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Shuffle Up and Deal: Should We Have Jokers Wild?

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Abstract

In the neighborhood poker games, one often hears of adding the Jokers as wild cards, or declaring deuces wild. In this talk, we'll explore what happens mathematically when two Jokers are added to the deck. The probabilities for different hands change, in ways that might surprise you. Further, we will explore the mathematical consequences of playing poker with more or fewer than five cards. Finally, we will look at the historical beginnings of poker, and a mathematical anomaly that occurred along the way.

1 The Basics

Neighborhood and online poker games' popularity has skyrocketed recently. Many of these games involve five card hands, with or without wild cards. In this paper, we will explore what happens mathematically when two Jokers are added as wild cards. We will then look at some reasons that five card poker hands are the most popular. Finally, we will examine the evolution of poker to find a mathematical anomaly in the game's history. Before we get there, however, we need to begin with some basic poker definitions.

The most popular poker games involve five card hands with the following rankings:

Best:	Straight Flush	5 cards in a row in one suit
	Four of a Kind	4 of one denomination, 1 of another
	Full House	3 of one denomination, 2 of another
	Flush	All cards same suit, but not a Straight
	Straight	All cards in sequence, but not a Flush
	Three of a Kind	3 of one denomination, 2 of distinct denominations
	Two pair	2 of one denomination, 2 of another, 1 of third
	Pair	2 of one denomination, 3 of distinct denominations
Worst:	Junk	None of the above

These rankings correspond nicely to mathematics. A Straight Flush is the least likely hand to get, and Junk is the most likely. How do we know? We count! In a standard fifty-two card deck, there are four suits and thirteen denominations. If we shuffle and deal five cards, there are $52C5 = 2,598,960$ possible results.

Before we add Jokers to the picture, let's practice with a few easy counting arguments.

To get a Straight Flush, we must first choose a starting denomination as the highest in the Straight. This can be a five, six, up to an Ace. Then, we choose a suit. Thus, there are $10C1 * 4C1 = 40$ ways to get a Straight Flush. Once the suit and starting card are determined, there is only one way to fill in the rest.

To get a Straight, we first choose a starting denomination (from 5 to Ace). Then, we choose the suit of each denomination separately ($4C1$ five times). Finally, we must make sure this is not Straight Flush, so we subtract those. The result is $10C1 * (4C1)^5 - 40 = 10,200$ Straights.

To get a Three of a Kind, we first choose the denomination to be tripled ($13C1$). Then, we choose three suits for that denomination ($4C3$). Next, we must fill in the rest of the cards to be neither the chosen denomination, nor a pair. Pick two denominations from the remaining twelve ($12C2$), and then the suits for those cards ($4C1^2$). This gives us $13C1*4C3*12C2*4C1*4C1 = 54,912$ Three of a Kinds.

As the probabilities go down, the rank of the hand goes up. Sometimes, in the neighborhood versions of poker, we add wild cards. These are cards that can take on any desired denomination or suit. In this talk, I want to specifically address adding the two Jokers as wild cards (as opposed to "deuces wild" or using any other of the fifty-two cards as wild cards). The Jokers are distinguishable, as one is red and the other is black. In the section that follows, the mathematics is a slight alteration of that found in Gadbois' paper [1].

2 Adding the Jokers

It makes sense that we want to win. So, if we get a hand with a Joker in it, we'll want to use that Joker to our best advantage. For example, a hand with three Kings, a nine, and a Joker can be made into a Full House by making the Joker a nine, or a Four of a Kind by declaring the Joker a King. Since Four of a Kind ranks higher, we would make the Joker a King. Notice that we have a new hand, now. With two Jokers added to the deck, it is now possible to get a five of a kind.

All the probabilities are going to change, because we've added two cards to the deck, so the total number of hands is now $54C5 = 3,162,510$. Also, by adding a wild card, certain hands such as Three of a Kind become easier to get.

For example, to get Two Pair we look at three disjoint cases:

a) No Joker: $13C2 * 4C2 * 4C2 * 11C1 * 4C1 = 123,552$

b) One Joker: This never happens. In order to get Two Pair here, we'd have to have at least one natural pair (and pair the Joker with another card). But, with this ranking, we'd always pair the Joker with the natural pair to make Three of a Kind.

c) Two Jokers: Again, this never happens.

