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2.1: Matrix Solution of the Continuity Equations

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2. Methods to Solve Hydraulic Networks

2.1. Matrix Solution of the Continuity Equations

In Section 1.2 - Conservation of Mass we indicated that a single continuity equation must be written for each junction node. Therefore, for a system with J junction nodes, we must write J continuity equations each having the form of Eq. (2) which is shown below.

$$F(\bar{Q}) = \sum_i Q_{in,j} - \sum_i Q_{out,j} = D_j \quad j = 1,2, \ldots J$$

Eq. (2)

Notice that Eq. (2) is linear in the pipeline flows. Thus we can easily find pipeline flows that satisfy continuity by writing a system of J continuity equations and then solving the system of equations using matrix or linear algebra solution techniques. This method works particularly well for branched hydraulic networks with only one reservoir or one tank (one source). Branched systems with only one source do not have any energy equations which influence pipeline flows.

Consider the simple branched system shown in the figure below where the units of flow are gallons per minute (Gpm). We can easily see from inspection that the flow in pipe P_2 is 520 Gpm and that the flow in pipe P_3 is 375 Gpm. Applying continuity at Node A we can see that the flow in pipe P_1 is 895 Gpm. We can also find these same flows by writing a system of three continuity equations – one for each of the three junction nodes.

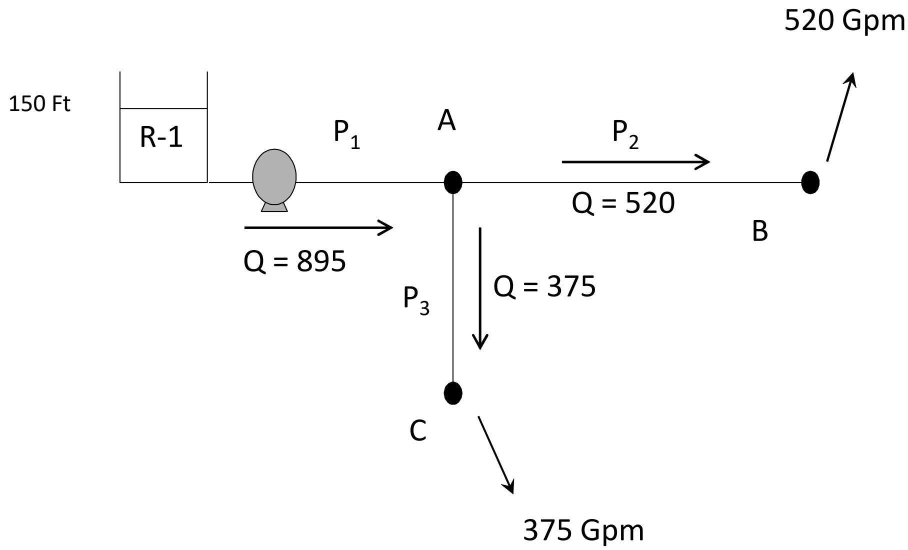

Figure 9 – Branched Hydraulic Network #1

Continuity equation for Node A:

$$\sum_i Q_{in,A} - \sum_i Q_{out,A} = D_A$$

$$Q_1 - (Q_2 + Q_3) = 0\ Gpm$$

Continuity equation for Node B:

$$\sum_i Q_{in,B} - \sum_i Q_{out,B} = D_B$$

$$Q_2 = 520\ Gpm$$

Continuity equation for Node C:

$$\sum_i Q_{in,C} - \sum_i Q_{out,C} = D_C$$

$$Q_3 = 375\ Gpm$$

Notice that there are three unknowns in this system Q_1, Q_2 and Q_3 and that there are three continuity equations. We have the same number of equations as we have unknowns. Therefore, we can solve for the unknowns. Our three continuity equations are:

Node A: $Q_1 - Q_2 - Q_3 = 0$

Node B; $Q_2 = 520$

Node C: $Q_3 = 375$

This formulation lends itself nicely to formation of a matrix as shown below.

$$\begin{matrix} Node\ A: & (1 * Q_1) + & (-1 * Q_2) + & (-1 * Q_3) & & 0 \\ Node\ B: & (0 * Q_1) + & (1 * Q_2) + & (0 * Q_3) & = & 520 \\ Node\ C: & (0 * Q_1) + & (0 * Q_2) + & (1 * Q_3) & & 375 \end{matrix}$$

