2004

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Electronic Proceedings of Undergraduate Mathematics Day
2004, No. 1, http://www.udayton.edu/∽mathdept/EPUMD/EPMUD.htm

Newton's Unfinished Business: Uncovering the Hidden Powers of Eleven in Pascal's Triangle

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Abstract. Sir Isaac Newton once observed that the first five rows of Pascal's Triangle, when concatenated, yield the corresponding powers of eleven. He claimed without proof that subsequent rows also generate powers of eleven. Was he correct? While not all rows can simply be concatenated, the powers of eleven can still be easily derived from each. We have uncovered an algorithm the supports Newton's claim and will prove its validity for all rows of the Triangle.

In Isaac Newton's *A Treatise of the Method of Fluxions and Infinite Series*, he makes a startling observation: "But these in the alternate Areas, which are given, I observed were the same with the Figures of which the several ascending Powers of the Number 11 consist, viz. $11^0, 11^1, 11^2, 11^3, 11^4$, etc. that is, first 1; the second $1, 1$; the third $1, 2, 1$; the fourth $1, 3, 3, 1$; the fifth $1, 4, 6, 4, 1$, &." [2] Essentially what Newton is stating is that there is a correlation between the powers of eleven and the first five rows of Pascal's Triangle. He never proved that it works for subsequent rows, and the problem was left as merely these observations. Lacke [1] introduces the use of the Binomial Theorem to solve this for the later rows of the Triangle.

It would seem that this pattern does not hold for further rows of the triangle since $11^5 = 161,051$ and this does not match the next row of the triangle, $1, 5, 10, 10, 5, 1$. However, if one assigns a place value to each of the individual terms in the triangle, the pattern **can** be seen again. In other words, to easily read the powers of 11 in subsequent rows, one simply needs to carry the digits. For example, the sixth row of the Triangle yields

$$\begin{array}{cccccc} 1 & 5 & 10 & 10 & 5 & 1 \\ \\ 1 & 6 & 1 & 0 & 5 & 1 \end{array}$$

In expanded notation

$$161051 = 1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10^1 + 1 \cdot 10^0, \tag{1}$$

or

$$\begin{array}{lr} & \underline{1}00000 \\ & \underline{5}0000 \\ & \underline{10}000 \\ & \underline{10}00 \\ & \underline{5}0 \\ + & \underline{1} \\ \hline \\ & 161051 \end{array}$$

It can thus be seen that the value of the digits in any particular row of Pascal's Triangle correspond to increasing powers of ten when read from right to left as shown in (1). The digits of that row are multiplied by their corresponding power of ten and are then added together to yield the power of eleven.

We generalize this principle as follows to prove that the powers of eleven are infinitely generated. Recall the rth entry of row n of Pascal's Triangle is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

where r is read from the left [3, p. 758]. In this paper, the following notation will be used:

$$\binom{p}{k} = \frac{p!}{k!(p-k)!},$$

where k represents a term in row p to be read from the right, rather than from the left. For example, for the row with terms

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1,$$

the p value would equal five and the exponents in (1) represent the k value for each specific term in the row. In order to prove that this pattern continues indefinitely, the induction method is utilized. It has already been demonstrated that the hypothesis holds true for $p = 0$ to $p = 5$. It thus remains to be shown that if

$$\sum_{k=0}^{p} \frac{p!}{k!(p-k)!} 10^k = 11^p, \tag{2}$$

then

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 11^{p+1}. \tag{3}$$

All entries (except for the first and the last) in each row of the triangle can be written as $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$[3, p. 26]. This can be rewritten in our notation as

$$\frac{(p+1)!}{k!(p+1-k)!} = \frac{p!}{k!(p-k)!} + \frac{p!}{(k-1)!(p-(k-1))!}. \tag{4}$$

Break off the $k = 0$ and $k = p + 1$ terms of the summation in (3) to obtain

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 10^{p+1} + \left(\sum_{k=1}^{p} \frac{(p+1)!}{k!(p+1-k)!} 10^k \right) + 1. \tag{5}$$

