurrence, it is convenient to abbreviate their norms thusly;

$$\|\cdot\|_s = \|\cdot\|_{H^s(0,1)}, \quad |\cdot|_{s,(a,b)} = \|\cdot\|_{H^s(a,b)}.$$

For $0 < T \le \infty$ and $1 \le p \le \infty$, $L^p(0,T;X)$ denotes the Banach space of all measurable functions $u:(0,T)\to X$ such that, $t \to \|u\|_X$ is in $L^p(0,T)$, with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}}, \quad \text{if } 1 \le p < \infty,$$

and the obvious modification if $p=\infty$. In addition, $C_b(0,T;X)$ denotes the Banach space of all bounded continuous functions $u:(0,T)\to X$, such that $t \to \|u\|_X$ is continuous and in $L^\infty(0,T)$, with the norm

$$\|u\|_{C_b(0,T;X)} = \sup_{0\le t\le T} \|u(t)\|_X.$$

2. **Preliminaries.** In this section we first consider the following IBVP of the linear KdV equation with homogeneous boundary conditions:

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \qquad u(x,0)=\phi(x), \qquad 0<x<1,\ t>0, \\ u(0,t) = u(1,t) = u_x(1,t) = 0. \end{cases} \tag{6}$$

Its solution u can be written in the form

$$u(x,t) = W(t)\phi$$

where $W(t)$ is the C^0-semigroup in the space $L^2(0,1)$ generated by the operator

$$Af = -f' - f'''$$

with its domain $D(A) = \{f \in H^3(0,1); \quad f(0)=f(1)=f'(1)=0\}$. The following results are well-known (cf. [15, 4, 12]).

Proposition 2.1. *Let $T>0$ be given. For any $\phi \in L^2(0,1)$, the IBVP (6) admits a unique solution $u \in C(R^+;L^2(0,1)) \cap L^2(0,T;H^1(0,1))$. Moreover, there exists a constant $C>0$ depending only on T such that*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_0 + \|u\|_{L^2(0,T;H^1(0,1))} \le C\|\phi\|_0. \tag{7}$$

In addition, there exists a constant $\beta > 0$ such that for any $\phi \in L^2(0,1)$,

$$\|u(\cdot,t)\|_0 \le Ce^{-\beta t}\|\phi\|_0 \tag{8}$$

for any $t \ge 0$ where $C > 0$ is a constant.

For given $T > 0$ and $t > 0$, let $Y_{t,T}$ be the space

$$Y_{t,T} = \{v \in C_b(t, t+T; L^2(0,1)) |\, v \in L^2(t, T+t; H^1(0,1))\}$$

which is a Banach space equipped with the norm

$$\|v\|_{Y_{t,T}} \equiv \sup_{t\le t'\le t+T} \|v(\cdot,t')\|_0 + \|v\|_{L^2(t,t+T;H^1(0,1))}.$$

In addition, let Y_T be the space

$$Y_T = \{f(x,t)|\ f \in Y_{t,T} \text{ for any } t \ge 0;\ \sup_{t\ge 0} \|f\|_{Y_{t,T}} < +\infty\}.$$

The space Y_T is also Banach space when equipped with the norm

$$\|f\|_{Y_T} \equiv \sup_{t\ge 0} \|f\|_{Y_{t,T}}.$$

Using the semigroup property of the IBVP (6) and Proposition 2.1, one can conclude that the following estimate holds for solutions of IBVP (6).

Theorem 2.2. *Let $T > 0$ be given. For any $\phi \in L^2(0,1)$, the unique solution u of (6) belongs to the space $Y_{t,T}$ for any $t \ge 0$. Moreover, there exist constants $C > 0$ and $\beta > 0$ independent of T and ϕ,*

$$\|u\|_{Y^s_{t,T}} \le C\|\phi\|_0 e^{-\beta t} \qquad \textit{for any } t > 0. \tag{9}$$

Note that, choosing $t = 0$, estimate (9) reduces to an improved version of estimate (7) since the constant C is estimate (9) is independent of T while the constant C in estimate (7) may depends on T.