Let's do a more complicated example. The number of ways to get a Three of a Kind has changed. We can,

a) Have no Jokers, and a regular Three of a Kind: choose no Jokers ($2C0$), choose denomination for triple ($13C1$), Choose three of that denomination ($4C3$), then from the remaining twelve denominations, choose two (so there is no pair: $12C2$) and then choose one suit from each ($4C1*4C1$). This gives $13C1 * 4C3 * 12C2 * 4C1 * 4C1 = 54,912$ possible hands with no Jokers.

b) Have one Joker and a regular pair: Choose one of the Jokers ($2C1$), choose denomination ($13C1$), choose two from that denomination ($4C2$), then from the remaining twelve denominations, choose two (so there is no pair: $12C2$) and then choose one suit from each ($4C1 * 4C1$). This adds $2C1 * 13C1 * 4C2 * 12C2 * 4C1 * 4C1 = 164,736$ more hands.

c) Have two Jokers and no regular pairs: Note here, we must also avoid a Straight, which would beat a Three of a Kind. We must also choose at least one card of a different suit so we don't have a Flush. Our procedure is to choose two Jokers ($2C2$). Next, choose three unique denominations, ($13C3$). Now, we have to subtract those that would give higher hands, namely a Straight or a Flush. First, we count those hands that would give a Straight. If the highest non-Joker card is a five through Ace, of the four denominations below that card, we'd need two of those denominations as non-Jokers, for a total of $10 * 4C2$. If the highest non-Joker card is a four, of the three spots below the four, we need two of those denominations as non-Jokers, for $3C2$ more. If the highest non-Joker card is a 3, there is only one way to get a Straight here.

Hence, to assign the denominations for Three of a Kind, we have $2C2 * [13C3 - 10 * 4C2 - 3C2 - 1]$

Next, we need to assign the suits to avoid a Flush. There are $4^3 - 4$ ways to do this.

Thus, the total number of two-Joker Three of a Kind hands is $2C2 * [13C3 - 10 * 4C2 - 3C2 - 1][4^3 - 4] = 13,320$.

These three methods give us disjoint solutions, for a total of $54,912 + 164,736 + 13,320 = 232,968$ Three of a Kinds.

Without going through the remaining calculations, the table of frequencies and probabilities is as follows.

Type	Frequency	Probability
Five of a Kind	78	0.000025
Straight Flush	624	0.000197
Four of a Kind	9360	0.002960
Full House	9360	0.002960
Flush	11,388	0.003601
Straight	34,704	0.010974
Three of a Kind	232,968	0.073666
Two Pair	123,552	0.039068
One Pair	1,437,936	0.454682
Junk	1,302,540	0.411869

Note that there are several problems with this table. The traditional hierarchy no longer works, since it is more common to get Three of a Kind than Two Pair. Also, it is more common to get One Pair than Junk.

We might try to fix this by making Two Pair a higher hand than Three of a Kind, and Junk higher than One Pair when we add the Jokers to the deck. That is, we'll re-write the rules of the game. But this gives us another problem. Certain hands that would have been Three of a Kind under the original hierarchy are now Two Pair! For example, a hand with a Joker, two Jacks, a ten and a three would now become Two Pair by declaring the Joker to be a ten.

Thus, the frequencies of the hands will change. For example, the number of ways to get Three of a Kind are now:

a) no Jokers – 54,912

b) one Joker – 0 (if we have a pair and one Joker, now we'd make the hand a two-pair hand)

c) two Jokers – again, this won't happen as we'll be able to make it a higher hand.

On the other hand, the number of ways to get Two Pairs is now:

a) no Jokers – 123,552

b) one Joker – $2C1 * 13C1 * 4C2 * 12C2 * 4C1 * 4C1 = 164,736$ (note that this follows the same logic as a one-Joker Three of a Kind above)

c) two Jokers – $2C2 * [13C3 - 10 * 4C2 - 3C2 - 1][4^3 - 4] = 13,320$ (note that this also follows the same logic as a two-Joker Three of a Kind above).

So the frequencies and probabilities of these two hands under the new hierarchy would as displayed in the table below.

Type	Frequency	Probability
Five of a Kind	78	0.000025
Straight Flush	624	0.000197
Four of a Kind	9360	0.002960
Full House	9360	0.002960
Flush	11,388	0.003601
Straight	34,704	0.010974
Two Pair	301,608	0.095370
Three of a Kind	54,912	0.017363
Junk	1,642,236	0.519282
One Pair	1,098,240	0.347268

Notice that we still have the same problems. Under the new hierarchy, Two Pair is more likely than Three of a Kind. Further, Junk is more likely than One Pair for similar reasons. There are more problems with the original hierarchy - note that Four of a Kind and Full House are equally likely. It turns out that switching the way these are ranked results in a worse situation than before.

These calculations lead us to conclude that the mathematically wise should not play five card poker with wild cards. But, the mathematically curious might ask, what about other-sized hands? If we have lots of players, does it make mathematical sense to play with four card hands? Six card hands offer even more possibilities, and more challenges. Does this style of play fit the mathematics? The next section uses calculations first published by Y.L. Cheung [2].

3 Four and Six Card Hands

To deal with the probabilities for four and six card hands, we first must agree on the rules for these hands. A Straight and a Flush must use all the cards in the hand. In four card hands, we no longer have a Full House. The other standard poker hands are all possible, but the probabilities change from the five card hand probabilities.