We can simplify the matrix shown above even further so that we obtain:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 520 \\ 375 \end{Bmatrix}$$

Eq. (24)

The system of equations shown in Eq. (24) above has the form:

$$[\boldsymbol{J}]\{\boldsymbol{Q}\} = \{\boldsymbol{D}\}$$

Eq. (25)

Where [**J**] is a coefficient matrix consisting of values of 1, 0 and -1. **Q** is a vector (one-dimensional array) representing pipeline flows and **D** is a vector representing nodal demands. We can solve for the vector of unknowns by inverting matrix [**J**] and multiplying the matrix inverse by **D** as shown below:

$$\{\boldsymbol{Q}\} = [\boldsymbol{J}]^{-1}\{\boldsymbol{D}\}$$

Eq. (26)

Using the matrix inverse Eq. (24) can now be shown as:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 520 \\ 375 \end{Bmatrix}$$

Eq. (27)

The solution to the system of equations shown in Eq. (27) is:

Q = $[875, 520, 375]^T$

Now let's apply this solution approach to the much more complicated branched system shown in Figure 10. Also shown below is a table of nodal demands. We write 14 continuity equations – one for each junction node in the system.

Table 8 – Node Data

Node	Elevation	Demand (Gpm)
R1	780	N/A
J1	880	0
J2	910	50
J3	900	120
J4	905	55
J5	905	30
J6	890	25
J7	880	50
J8	905	120
J9	910	75
J10	925	50
J11	930	200
J12	930	50
J13	935	75
J14	935	100

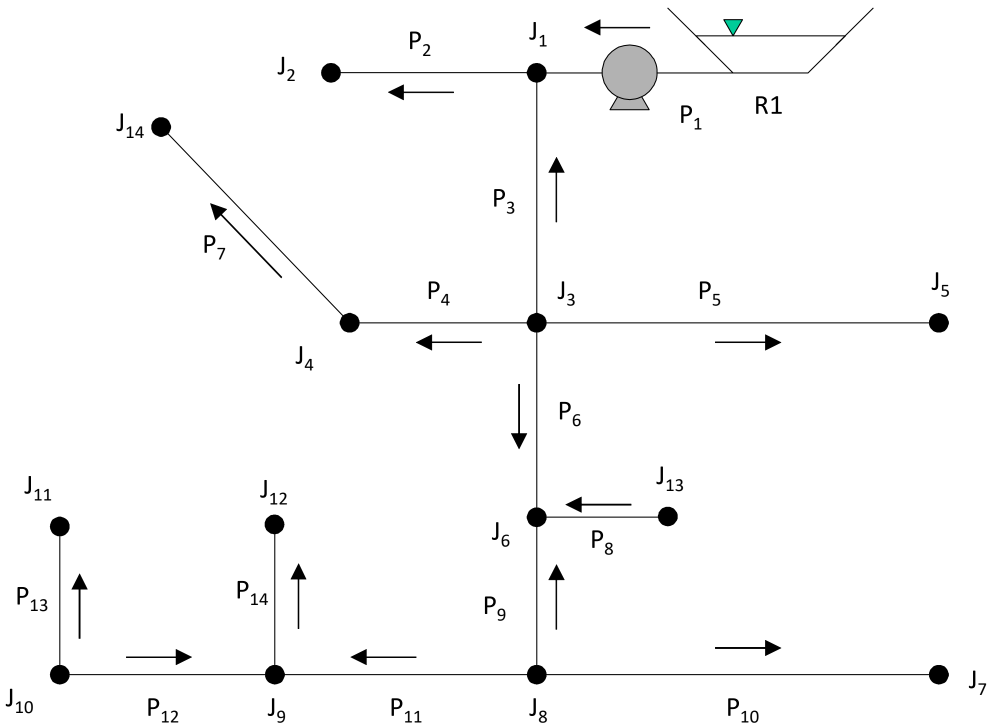

Figure 10 – Branched Hydraulic Network #2

Before we write the continuity equations, let's notice the flow directions in the pipes. The flow directions do not necessarily have to be in the correct direction – at least initially. For example, we can see that the flow direction in pipe P_3 is incorrect. If the flow in pipe P_3 is indeed from J_3 to J_1, then there would be no way to meet the demands downstream of pipe P_3. We will see in a moment how the solution to the system of continuity equations addresses the flow direction issue.