Proof: Multiplying the both sides of the equation in (6) by $2xu$ and integrating over $(0,1)$ with respect to x,

$$\frac{d}{dt}\int_0^1 xu^2(x,t)dx + \int_0^1 u_x^2(x,t)dx = \int_0^1 u^2(x,t)dx.$$

Thus, for any $T > 0$,

$$\begin{aligned} \int_0^T \int_0^1 u_x^2(x,t)dx &\le \int_0^1 \phi^2(x)dx + \int_0^T \int_0^1 u^2(x,t)dx \\ &\le \|\phi\|_0^2 + \|\phi\|_0^2 \int_0^T e^{-2\beta t}dt \\ &\le (1+\beta^{-1})\|\phi\|_0^2. \end{aligned}$$

Consequently,

$$\|u\|_{Y_{0,T}} \le C\|\phi\|_0$$

for any $T > 0$ where the constant C is independent of T. Furthermore, using the semigroup property of (6) and Proposition 2.1, for any $t > 0$ and $T > 0$,

$$\|u\|_{Y_{t,t+T}} \le C\|u(\cdot,t)\|_0 \le Ce^{-\beta t}\|\phi\|_0.$$

The proof is complete. □

Next we consider the nonhomogeneous boundary value problem

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & u(x,0) = 0, \\ \\ u(0,t) = h(t), & u(1,t) = u_x(1,t) = 0. \end{cases} \tag{10}$$

According to [4], the solution u of (10) is given by

$$u(t) = W_b(t)h,$$

where $W_b(t)$ is the boundary integral operator associated with (10) (cf. [4]). The proof of the following proposition can be found in [4].

Proposition 2.3. *For given $T > 0$ and $h \in H^{\frac{1}{3}}(0,T)$, the IBVP (10) admits a unique solution $u \in Y_{0,T}$ satisfying*

$$\|u\|_{Y_{0,T}} \leq C|h|_{\frac{1}{3},(0,T)}$$

where $C > 0$ is a constant depending only on T.

We now turn to consider the IBVP

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & u(x,0) = \phi, \\ \\ u(0,t) = h(t), & u(1,t) = u_x(1,t) = 0 \end{cases} \tag{11}$$

for long time behavior of its solutions. The following theorem is an extension of Theorem 2.2 to the IBVP (11).

Theorem 2.4. *Let $T > 0$ be given. For a given pair $(\phi, h) \in L^2(0,1) \times H^{\frac{1}{3}}_{loc}(R^+)$, then corresponding solution u of (11) satisfies*

$$\|u(\cdot,t)\|_{Y_{t,t+T}} \leq C_1 e^{-\beta t}\|\phi\|_0 + C_2 \sup_{0\leq t'\leq t} |h|_{\frac{1}{3},(t',t'+T)}$$

for any $t \geq 0$ where, $\beta > 0$ is as given in Theorem 2.2, C_1 is a constant independent of T and C_2 is a constant depending on T.

Proof: Note the solution u can be written as

$$u(t) = W(t)\phi + W_b(t)h := u_1 + u_2.$$

By Theorem 2.2 and Proposition 2.3,

$$\|u_1\|_{Y_{t,t+T}} \leq C_1 e^{-\beta t}\|\phi\|_0$$

for any $t \geq 0$,

$$\begin{aligned} \|u_2\|_{Y_{t,t+T}} &\leq C_1 e^{-\beta T}\|u_2(\cdot,t)\|_0 + C_1^*|h|_{\frac{1}{3},(t,t+T)}, \\ &\leq C_1 e^{-2\beta T}\|u_2(\cdot,t-T)\|_0 + C_1^*\left(C_1 e^{-\beta T}|h|_{\frac{1}{3},(t-T,t)} + |h|_{\frac{1}{3},(t,t+T)}\right), \\ &\leq \cdots\cdots \\ &\leq \frac{C_1 C_1^*}{1 - e^{-\beta T}} \sup_{0\leq t'\leq t} |h|_{\frac{1}{3},(t',t'+T)}. \end{aligned}$$

The proof is complete. □

Next, consideration is given to an abstract result about a sequence in a Banach space X generated by iteration as follows:

$$y_{n+1} = Ay_n + F(y_n)\,, \quad n = 0, 1, 2, \cdots . \tag{12}$$

Here, the linear operator A is bounded from X to X with

$$\|Ay_n\|_X \le \gamma \|y_n\|_X \tag{13}$$

for some finite value γ and all $n \ge 0$. The nonlinear function F mapping X to X is such that there are constants β_1 and β_2 and a sequence $\{b_n\}_{n\ge 0}$ for which

$$\|F(y_n)\|_X \le \beta_1 \|y_n\|_X + \beta_2 \|y_n\|_X^2 + b_n \tag{14}$$

for all $n \ge 0$. The following two lemmas apply to such a sequence whose proofs can found in [3]. These lemmas will find use in Sections 4.

