Totally Geodesic Surfaces in Arithmetic Hyperbolic 3-Manifolds

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Totally geodesic surfaces in arithmetic hyperbolic 3-manifolds

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Definition: A hyperbolic 3-manifold is a quotient \( M = \mathbb{H}^3 / \Gamma \) of three dimensional hyperbolic space \( \mathbb{H}^3 \) by a discrete subgroup \( \Gamma \) of \( \text{PSL}_2(\mathbb{C}) \) acting freely and properly discontinuously.

The Kleinian group \( \Gamma \) is isomorphic to the fundamental group \( \pi_1(M) \).

If we relax the requirement that \( \Gamma \) acts freely, allowing \( \Gamma \) to contain torsion, then we obtain a hyperbolic 3-orbifold.

Theorem (Mostow-Prasad Rigidity, 1974)

If \( M_1 \) and \( M_2 \) are complete finite volume hyperbolic \( n \)-manifolds with \( n > 2 \) then any isomorphism \( f : \pi_1(M_1) \rightarrow \pi_1(M_2) \) is induced by a unique isometry from \( M_1 \) to \( M_2 \).
The Length Spectrum of Hyperbolic 3-Manifolds

Fundamental domains of a pair of isospectral hyperbolic 3-orbifolds

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The Length Spectrum of Hyperbolic 3-Manifolds

What is an arithmetic hyperbolic 3-manifold?

The **commensurator** $C_\Gamma$ of a Kleinian group $\Gamma \subset \text{PSL}_2(\mathbb{C})$ is

$$C_\Gamma = \{g \in \text{PSL}_2(\mathbb{C}) : g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}.$$  

**Theorem (Margulis)**

$\Gamma$ is arithmetic if and only if $C_\Gamma$ is dense in $\text{PSL}_2(\mathbb{C})$. 

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**Background**

Recall the classification of elements $\gamma \in \text{PSL}_2(\mathbb{C}) \setminus \{\text{Id}_2\}$:

- $\gamma$ is *elliptic* if $\text{Tr}(\gamma) \in \mathbb{R}$ and $|\text{Tr}(\gamma)| < 2$.
- $\gamma$ is *parabolic* if $\text{Tr}(\gamma) = \pm 2$.
- $\gamma$ is *loxodromic* otherwise.

We will typically abuse notation and consider the eigenvalues (up to sign) of a lift of $\gamma$ to $\text{SL}_2(\mathbb{C})$. These are the roots of

$$p_\gamma(x) = x^2 - \text{Tr}(\gamma)x + 1;$$

that is,

$$\lambda_\gamma = \frac{\text{Tr}(\gamma) \pm \sqrt{\text{Tr}(\gamma)^2 - 4}}{2}.$$
When $\gamma$ is loxodromic it has a pair of fixed points and the unique geodesic in $H^3$ joining these points is the axis of $\gamma$.

Let $M = H^3/\Gamma$ be a finite-volume hyperbolic 3-manifold. The axis of $\gamma$ projects onto a closed geodesic in $M$ whose length is the translation length $\ell_0(\gamma)$ of $\gamma$, where

$$\ell_0(\gamma) = 2 \log |\lambda_\gamma|.$$ 

The element $\gamma$ also rotates around its axis as it translates along it. If $\theta(\gamma)$ is the angle incurred it translating along the axis by $\ell_0(\gamma)$, then the complex translation length of $\gamma$ is

$$\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma).$$
The length spectrum $L(M)$ of a hyperbolic 3-manifold $M$ is the set of all complex translation lengths of all closed geodesics in $M$, considered with multiplicities.

The length spectrum of $M$ determines the Laplace spectrum of $M$, hence determines spectral invariants like dimension and volume.

It is known however, that the length spectrum $L(M)$ of $M$ does not determine the isometry class of $M$. 
Theorem (Vigneras, 1980)

There exist non-isometric hyperbolic 3-manifolds with the same length spectra.

By the Mostow Rigidity Theorem, this shows that the isomorphism class of the fundamental group of a hyperbolic 3-manifold is not a spectral invariant.
It is in general unknown whether $L(M)$ determines the commensurability class of $M$. This is known to be the case when $M$ is arithmetic however.

**Theorem (Chinburg, Hamilton, Long and Reid, 2008)**

*If two arithmetic hyperbolic 3-manifolds have the same length spectra then they are commensurable.*
On the other hand non-commensurable hyperbolic 3-manifolds may share arbitrarily large portions of their length spectra.

**Theorem (Futer and Millichap, 2016)**

For every sufficiently large \( n > 0 \) there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds \( \{ N_n, N_n^\mu \} \) such that:

1. \( \text{vol}(N_n) = \text{vol}(N_n^\mu) \), where this volume grows coarsely linearly with \( n \).
2. The (complex) length spectra of \( N_n \) and \( N_n^\mu \) agree up to length \( n \).
3. \( N_n \) and \( N_n^\mu \) have at least \( e^n/n \) closed geodesics up to length \( n \).
4. \( N_n \) and \( N_n^\mu \) are not commensurable.

This builds on previous work of Millichap.
The Length Spectrum of Hyperbolic 3-Manifolds

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One of the major open problems in the study of arithmetic hyperbolic 3-manifolds is the following.