The system of continuity equations for the network shown in Figure 10 based on the flow directions shown is:

Node J1: $$Q_1 - Q_2 + Q_3 = 0$$

Node J2: $Q_2 = 50$
Node J3: $-Q_3 - Q_4 - Q_5 - Q_6 = 120$
Node J4: $Q_4 - Q_7 = 55$
Node J5: $Q_5 = 30$
Node J6: $Q_6 + Q_8 + Q_9 = 25$
Node J7: $Q_{10} = 50$
Node J8: $-Q_9 - Q_{10} - Q_{11} = 120$
Node J9: $Q_{11} + Q_{12} - Q_{14} = 75$
Node J10: $-Q_{12} + Q_{13} = 50$
Node J11: $Q_{13} = 200$
Node J12: $Q_{14} = 50$
Node J13: $-Q_8 = 75$
Node J14: $Q_7 = 100$

We can express the system of continuity equation in matrix form as shown in Eq. (28). Solving this system of equations using linear algebra produces the vector of flows.

The solution to the system of continuity equations is presented in Eq. (29). Notice that some of the flows are negative. What this means is that the final flow direction in the pipe is opposite the assumed initial flow direction. Specifically, the flow direction in pipes P_3, P_8, P_9, and P_{12} is opposite the initial flow direction. The final flow directions for all pipes is shown in Figure 11.

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \\ Q_{11} \\ Q_{12} \\ Q_{13} \\ Q_{14} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 50 \\ 120 \\ 55 \\ 30 \\ 25 \\ 50 \\ 120 \\ 75 \\ 50 \\ 200 \\ 50 \\ 75 \\ 100 \end{Bmatrix}$$

Eq. (28)

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \\ Q_{11} \\ Q_{12} \\ Q_{13} \\ Q_{14} \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 50 \\ -950 \\ 155 \\ 30 \\ 645 \\ 100 \\ -75 \\ -545 \\ 50 \\ 375 \\ -250 \\ 200 \\ 50 \end{Bmatrix}$$

Eq. (29)

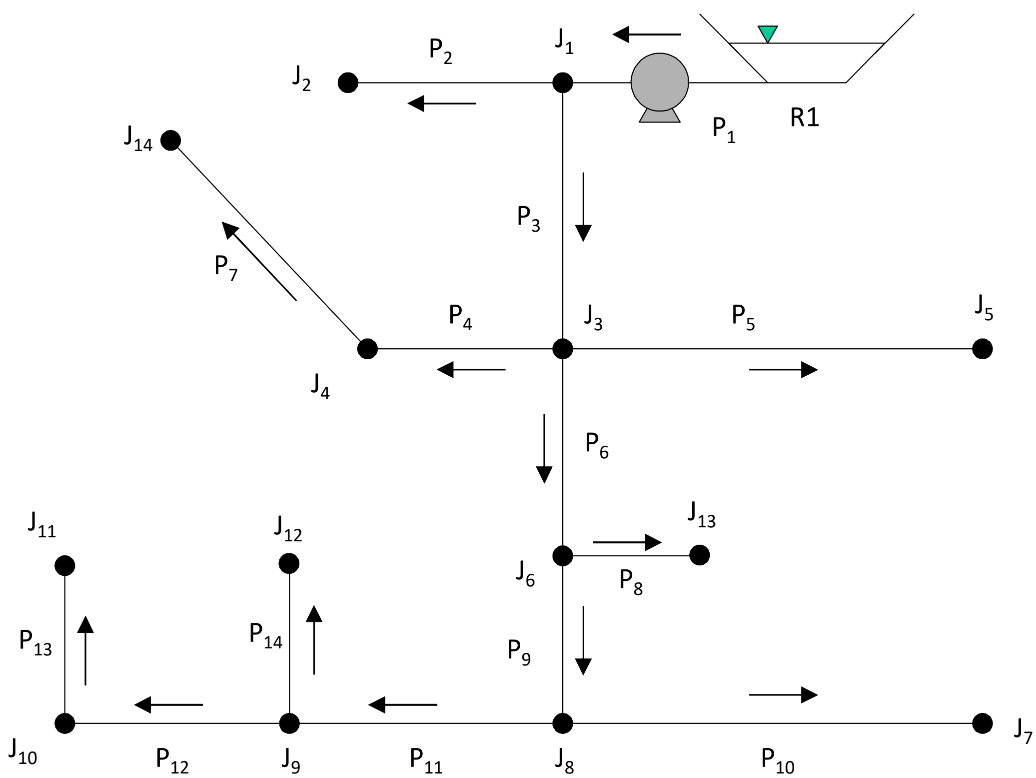

Figure 11 – Branched Hydraulic Network #2 with Correct Flow Directions

We are particular interested in the pressure at each junction node. Pressures tell us if the system is operating at an acceptable level of service. Many water utilities do not wish for system-wide pressures to fall below 30-35 psi nor do they wish to operate at excessively high pressures, say, above 100-110 psi. We can compute system-wide pressures once we know the flow in each pipe. We do so by starting at some known head source such as Reservoir R-1 and applying the energy equation to each junction node accounting for the head losses between the source and the junction node.