Lemma 2.5. *If $\beta_2 = 0$ in (14) and $r = \gamma + \beta_1 < 1$, then the sequence $\{y_n\}_{n=0}^{\infty}$ defined by (12) satisfies*

$$\|y_{n+1}\|_X \le r^{n+1} \|y_0\|_X + \frac{b^*}{1-r} \tag{15}$$

for any $n \ge 1$, where $b^* = \sup_{n\ge 0} b_n$.

Lemma 2.6. *If $\beta_2 \ne 0$ in (14) and $r = \gamma + \beta_1 < 1$, then there exist r_1 with $0 < r_1 < 1$, $\delta_1 > 0$ and $\delta_2 > 0$ such that if $\|y_0\|_X < \delta_1$ and $b_n \le \delta_2$ for all $n \ge 0$, the sequence $\{y_n\}_{n=0}^{\infty}$ defined by (12) satisfies*

$$\|y_{n+1}\|_X \le r_1^{n+1} \|y_0\|_X + \frac{b^*}{1-r} \tag{16}$$

for any $n \ge 1$, where $b^* = \max_n \{b_n\}$.

Finally, we consider the initial-boundary-value problem for a linearized KdV–equation with a variable coefficient, namely

$$\begin{cases} u_t + u_x + (au)_x + u_{xxx} = 0, \qquad u(x,0) = \phi, \\ \\ u(0,t) = h(t), \qquad u(1,t) = u_x(1,t) = 0 \end{cases} \tag{17}$$

where $a = a(x,t)$ is a given function. The following result is known (cf. [4]).

Proposition 2.7. *Let $T > 0$ be given. Assume that $a \in Y_{0,T}$. Then for any $(\phi, h) \in X_{0,T} := L^2(0,1) \times H^{\frac{1}{3}}(0,T)$, the IBVP (17) admits a unique solution $u \in Y_{0,T}$ satisfying*

$$\|u\|_{Y_{0,T}} \le \mu(\|a\|_{Y_{0,T}}) \|(\phi,h)\|_{X_{0,T}}$$

where $\mu : R^+ \to R^+$ is a $T-$dependent continuous nondecreasing function independent of ϕ and h.

Our next theorem presents an asymptotic estimate for solutions of the IBVP (17).

Theorem 2.8. *There exists a $T > 0$, $r > 0$ and $\delta > 0$ such that if $a \in Y_T$ and*

$$\|a\|_{Y_T} \le \delta,$$

then for any $(\phi, h) \in L^2(0,1) \times H^{\frac{1}{3}}_{loc}(R^+)$, the corresponding solution u of (17) satisfies

$$\|u(\cdot,t)\|_0 \le C_1 e^{-rt} \|\phi\|_0 + C_2 \sup_{n\ge 0} |h|_{\frac{1}{3},(nT,(n+1)T)}$$

for any $t \ge 0$ where C_1 and C_2 are constants independent of ϕ and h.