Substituting (4) into (5) yields

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 10^{p+1} + \left(\sum_{k=1}^{p} \Big(\frac{p!}{k!(p-k)!} + \frac{p!}{(k-1)!(p-(k-1))!} \Big) 10^k \right) + 1. \tag{6}$$

If the $k = 0$ term is broken off of the summation in (2) and subtracted from both sides of the equation, then

$$\sum_{k=1}^{p} \frac{p!}{k!(p-k)!} 10^k = \left(\sum_{k=0}^{p} \frac{p!}{k!(p-k)!} 10^k \right) - 1 = 11^p - 1.$$

Substituting this into (6) yields

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 10^{p+1} + (11^p - 1) + \left(\sum_{k=1}^{p} \frac{p!}{(k-1)!(p-(k-1))!} 10^k \right) + 1. \tag{7}$$

The -1 and $+1$ cancel. In order to make the summation more similar to (2), the general term of the summation is adjusted to force $k = 0$:

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 10^{p+1} + 11^p + \sum_{k=0}^{p-1} \frac{p!}{(k)!(p-k)!} 10^{k+1}.$$

The summation is factored to lower the power of 10 from $k+1$ to k:

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k = 10^{p+1} + 11^p + 10 \sum_{k=0}^{p-1} \frac{p!}{(k)!(p-k)!} 10^k. \tag{8}$$

Break off the $k = p$ term and subtract from both sides of (2) to obtain

$$\sum_{k=0}^{p-1} \frac{p!}{k!(p-k)!} 10^k = 11^p - 10^p. \tag{9}$$

By substituting (9) into (8) and simplifying, the desired result is obtained:

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} 10^k$$

$$\begin{aligned} &= 10^{p+1} + 11^p + 10(11^p - 10^p) \\ &= 10^{p+1} + 11^p + 10 \cdot 11^p - 10^{p+1} \\ &= 11^p(1+10) \\ &= 11^p \cdot 11 \\ &= 11^{p+1}. \end{aligned}$$

QED

We can further generalize this result by replacing 10 and 11 with x and $x+1$. In other words, the terms in Pascal's Triangle can be used to generate powers of $x+1$. However, it is only easy to read the powers of $x+1$ from Pascal's Triangle when it is expressed in base x.

Theorem 1 *Let p be a nonnegative integer and let x be a positive integer. Then*

$$\sum_{k=0}^{p} \frac{p!}{k!(p-k)!} x^k = (x+1)^p.$$

Proof The proof is by induction and we outline the details. Note that

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} x^k = (x+1)^p + \sum_{k=1}^{p} \frac{(p+1)!}{k!(p+1-k)!} x^k + 1$$

and recall (4). Then (analogous to (7))

$$\sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p+1-k)!} x^k =$$

$$= x^{p+1} + ((x+1)^p - 1) + \left(\sum_{k=1}^{p} \frac{p!}{(k-1)!(p-(k-1))!} x^k\right) + 1$$

$$= x^{p+1} + (x+1)^p + \sum_{k=0}^{p-1} \frac{p!}{k!(p-k)!} x^{k+1}$$

$$= x^{p+1} + (x+1)^p + x \sum_{k=0}^{p-1} \frac{p!}{k!(p-k)!} x^k$$

$$= x^{p+1} + (x+1)^p + x\big((x+1)^p - x^p\big)$$

$$= x^{p+1} + (x+1)^p + x(x+1)^p - x^{p+1}$$

$$= (x+1)^p(1+x)$$

$$= (x+1)^{p+1}.$$

QED

References

[1] C. J. Lacke, *Powers of Eleven in Pascal's Triangle*, Mathematics and Computer Education (2002), 28–30.

[2] I. Newton, "A Treatise of the Method of Fluxions and Infinite Series", *The Mathematical Works of Isaac Newton, Vol. 1*, Derek T. Whiteside, ed., 31–33, Johnson Reprint Corp., New York, NY, 1964.

[3] A. F. Coxford and J. N. Payne, *Advanced Mathematics: A Preparation for Calculus, 2nd Edition*, Harcourt Brace Jovanovich, Inc., Orlando, 1978.