In order to compute head losses, we must have information on pipe and pump characteristics. Specifically, for each pipe we must know the length, diameter and roughness of the pipe. We must also know the minor loss coefficient, if there is one, for each pipe. We must also know the pump head-discharge curve for each pump. Finally, if we wish to compute pressures, we must know the elevation of each junction node. Nodal elevations are given in Table 8 for the system presented in Figure 10. Let’s compute system-wide pressures given the pipe characteristics in Table 9 and the pump head-discharge relationship in Table 10.

Table 9 – Pipe Data

Pipe	Start Node	End Node	Length (Ft)	Diameter (In)	C-Factor	ΣK_L
P-1	R-1	J-1	500	16	120	5
P-2	J-1	J-2	1200	8	110	0
P-3	J-3	J-1	1500	12	120	0
P-4	J-3	J-4	350	6	110	0
P-5	J-3	J-5	2500	6	110	25
P-6	J-3	J-6	750	4	120	0
P-7	J-4	J-14	850	6	120	0
P-8	J-13	J-6	500	8	110	0
P-9	J-8	J-6	800	10	120	0
P-10	J-8	J-7	1200	8	105	0
P-11	J-8	J-9	1250	8	120	0
P-12	J-10	J-9	1050	6	120	0
P-13	J-10	J-11	750	6	110	0
P-14	J-9	J-12	750	6	110	0

Table 10 – Pump Data

Pump Flow (Gpm)	Pump Head (Ft)
0	325
500	295
1200	210

We begin by computing the pipe resistance and minor loss resistance coefficients for each pipe. When using the Hazen-Williams head loss formula, the pipe resistance coefficient for pipe i becomes:

$$K_{p,i} = \frac{4.73 L_i}{C_i^{1.852} D_i^{4.87}}$$

For pipe P-1 the pipe resistance coefficient is:

$$K_{p,1} = \frac{4.73L_1}{C_1^{1.852}D_1^{4.87}} = \frac{4.73(500)}{(120)^{1.852}\left({}^{16}/_{12}\right)^{4.87}} = 0.082$$

The minor loss resistance coefficient for pipe i can be found from:

$$K_{m,i} = \frac{\sum K_{L,i}}{2gA_i^2}$$

For pipe P-1 the minor loss resistance coefficient is:

$$K_{m,i} = \frac{\sum K_{L,1}}{2gA_1^2} = \frac{5}{2g\left(0.25\pi\left({}^{16}/_{12}\right)^2\right)^2} = 0.040$$

The discharge, pipe resistance coefficient and minor loss resistance coefficient for each pipe in the 14-pipe branched system is shown in
Table 11. Recall that we must use the pipeline flow in units of Cfs when computing head losses with the Hazen-Williams equation.

Table 11 – Pipe Resistance and Minor Loss Resistance Terms

Pipe	Flow (Gpm)	Flow (Cfs)	Area (Ft^2)	K_P	K_M
P-1	1000	2.228	1.396	0.082	0.040
P-2	50	0.111	0.349	6.776	0.000
P-3	950	2.117	0.785	1.001	0.000
P-4	155	0.345	0.196	8.022	0.000
P-5	30	0.067	0.196	57.300	10.069
P-6	645	1.437	0.087	105.405	0.000
P-7	100	0.223	0.196	16.583	0.000
P-8	75	0.167	0.349	2.823	0.000
P-9	545	1.214	0.545	1.297	0.000
P-10	50	0.111	0.349	7.385	0.000
P-11	375	0.835	0.349	6.007	0.000
P-12	250	0.557	0.196	20.484	0.000
P-13	200	0.446	0.196	17.190	0.000
P-14	50	0.111	0.196	17.190	0.000

Developing a Pump Head-Discharge Curve:

We will use a pump head-discharge equation having the form shown below. This happens to be the form of the default pump head-discharge equation used in EPANET.

$$E(Q) = H_o - cQ^N$$

Where: E(Q) – pump head at discharge Q
H_o – cutoff head, E(Q=0)
C,N – coefficients

From the data presented in Table 10, the cutoff head – that is the pump head at zero flow – is H_0 = 325 ft. Now we can write two equations and solve for the two remaining unknowns C and N.