Proof: Rewrite (17) in its integral form

$$u(t) = W(t)\phi + W_b(t)h - \int_0^t W(t-\tau)(au)_x(\tau)d\tau.$$

Thus, for any $T > 0$, using Theorem 2.2 and Proposition 2.3,

$$\begin{aligned}
\|u(\cdot,T)\|_0 &\leq Ce^{-\beta T}\|\phi\|_0 + C_T|h|_{\frac{1}{3},(0,T)} + \int_0^T \|(au)_x\|_0(\tau)d\tau \\
&\leq Ce^{-\beta T}\|\phi\|_0 + C_T|h|_{\frac{1}{3},(0,T)} + C_T\|a\|_{Y_{0,T}}\|u\|_{Y_{0,T}} \\
&\leq Ce^{-\beta T}\|\phi\|_0 + C_T|h|_{\frac{1}{3},(0,T)} + C_T\|a\|_{Y_{0,T}}\mu(\|a\|_{Y_{0,T}})\left(\|\phi\|_0 + |h|_{\frac{1}{3},(0,T)}\right) \\
&\leq Ce^{-\beta T}\|\phi\|_0 + C_T\|a\|_{Y_{0,T}}\mu(\|a\|_{Y_{0,T}})\|\phi\|_0 + C_T(1+\|a\|_{Y_{0,T}}\mu(\|a\|_{Y_{0,T}}))|h|_{\frac{1}{3},(0,T)}.
\end{aligned}$$

Note that in the above estimate, the constant C is independent of T. Let

$$y_n = u(\cdot, nT) \quad \text{for } n = 0,1,2,\cdots$$

and let v be the solution of the IBVP

$$\begin{cases} v_t + v_x + (av)_x + v_{xxx} = 0, & v(x,0) = y_n(x), \\ v(0,t) = h(t+nT), & v(1,t) = v_x(1,t) = 0 \end{cases} \tag{18}$$

Thus $y_{n+1}(x) = v(x,T)$ by the semigroup property of the system (17). Consequently, we have the following estimate for y_{n+1} :

$$\begin{aligned}
\|y_{n+1}\|_0 &\leq Ce^{-\beta T}\|y_n\|_0 + C_T\|a\|_{Y_{nT,(n+1)T}}\mu(\|a\|_{Y_{nT,(n+1)T}})\|y_n\|_0 \\
&+ C_T(1+\|a\|_{Y_{nT,(n+1)T}}\mu(\|a\|_{Y_{nT,(n+1)T}}))|h|_{\frac{1}{3},(nT,(n+1)T)}
\end{aligned}$$

for $n = 0,1,2,\cdots$. Choose T and δ such that

$$Ce^{-\beta T} = \gamma < 1, \quad \gamma + C_T\delta\mu(\delta) := r < 1.$$

Then,

$$\|y_{n+1}\|_0 \leq r\|y_n\|_0 + b_n$$

for all $n \geq 0$ if $\|a\|_{Y_T} \leq \delta$ where

$$b_n = C_T(1+\|a\|_{Y_{nT,(n+1)T}}\mu(\|a\|_{Y_{nT,(n+1)T}}))|h|_{\frac{1}{3},(nT,(n+1)T)}.$$

It follows from Lemma 2.5 that

$$\|y_{n+1}\|_0 \leq r\|y_n\|_0 + \frac{b^*}{1-r}$$

for any $n \geq 0$ where $b^* = \sup_{n\geq 0} b_n$. This inequality implies the conclusion of Theorem 2.8. $\square$

3. **Long time asymptotic behavior.** In this section, attention is turned to the long-time behavior of solutions of the IBVP of the KdV equation posed on the finite interval $(0, 1)$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0, 1), \\ u(x, 0) = \phi(x), \quad x \in (0, 1), \\ u(0, t) = h(t), \ u(1, t) = 0, \quad u_x(1, t) = 0. \end{cases} \tag{19}$$

This problem is known to be locally well-posed in the space $L^2(0, 1)$ (cf. [4]).

Proposition 3.1. [1] *Let $T > 0$ be given. For any any $(\phi, h) \in X_{0,T}$ there exists a $0 < T^* \le T$ such that (19) admits a unique solution $u \in Y_{0,T^*}$.*

This well-posedness result is temporally local in the sense that given s-compatible auxiliary data ϕ and h, the corresponding solution u is only guaranteed to exist on the time interval $(0, T^*)$, where T^* depends on the norm of (ϕ, h) in the space $X_{0,T}$. The next proposition presents an alternative view of local well-posedness for the IBVP (19). If the norm of (ϕ, h) in $X_{0,T}$ is not too large, then the corresponding solution is guaranteed to exist over the entire time interval $(0, T)$.