Specifically, let us examine the probabilities for Two Pair, Flush, and Straight hands.

The number of ways to get a Two Pair hand is $13C2 * 4C2 * 4C2 = 2808$.

The number of ways to get a Flush is $4C1 * 13C4 - 44 = 2816$, calculated by finding all hands of the same suit and subtracting the number of Straight Flushes.

Similarly, the number of ways to get a Straight is $11 * (4C1)^4 - 44 = 2772$.

Dividing these hands by $52C4$ gives us the probabilities below.

Type	Probability
Two Pair	0.01035
Flush	0.01042
Straight	0.01025

Notice how very close these probabilities are. There isn't a practical difference in the rankings of these hands, and this is a big problem for poker players. Four card poker isn't mathematically sound because of this problem.

In six card poker, we get three more possible hands. In addition to the standard hands, we now have Three Pair, two sets of Three of a Kind, and a Four of a Kind plus a Pair. What are the rankings of the six card hands?

Six Card Hand	Probability
Straight Flush	0.0000018
One Pair plus Four of a Kind	0.0000460
Two sets of Three of a Kind	0.0000613
Flush	0.0003354
Four of a Kind	0.0006743
Straight	0.0018090
Three Pair	0.0030344
One Pair plus Three of a Kind	0.0080917
Three of a Kind	0.0359633
Two Pair	0.1213762
Junk	0.3431018
One Pair	0.4855048

As it turns out, there is a good ranking system. It just isn't consistent with the five card ranking system. Note where the new hands fall. Also, note that it is easier in six card poker to get One Pair than to get a Junk hand. So, One Pair is the worst hand. This may seem counterintuitive, and for some that is enough reason not to play six card poker hands.

4 An Historical Anomaly

One of my students wanted to know if the game of poker was invented to fit these probabilities so nicely, or is it just a coincidence that the rules and the probabilities line up? In searching for an answer to her question, I discovered a mathematical anomaly in the development of modern poker.

The origins of poker are generally accepted to have come from 15th century Germany and 16th century France (predating the probability theorists Laplace, Pascal, and the Bernoullis). However, the modern version of poker developed in America throughout the 1800s. In the early 1800s, New Orleans made popular a game using twenty cards, with four players each getting five cards. In this early version, there were no Straights or Flushes - only "of a kinds" and "full" (the only hand to use all five cards). Under these rules there are two hands that are unbeatable. Four Aces with any other card is known to be unbeatable. Also, a player holding four Kings with one Ace knows this is the best hand of the four, as no other player can have four Aces.

As poker increased in popularity, the British deck of fifty-two cards was adopted. This allowed more players to play at once, and also provided enough cards for the "draw" as it was introduced in the mid 1800s. The Flush was adopted about this time, and put in its rightful place in the hierarchy of hands.

The wild card, so critical to our explorations, was introduced after the Civil War. About the same time, Straights were introduced as a poker hand, and it is here the anomaly occurs. As we have seen, some hands are both a Straight and a Flush. These hands are particularly rare, but when they were first introduced, they were not recognized as the best hands. The players didn't want to give up the unbeatable hands described above. The rules kept the Four of a Kind above a Straight Flush for about twenty years, so for a while, the rules of poker were not compatible with mathematics. It took much of arguing before the Straight Flush was recognized as a separate, superior hand. One argument appeals to the immorality of having an unbeatable hand in poker. This person (John Keller) thought it was "ungentlemanly" to bet on a sure thing. Better to rank the Straight Flush highest, as there is always the outside chance of a tie hand. [3]

So, the long answer to my student's question is that poker seems to have evolved to the mathematically correct order of hands, rather than to have been developed according to the mathematical wisdom of the day.

5 Questions

Many questions arise from today's topics. One question I came across is historical in basis. The word "jackpot" has its origins in poker. With a Jack Pot, one must have Jacks or better to open the betting, which takes some of the bluffing out of the game. The author of this history article [3] contends that, at a table of five players, it is normally true that at least one player is dealt a hand consisting of a pair of Jacks or better. It turns out to be a very complicated computation, as the dealt hands are not independent. (Assuming independence, we get about a 70% probability of Jacks or better. Empirical studies indicate a higher probability. However, even trying to prove that 70% is a lower bound is not trivial.)

Other questions: what happens if you make part of the deck into wild cards (deuces wild? One-eyed Jacks wild)?

In games of this sort, variations abound. What happens to a Joker introduced into Texas Hold em (or other games with communal cards)?

You can play and play - and that's what mathematicians do!

References

[1] S. Gadbois, *Poker With Wild Cards – A Paradox?*, Mathematics Magazine, 69(4) (1996), 283-285.

[2] Y.L. Cheung, *Why Poker is Played with Five Cards*, The Mathematical Gazette, 73(1989), 313-315.

[3] D. Parlett, *The Oxford Guide to Card Games*, Oxford University Press, Oxford, 1990.