The first of the two equations is presented below. Note that the flows have been converted from Gpm to Cfs since the Hazen-Williams head loss equation requires flow to be in units of Cfs.

Eq. 1: $$295 = 325 - c\left(\frac{500}{448.84}\right)^N$$

Eq. 2: $$210 = 325 - c\left(\frac{1200}{448.84}\right)^N$$

Simplifying Eq. 1 and Eq. 2 above we obtain:

Eq. 1: $$30 = c(1.114)^N$$

Eq. 2: $$115 = c(2.674)^N$$

Dividing Eq. 2 by Eq. 1 yields:

$$\frac{115}{30} = 3.833 = \frac{c(2.674)^N}{c(1.114)^N} = \left(\frac{2.674}{1.114}\right)^N = 2.400^N$$

Solving for N we obtain:

$$3.833 = 2.400^N$$

$$\log(3.833) = \log(2.400^N) = N\log(2.400)$$

$$\mathrm{N} = \frac{\log(3.833)}{\log(2.400)} = 1.535$$

Now that we know the value of N, we can use either Eq. 1 or Eq. 2 above to find the value of C. Let's use Eq. 1:

Eq. 1: $30 = c(1.114)^{1.535}$

$$c = \frac{30}{(1.114)^{1.535}} = 25.419$$

Now our pump head-discharge equation with discharge, Q, expressed in Cfs becomes:

$$E(Q) = 325 - 25.419Q^{1.535}$$

We now have all the information necessary to compute the head (hydraulic grade line) and the pressure at each junction node. We start by finding the head and pressure at J_1 as shown below:

Node J_1:

$$H_{J1} = H_{R-1} + E(Q)_{P1} - h(f)_{P1} - h(m)_{P1}$$

The equation above states that the head at node J_1 is equal to the head at reservoir R-1 plus the pump head minus the friction loss in pipe P_1 minus the minor loss in pipe P_1. The equation below makes use of the pipe resistance and minor loss resistance terms.

$$H_{J1} = H_{R-1} + \left(325 - 25.419Q_1^{1.535}\right) - K_{p,1}Q_1^{1.852} - K_{m,1}Q_1^2$$
$$= 780 + (325 - 25.419(2.228)^{1.535}) - 0.082(2.228)^{1.852} - 0.040(2.228)^2$$

$$H_{J1} = 780 + 238.062 - 0.362 - 0.198 = 1017.502\,ft$$

Now that we have the head at node J_1, we can compute the pressure at the node – assuming, of course, that we have the nodal elevation. From the definition of total head at a point:

$$H_{J1} = Z_{J1} + \frac{P_{J1}}{\gamma} + \frac{V^2}{2g}$$

We ignore the velocity head since it's value is so small it does not contribute significantly to the total head. The pressure at node J_1 can now be found from:

$$P_{J1} = \left(H_{J1} - Z_{J1}\right)\gamma = (1017.502 - 880)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 8{,}580.125\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 59.58\,psi$$

Now that we know the head at node J_1, we can find the head and pressure at node J_2 and node J_3. The steps below illustrated how this is done.

Node J_2:

$$H_{J2} = H_{J1} - h(f)_{P2} - h(m)_{P2} = H_{J1} - K_{p,2}Q_2^{1.852} - K_{m,2}Q_2^2$$

$$H_{J2} = 1017.502 - 6.776(0.111)^{1.852} - 0.00(0.111)^2$$
$$= 1017.502 - 0.116 = 1017.386\, ft$$

Now we find the pressure at node J_2:

$$P_{J2} = \left(H_{J2} - Z_{J2}\right)\gamma = (1017.386 - 910)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 6{,}700.886\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 46.53\, psi$$

Node J_3:

$$H_{J3} = H_{J1} - h(f)_{P3} - h(m)_{P3} = H_{J1} - K_{p,3}Q_3^{1.852} - K_{m,3}Q_3^2$$

$$H_{J3} = 1017.502 - 1.001(2.117)^{1.852} - 0.00(2.117)^2$$
$$= 1017.502 - 4.015 = 1013.487\, ft$$

Now we find the pressure at node J_3:

$$P_{J3} = \left(H_{J3} - Z_{J3}\right)\gamma = (1013.487 - 900)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 7{,}081.651\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 49.18\, psi$$

Now that we know the head at node J_3, we can compute the head and pressure at nodes J_4, J_5 and J_6. Notice that J_3 is the node that is upstream of nodes J_4, J_5 and J_6. Thus the head at node J_3 is the boundary head for nodes J_4, J_5 and J_6.