Proposition 3.2. *Let $T > 0$ be given. There exists a $\delta > 0$ depending on T such that if $(\phi, h) \in X_{0,T}$ satisfying $\|(\phi, h)\|_{X_{0,T}} \le \delta$, then (19) admits a unique solution $u \in Y_{0,T}$. Moreover there exists a constant $C > 0$ independent of T such that*

$$\|u\|_{Y_{0,T}} \le C\|(\phi, h)\|_{X_{0,T}}.$$

Proof: The proof is based on the contraction mapping principle and is similar to that of Proposition 3.1 with a slight modification. □

Next we show that if δ is small enough, then the corresponding solution u of (19) exists for any time $t > 0$ and its norm in $L^2(0, 1)$ is uniformly bounded.

Theorem 3.3. *There exist positive constants T, δ_j, $j = 1, 2$ and r such that if*

$$\|\phi\|_0 \le \delta_1, \qquad \sup_{n \ge 0} |h|_{\frac{1}{3},(nT,(n+1)T)} \le \delta_2, \tag{20}$$

then the corresponding solution u of (19) is globally defined and belongs to the space $C_b(0, \infty; L^2(0, 1))$. Moreover,

$$\|u(\cdot, t)\|_0 \le C_1 e^{-rt}\|\phi\|_0 + C_2 \sup_{n \ge 0} |h|_{\frac{1}{3},(nT,(n+1)T)}$$

for any $t \ge 0$ where $C_1 > 0$ and $C_2 > 0$ depend only on δ_1 and δ_2.

Proof: For given $\phi \in L^2(0, 1)$ and $h \in H^{\frac{1}{3}}_{loc}(R^+)$, rewrite the IBVP (19) in its integral form

$$u(t) = W(t)\phi + W_b(t)h - \int_0^t W(t - \tau)(uu_x)(\tau)d\tau.$$

For given $T > 0$, according to Theorem 2.4, there exist $C_1 > 0$ independent of T and C_2, C_3 depending only on T such that for any $0 \le t \le T$,

$$\|u(\cdot, t)\|_0 \le C_1 e^{-rt}\|\phi\|_0 + C_2\|u\|^2_{Y_{0,T}} + C_3|h|_{\frac{1}{3},(0,T)}. \tag{21}$$

[1] If $h \in H^{\frac{1}{3}+\epsilon}_{loc}(R^+)$ for some $\epsilon > 0$, then the IBVP (19) is globally well-posed, i.e., $T^* = T$. In fact, the IBVP (19) is known to be globally well-posed in the space $H^s(0, 1)$ $(s \ge 0)$ with $s-$compatible pair $(\phi, h) \in H^s(0, 1) \times H^{\frac{1+s}{3}+\eta(s)}_{loc}(R^+)$ where $\eta(s) = \epsilon$ if $0 \le s < 3$ and $\eta(s) = 0$ if $s \ge 3$.

By Proposition 3.2, there exists a $\delta > 0$, if

$$\|(\phi, h)\|_{X_{0,T}} \leq \delta, \tag{22}$$

then

$$\|u\|_{Y_{0,T}} \leq C_4 \|(\phi, h)\|_{X_{0,T}}.$$

Thus, if (22) holds and (21) is evaluated at $t = T$,

$$\|u(\cdot, T)\|_0 \leq C_1 e^{-rT} \|\phi\|_0 + C_5(\|\phi\|_0^2 + |h|^2_{\frac{1}{3},(0,T)}) + C_3 |h|_{\frac{1}{3},(0,T)}$$

with $C_5 = 2C_2C_4$. Choose $T > 0$ so that $C_1 e^{-rT} = \gamma < 1$. Let $\omega = \frac{1}{1-\gamma}$. Choose δ_1 and δ_2 such that

$$\delta_1 + \delta_2 \leq \delta$$

and

$$\omega[C_5(\delta_1^2 + \delta_2^2) + C_3\delta_2] \leq \delta_1.$$

For such values of δ_1 and δ_2, we have that

$$\|u(\cdot, T)\|_0 \leq \delta_1,$$

and, in addition, by the assumption,

$$|h|_{\frac{1}{3},(T,2T)} \leq \delta_2.$$

Hence repeating the argument, we have that

$$\sup_{T \leq t \leq 2T} \|u(\cdot, t)\|_0 \leq \delta_1.$$

Continuing inductively, it is adduced that

$$\sup_{t \geq 0} \|u(\cdot, t)\|_0 \leq \delta_1.$$

Let $y_n = u(\cdot, nT)$ for $n = 1, 2, \cdots$. Using the semigroup property of (19), one obtains constants C_1, C_2 and C_3 which are independent of T and positive parameters δ_1 and δ_2 such that