Conjecture (Short Geodesic Conjecture)
There is a positive universal lower bound for the length of closed geodesics on an arithmetic hyperbolic 3-orbifold.

It is known that the Short Geodesic Conjecture would follow from Lehmer’s Conjecture on Mahler measures of polynomials.

This conjecture has long been known to be false in the context of non-arithmetic hyperbolic 3-orbifolds. In 2006 Agol showed that closed hyperbolic 4-manifolds may also have arbitrarily short closed geodesics.
Let $M$ be a closed hyperbolic 3-manifold.

The length spectrum of $M$ encodes isometric immersions of $S^1$ into $M$.

It turns out to be useful to consider the two-dimensional case; that is, totally geodesic immersions of orientable, finite type surfaces into $M$.

Let $GS(M)$ denote the set of isometry classes of finite area, properly immersed, totally geodesic surfaces of $M$, considered up to free homotopy.

$GS(M)$ is called the Geometric Genus Spectrum of $M$. 
The geometric genus spectrum was introduced by McReynolds and Reid.

**Theorem (McReynolds and Reid, 2009)**

If two arithmetic hyperbolic 3-manifolds have the same geometric genus spectra then they are commensurable.
Recently Jeff Meyer and I have proven that non-commensurable hyperbolic 3-manifolds may share arbitrarily large portions of their geometric genus spectra.

This is a two-dimensional analog of Futer and Millichap’s result.

Given $N \geq 1$, define $GS(M)[N]$ to be the first $N$ terms of $GS(M)$ (which we consider as being ordered by area).
The geometric genus spectrum of a hyperbolic 3-manifold

Theorem (L. and Meyer, 2016)

Let $N \geq 1$. There exists an infinite sequence of incommensurable arithmetic $M_1, M_2, \ldots$ such that:

1. $\text{GS}(M_i)[N] = \text{GS}(M_j)[N]$ for all $i, j$,
2. $\text{vol}(M_n) < c_1(n \log(2n))^{3/2}$, and
3. $\text{sys}_1(M_n) < c_2 \log(n)$.
Define $\text{Sys}^{TG}_2(M)$ to be the totally geodesic 2-systole of $M$. That is, the minimal area of an immersed totally geodesic surface.

In analogy with the Short Geodesic Conjecture, one may ask whether there is a universal lower bound for $\text{Sys}^{TG}_2(M)$ as $M$ varies over all arithmetic hyperbolic 3-orbifolds.

This turns out to be trivially true, as it has long been known that the $(2, 3, 7)$ triangle group has minimal co-area amongst all Fuchsian groups.
Theorem (L. and Meyer, 2016)

Let $M$ be an arithmetic hyperbolic 3-manifold which has minimal volume in its commensurability class and contains a finite area, properly immersed, totally geodesic surface. Then

$$\text{Sys}_{2}^{TG}(M) > c \cdot \text{vol}(M)^{1/2},$$

where $c$ is a positive constant.

Corollary

For every $X > 0$ there exist infinitely many arithmetic hyperbolic 3-manifolds $M$ such that $\text{Sys}_{2}^{TG}(M) > X$. 

Let $\text{sysg}(M)$ denote the **systolic genus** of $M$; that is, the minimal genus of a $\pi_1$-injective surface of $M$.

Denote by $N(V)$ the number of commensurability classes of arithmetic hyperbolic 3-manifolds which have a representative with volume less than $V$.

Denote by $N^g(V; x)$ the number of commensurability classes of arithmetic hyperbolic 3-manifolds which have a representative $M$ with volume less than $V$ and $\text{sysg}(M) < x$. 
Theorem (L. and Meyer)

For all sufficiently large \( x \) we have

\[
\lim_{{V \to \infty}} \frac{N^g(V; x)}{N(V)} = 0.
\]

Proof.

Our proof has two main ingredients. The first is a strengthening of Gromov’s high genus systole inequality and is due to Belolieptskey.

The second is a systole counting result which is joint work with Ben McReynolds, Paul Pollack and Lola Thompson.
The geometric genus spectrum of a hyperbolic 3-manifold

**Theorem (Belolipetsky)**

Let $M$ be a closed hyperbolic 3-manifold. For any $\epsilon > 0$, if $\text{sys}_1(M)$ is sufficiently large, then

$$\text{sys}_g(M) \geq e^{(1/2 - \epsilon)\text{sys}_1(M)}.$$  

Choose $x_0$ large enough so that Belolipetsky’s bound holds with $\epsilon = 1/4$ whenever $\text{sys}_1(M) > x_0$.

It is now straightforward to show that $N^g(V; x)$ is at most the number of commensurability classes of arithmetic hyperbolic 3-manifolds having a representative with volume less than $V$ and systole at most $\max\{x_0, 4 \log(x)\}$. 

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The result now follows from the following.

**Theorem (L., McReynolds, Pollack and Thompson, 2015)**

Let $F(V, X)$ denote the number of commensurability classes of arithmetic hyperbolic 3-manifolds with volume less than $V$ and systole less than $X$. Then

$$\lim_{V \to \infty} \frac{F(V, X)}{N(V)} = 0.$$
Thanks!