Node J_4:

$$H_{J4} = H_{J3} - h(f)_{P4} - h(m)_{P4} = H_{J3} - K_{p,4}Q_4^{1.852} - K_{m,4}Q_4^2$$

$$H_{J4} = 1013.487 - 8.022(0.345)^{1.852} - 0.00(0.345)^2$$
$$= 1013.487 - 1.118 = 1012.369\, ft$$

Now we find the pressure at node J_4:

$$P_{J4} = \left(H_{J4} - Z_{J4}\right)\gamma = (1012.369 - 905)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 6{,}699.826\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 46.53\ psi$$

Node J_5:

$$H_{J5} = H_{J3} - h(f)_{P5} - h(m)_{P5} = H_{J3} - K_{p,5}Q_5^{1.852} - K_{m,5}Q_5^2$$

$$H_{J5} = 1013.487 - 57.300(0.067)^{1.852} - 10.069(0.067)^2$$
$$= 1013.487 - 0.384 - 0.045 = 1013.058\ ft$$

Now we find the pressure at node J_5:

$$P_{J5} = \left(H_{J5} - Z_{J5}\right)\gamma = (1013.058 - 905)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 6{,}742.819\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 46.83\ psi$$

Node J_6:

$$H_{J6} = H_{J3} - h(f)_{P6} - h(m)_{P6} = H_{J3} - K_{p,6}Q_6^{1.852} - K_{m,6}Q_6^2$$

$$H_{J6} = 1013.487 - 105.405(1.437)^{1.852} - 0.000(1.437)^2$$
$$= 1013.487 - 206.287 = 807.200\ ft$$

Now we find the pressure at node J_6:

$$P_{J6} = \left(H_{J6} - Z_{J6}\right)\gamma = (807.200 - 890)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -5{,}166.720\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -35.88\ psi$$

Knowing the head at node J_6 permits us to find the head at nodes J_8 and J_{13}.

Node J_8:

$$H_{J8} = H_{J6} - h(f)_{P9} - h(m)_{P9} = H_{J6} - K_{p,9}Q_9^{1.852} - K_{m,9}Q_9^2$$

$$H_{J8} = 807.200 - 1.297(1.214)^{1.852} - 0.000(1.214)^2$$
$$= 807.200 - 1.857 = 805.343\ ft$$

Now we find the pressure at node J_8:

$$P_{J8} = \left(H_{J8} - Z_{J8}\right)\gamma = (805.343 - 905)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -6{,}218.597\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -43.18\ psi$$

Node J_{13}:

$$H_{J13} = H_{J6} - h(f)_{P8} - h(m)_{P8} = H_{J6} - K_{p,8}Q_8^{1.852} - K_{m,8}Q_8^2$$

$$H_{J13} = 807.200 - 2.823(0.167)^{1.852} - 0.000(0.167)^2$$
$$= 807.200 - 0.103 = 807.098\ ft$$

Now we find the pressure at node J_{13}:

$$P_{J13} = \left(H_{J13} - Z_{J13}\right)\gamma = (807.098 - 935)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -7{,}981.085\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -55.42\ psi$$

Knowing the head at node J_8 permits us to find the head at nodes J_7 and J_9.

Node J_7:

$$H_{J7} = H_{J8} - h(f)_{P10} - h(m)_{P10} = H_{J8} - K_{p,10}Q_{10}^{1.852} - K_{m,10}Q_{10}^{2}$$

$$H_{J7} = 805.343 - 7.385(0.111)^{1.852} - 0.000(0.111)^{2}$$
$$= 805.343 - 0.126 = 805.217\,ft$$

Now we find the pressure at node J_7:

$$P_{J7} = \left(H_{J7} - Z_{J7}\right)\gamma = (805.217 - 880)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -4{,}666.459\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -32.41\,psi$$

Node J_9:

$$H_{J9} = H_{J8} - h(f)_{P11} - h(m)_{P11} = H_{J8} - K_{p,11}Q_{11}^{1.852} - K_{m,11}Q_{11}^{2}$$

$$H_{J9} = 805.343 - 6.007(0.835)^{1.852} - 0.000(0.835)^{2}$$
$$= 805.343 - 4.302 = 801.041\,ft$$

Now we find the pressure at node J_9:

$$P_{J9} = \left(H_{J9} - Z_{J9}\right)\gamma = (801.041 - 910)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -6{,}799.042\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -47.22\,psi$$

Knowing the head at node J_9 permits us to find the head at nodes J_{10} and J_{12}.