$$\|y_{n+1}\|_0 \leq C_1 e^{-rt} \|y_n\|_0 + C_2 \|y_n\|_0^2 + C_3 |h|_{\frac{1}{3},(nT,(n+1)T)}$$

for any $n \geq 1$ provide $\|y_0\|_0 \leq \delta_1$ and

$$\sup_{n \geq 0} |h|_{\frac{1}{3},(nT,(n+1)T)} \leq \delta_2.$$

By Lemma 2.6, there exists $0 < \nu < 1$, $\delta_1^* > 0$ and $\delta_2^* > 0$ such that if

$$\|y_0\|_0 \leq \delta_1^*, \quad b_n = C_3 |h|_{\frac{1}{3},(nT,(n+1)T)} \leq \delta_2 \leq \delta_2^*$$

for all $n \geq 0$, then

$$\|y_{n+1}\|_0 \leq \nu^{n+1} \|y_0\|_0 + \frac{b^*}{1-\nu}$$

for all $n \geq 1$, where $b^* = \max_n\{b_n\}$. This leads by standard arguments to the conclusion of Theorem 3.3. □

4. **Forced oscillations and their stability.** In this section, we consider first the pure boundary value problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & x \in (0,1),\ t > 0, \\ u(0,t) = h(t), & u(1,t) = u_x(1,t) = 0. \end{cases} \tag{23}$$

The boundary forcing h is now assumed to be periodic with period τ. We are concerned with whether or not this periodic forcing generate a time-periodic solution of (23).

Theorem 4.1. *There exists a $\delta > 0$ such that if $h \in H^{\frac{1}{3}}_{loc}(R^+)$ is a time -periodic function of period τ satisfying $\|h\|_{H^{\frac{1}{3}}(0,\tau)} \le \delta$, then (23) admits a solution*

$$u^* \in C_b(0,\infty; L^2(0,1)) \cap L^2_{loc}(0,\infty; H^1(0,1)),$$

which is a time-periodic function of period τ. In addition, there exist a $\eta > 0$ such that if $u_1^* \in C_b(0,\infty; L^2(0,1)) \cap L^2_{loc}(0,\infty; H^1(0,1))$ is a time-periodic solution of (23) and

$$\|u^*(\cdot,0) - u_1^*(\cdot,0)\|_{L^2(0,1)} \le \eta,$$

then $u^*(x,t) \equiv u_1^*(x,t)$ for any $x \in (0,1)$ and $t \ge 0$.

Proof: Choose $\phi \in L^2(0,1)$ and consider the IBVP

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = \phi(x), \\ u(x,0) = h(t), & u(1,t) = u_x(1,t) = 0. \end{cases} \tag{24}$$

For the solution u of (24), let

$$w(x,t) = u(x,t+\tau) - u(x,t).$$

Then w solves the IBVP

$$\begin{cases} w_t + w_x + (aw)_x + w_{xxx} = 0, & x \in (0,1),\ t > 0, \\ w(x,0) = \phi^*(x), \\ w(x,0) = 0, & w(1,t) = w_x(1,t) = 0 \end{cases} \tag{25}$$

where $\phi^*(x) = u(x,\tau) - \phi(x)$, and $a(x,t) = \frac{1}{2}(u(x,t+\tau) + u(x,t))$.

By Theorem 2.8, there exist $T > 0,\ r > 0$ and $\delta > 0$,

$$\text{if } a \in Y_T \text{ satisfying } \|a\|_{Y_T} \le \delta, \tag{26}$$

then

$$\|w(\cdot,t)\|_0 \le Ce^{-rt}\|\phi^*\|_0$$

for any $t \ge 0$.

To see condition (26) is satisfied, note that $a(x,t) = \frac{1}{2}(u(x,t+\tau) + u(x,t))$. According to Proposition 3.2 and Theorem 3.3, there exist $T > 0,\ \delta_j > 0, j = 1,2$ and $r > 0$ such that if

$$\|\phi\|_0 \le \delta_1,\ \sup_{n\ge 0} |h|_{\frac{1}{3},(nT,(n+1)T)} \le \delta_2,$$

then

$$\|u(\cdot,t)\|_0 \le C_1 e^{-rt}\|\phi\|_0 + C_2 \sup_{n\ge 0} |h|_{\frac{1}{3},(nT,(n+1)T)}$$

and

$$\|u\|_{Y_{t,t+T}} \le C_2\|u(\cdot,t)\|_0 + C_2|h|_{\frac{1}{3},(t,t+T)}$$

for any $t \geq 0$. Consequently, $u \in Y_T$ and

$$\|u\|_{Y_T} \leq C\left(\|\phi\|_0 + \sup_{n\geq 0} |h|_{\frac{1}{3},(nT,(n+1)T)}\right)$$

for some $C > 0$ depending only on T. Similarly, $u^*(x,t) = u(x,t+\tau)$ also satisfies

$$\|u^*\|_{Y_T} \leq C\left(\|\phi\|_0 + \sup_{n\geq 0} |h|_{\frac{1}{3},(nT,(n+1)T)}\right).$$

Condition (26) is therefor satisfied so long as δ_1 and δ_2 is small enough. Furthermore since h is a periodic function of period τ,

$$\sup_{n\geq 0} |h|_{\frac{1}{3},(nT,(n+1)T)} \leq C|h|_{\frac{1}{3},(0,\tau)}$$

for some constant $C > 0$ depending only on T. We have thus proved that there exist $T > 0$, $\delta > 0$, and $r > 0$ such that if

$$\|\phi\|_0 + |h|_{\frac{1}{3},(0,\tau)} \leq \delta,$$

then

$$\|w(\cdot,t)\|_0 \leq Ce^{-rt}\|\phi^*\|_0 \quad \text{ for any } t \geq 0.$$

With this fact in hand, we can show that (23) possesses a time-periodic solution of period τ.

Let $u_n = u(x, n\tau)$ for $n \geq 1$. For any positive integers n and m,

$$\begin{aligned}
\|u_{n+m} - u_n\|_0 &= \left\|\sum_{k=1}^{m} u_{n+k} - u_{n+k-1}\right\|_0 \\
&\leq \sum_{k=1}^{m} \|w(\cdot,(n+k-1)\tau)\|_0 \\
&\leq \frac{Ce^{-n\tau}}{1-e^{-\tau}}\|\phi^*\|_0
\end{aligned}$$

for any $m \geq 1$. Thus, $\{u_n\}$ is a Cauchy sequence in $L^2(0,1)$. Let $\psi \in L^2(0,1)$ be the limit of u_n as $n \to \infty$. By Proposition 3.2, $\|\psi\|_0 \leq C\delta$. Taking ψ as an initial data together with the boundary forcing h for IBVP (24), we claim that its solution u^* is the desired time-periodic solution of period τ.

Indeed, note that while $u_n(\cdot) = u(\cdot, n\tau)$ converges strongly to ψ and $u_{n+1}(\cdot) = u(\cdot, n\tau+\tau)$ converges strongly to $u^*(\cdot,\tau)$ in $L^2(0,1)$ as $n \to \infty$ because of the continuity of the associated solution map. Observing that

$$\begin{aligned}
\|u^*(\cdot,\tau) - u^*(\cdot,0)\|_0 \leq{}& \|u^*(\cdot,\tau) - u^*(\cdot,n\tau+\tau)\|_0 \\
&+ \|u^*(\cdot,n\tau+\tau) - u^*(\cdot,n\tau)\|_0 + \|u^*(\cdot,0) - u^*(\cdot,n\tau)\|_0
\end{aligned}$$

for any $n \geq 1$, it is concluded that

$$u^*(\cdot,\tau) = u^*(\cdot,0) \qquad \text{in } L^2(0,1)$$

and therefore u^* is a time-periodic function of period τ.