Node J_{10}:

$$H_{J10} = H_{J9} - h(f)_{P12} - h(m)_{P12} = H_{J9} - K_{p,12}Q_{12}^{1.852} - K_{m,12}Q_{12}^{2}$$

$$H_{J10} = 801.041 - 20.484(0.557)^{1.852} - 0.000(0.557)^{2}$$
$$= 801.041 - 6.930 = 794.111\ ft$$

Now we find the pressure at node J_{10}:

$$P_{J10} = \left(H_{J10} - Z_{J10}\right)\gamma = (794.111 - 925)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -8{,}167.474\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -56.72\ psi$$

Node J_{12}:

$$H_{J12} = H_{J9} - h(f)_{P14} - h(m)_{P14} = H_{J9} - K_{p,14}Q_{14}^{1.852} - K_{m,14}Q_{14}^{2}$$

$$H_{J12} = 801.041 - 17.190(0.111)^{1.852} - 0.000(0.111)^{2}$$
$$= 801.041 - 0.293 = 800.748\ ft$$

Now we find the pressure at node J_{12}:

$$P_{J12} = \left(H_{J12} - Z_{J12}\right)\gamma = (800.748 - 930)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -8{,}065.325\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -56.01\ psi$$

The pressure at Node J_{11} can be found by applying an energy balance, the same as we have done above, between node J_{10} and node J_{11} as shown below.

Node J_{11}:

$$H_{J11} = H_{J10} - h(f)_{P13} - h(m)_{P13} = H_{J10} - K_{p,13}Q_{13}^{1.852} - K_{m,13}Q_{13}^{2}$$

$$H_{J11} = 794.111 - 17.190(0.446)^{1.852} - 0.000(0.446)^2$$
$$= 794.111 - 3.853 = 790.258\,ft$$

Now we find the pressure at node J_{11}:

$$P_{J11} = \left(H_{J11} - Z_{J11}\right)\gamma = (790.258 - 930)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= -8{,}719.901\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = -60.55\,psi$$

Finally, we can find the pressure at Node J_{14} by applying an energy balance between node J_4 and node J_{14} as shown below.

Node J_{14}:

$$H_{J14} = H_{J4} - h(f)_{P7} - h(m)_{P7} = H_{J4} - K_{p,7}Q_7^{1.852} - K_{m,7}Q_7^2$$

$$H_{J14} = 1012.369 - 16.583(0.223)^{1.852} - 0.000(0.223)^2$$
$$= 1012.369 - 1.030 = 1011.340\,ft$$

Now we find the pressure at node J_{14}:

$$P_{J14} = \left(H_{J14} - Z_{J14}\right)\gamma = (1011.340 - 935)ft * 62.4\ {}^{lb}/_{ft^3}$$
$$= 4{,}763.616\ {}^{lb}/_{ft^2} * {}^{ft^2}/_{144\ in^2} = 33.08\,psi$$

We summarize the nodal results in the table below. We have now found the flow in each pipeline and the pressure at each junction node. This gives us the information necessary to make informed engineering decisions such as pressures downstream of pipe P_6 are unacceptably low. Of course, the reason for this is that the diameter of pipe P_6 is too small as evidenced by a friction loss of 206.287 ft (which is equivalent to a 89.38 psi pressure drop).

Table 12 – Node Results

Node	Hydraulic Grade Line	Pressure (Psi)
R1	780.000	N/A
J1	1017.502	59.58
J2	1017.386	46.53
J3	1013.487	49.18
J4	1012.369	46.53
J5	1013.058	46.83
J6	807.200	-35.88
J7	805.217	-32.41
J8	805.343	-43.18
J9	801.041	-47.22
J10	794.111	-56.72
J11	790.258	-60.55
J12	800.748	-56.01
J13	807.098	-55.42
J14	1011.340	33.08