To show the uniqueness, let u_1^* be another time-periodic solution with the same boundary forcing. Let $z(x,t) = u^*(x,t) - u_1^*(x,t)$. Then z solves the IBVP (25) with $a = \frac{1}{2}(u_1^* + u^*)$ and $\phi^*(x) = u_1^*(x,0) - \psi(x)$. One can show that z decays exponentially in the space $L^2(0,1)$ as $t \to \infty$. Thus $u_1^*(\cdot,t) = u^*(\cdot,t)$ in the space $L^2(0,1)$ for any $t \geq 0$. The proof is complete. □

5. **Conclusion remarks.** There have been many studies concerning with time-periodic solutions of partial differential equations in the literature. Early work on this subject include for example, Brézis [5], Vejvoda et al [17], Keller and Ting [10] and Rabinowitz [13, 14]. For recent theory, see Craig and Wayne [7] and Wayne [19]. In particular, the interested readers are referred to article [19] in which Wayne has provided a very helpful review of theory pertaining to time-periodic solutions of nonlinear partial differential equations.

In this paper, motivated by laboratory experiments, we have studied the initial-boundary-value problem of the KdV equation posed on the finite interval $(0,1)$:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \qquad 0 < x < 1,\ t > 0, \\ \\ u(x,0) = \phi(x), \quad u(0,t) = h(t), \quad u(1,t) = u_x(1,t) = 0, \qquad 0 < x < 1,\ t > 0. \end{cases} \tag{27}$$

We have proved that if the boundary forcing is periodic of small amplitude, then any solution of (19) is asymptotically time periodic so long as the norm of its initial value ϕ in the space $L^2(0,1)$ is small. In addition, the equation in (27) admits a unique small amplitude time-periodic solution satisfying the boundary conditions in (19), which is locally exponentially stable in $L^2(0,1)$.

We conclude the paper with a list of open questions with remarks for the interested readers to conduct further investigations.

Open problems

(1) The results presented in the paper for the IBVP (27) are local in the sense that both amplitudes of the initial value and the boundary forcing are required to be small. In particular, both uniqueness and existence of time-periodic solution as well as its stability are local in nature. Do those results hold globally? In particular, is the small amplitude time-periodic solution globally exponentially stable in the space $L^2(0,1)$? A more challenging question is whether a periodic boundary forcing of large amplitude produce a large amplitude time-periodic solution?

(2) Study IBVP (4) of the BBM equation posed on the finite interval $(0,1)$ with both the damping coefficients $\alpha = 0$ and $\gamma = 0$, i.e.,

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \qquad 0 < x < 1,\ t > 0, \\ \\ u(x,0) = \phi(x), \quad u(0,t) = h(t), \quad u(1,t) = 0, \qquad 0 < x < 1,\ t > 0 \end{cases} \tag{28}$$

If the boundary forcing h is a periodic function, does the equation in (28) admit a time-periodic solution satisfying the boundary conditions. If such a time-periodic solution exists, how many are there and what are their stability?

The interested readers should be reminded that while the BBM equation is usually easier to study mathematically than the KdV equation, it is the other way around, however, for the problems proposed here. This is because the system described by IBVP (28) of the BBM equation is conserved in the sense the H^1−norm of its solution is conserved:

$$\frac{d}{dt}\int_0^1 (u_t^2(x,t) + u_x^2(x,t))dx = 0 \qquad \text{for any } t \geq 0$$

if the boundary forcing $h \equiv 0$. By contrast, if the boundary forcing $h \equiv 0$, the small amplitude solution of IBVP (19) of the KdV equation (with $\alpha = 0$ and $\beta = 0$) decays exponentially to zero in the space $L^2(0,1)$. The system described by IBVP (27) possesses a built-in dissipative mechanism through imposing the boundary condition at $x = 0$ although there is no damping term appeared in the equation. A different approach is needed to study time-periodic solution of system (28).

(3) Study the following IBVP of the KdV type equation posed on the half line R^+:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_x - \gamma u = 0, & 0 < x,\ t < \infty, \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t), & 0 < x,\ t < \infty \end{cases} \tag{29}$$

where $\alpha \geq 0$ and $\gamma \geq 0$ are constants.

As we have pointed out in the introduction, Bona, Sun and Zhang [3] have shown that if the damping coefficient $\gamma > 0$ and the boundary forcing is periodic with small amplitude, then system (29) admits a unique amplitude time-periodic solution which is also globally exponentially stable in the space $H^s(R^)$ with $s \geq 1$. In their proof, the condition $\gamma > 0$ is crucial. A question arises naturally what will happen if $\gamma = 0$? A more challenging case is when both damping coefficients α and γ are zero.

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