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A GENERALIZATION OF THE POINCARÉ-CARTAN INTEGRAL INVARIANT FOR A NONLINEAR NONHOLONOMIC DYNAMICAL SYSTEM

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Abstract. Based on the d'Alembert-Lagrange-Poincaré variational principle, we formulate general equations of motion for mechanical systems subject to nonlinear nonholonomic constraints, that do not involve Lagrangian undetermined multipliers. We write these equations in a canonical form called the Poincaré-Hamilton equations, and study a version of corresponding Poincaré-Cartan integral invariant which are derived by means of a type of asynchronous variation of the Poincaré variables of the problem that involve the variation of the time. As a consequence, it is shown that the invariance of a certain line integral under the motion of a mechanical system of the type considered characterizes the Poincaré-Hamilton equations as underlying equations of motion. As a special case, an invariant analogous to Poincaré linear integral invariant is obtained.

Keywords. Poincaré-Cartan integral invariant, nonlinear constraints, nonholonomic, asynchronous variation, equations of motion, Poincaré-Hamiltonian Systems.

AMS (MOS) subject classification: 70F25, 70I125, 37J99, 53A55.

1 Introduction

The theory of integral invariants (relative or absolute) introduced by Poincaré in his pioneering work [20] and extended by Cartan [6] has found many applications in celestial mechanics and various other branches of science including analytical mechanics [4, 5, 8, 9, 11, 14, 15, 19, 23, 31, 33], mathematical physics [10], contact systems [24], fluid mechanics [27], optics and lasers [22] etc. In particular, the relative integral invariants are important for oscillatory systems. An essential tool for the study of the time evolution of the phase of a Lagrangian manifold under the action of Hamiltonian flows is the Poincaré-Cartan form ($pdx - Hdt$): which is a relative integral invariant.

Cartan demonstrated that it is possible to characterize a Hamiltonian system by means of the relative integral invariants. Poincaré-Cartan integral invariants also provide tools that can be used with surface-of-section maps [26].

In this work authors have used the analytical mechanics setting rather than the differential geometric one. Let us consider here a dynamical system subject to nonlinear nonholonomic constraint. An example of nonholonomic constraint is a particle moving subject to a constraint expressed by a relation which is nonlinear in the velocity component [16, 25]:

$$\dot{z}^2 = \frac{a^2}{b^2}(\dot{x}^2 + \dot{y}^2).$$

In [18], the reader will find analytical and physical reasons for studying nonlinear nonholonomic constraints. It is known [19] that the motion of many mechanical systems is restricted by holonomic constraints so that every position in space is not accessible and in general, the variation of the velocity and velocity of variation are not identical. As a consequence, there are two viewpoints concerning the commutativity of the operation d (actual variation) and δ (virtual variation) of kinematical quantities. The first viewpoint is due to Hölder, Volterra and Hamel [16] which presumes that $d - \delta$ operation's commutativity is universally applicable for all velocities whereas the second viewpoint, due to Amaldi, Appell, Levi-Civita and Suslov [12, 16], assumes they commute only for independent generalized velocities.

In [3], the author has studied the Poincaré-Cartan integral invariant for linear nonholonomic systems in terms of Lagrangian coordinates and in [2] author has employed the Poincaré formalism based on the Lie theory of continuous groups of transformations to generalize these results for holonomic systems. The aim of the present investigation is twofold: (i) to extend the Poincaré formalism to include the generalized variation involving the variation of the time and (ii) to examine the theory, without appealing to the method of Lagrange undetermined multipliers, from the second viewpoint [13] for nonlinear nonholonomic dynamical systems. To achieve this we present a precise definition of the virtual displacement in terms of the newly introduced parameters [2] corresponding to the asynchronous variation and obtain a generalization of the Poincaré-Cartan integral invariant for such systems.

2 Generalized Variation

Consider the motion of a holonomic dynamical system whose configuration at any time t along the actual (real) trajectory is determined by the set of functions

$$x_p(t), \quad p = 1, 2, \dots, n, \quad (2.1)$$

of the time t of at least class C^2 and define the actual displacement in time ' dt ' by the relation

$$dx_p = \dot{x}_p dt, \quad p = 1, 2, \dots, n. \quad (2.2)$$

Any one of configurations infinitely close to the actual trajectory and compatible with the constraints along a neighboring curve is defined by

$$x_p^*(t) = x_p(t) + \delta x_p. \quad (2.3)$$

The quantities δx_p , arbitrary functions of the time t of class C^1 , are the variations of the variables $x_p(t)$ that are obtained by keeping the time t fixed. The difference

$$x_p^*(t) - x_p(t) = \delta x_p, \quad (2.4)$$

is called the synchronous (or simultaneous) virtual variation of the variable $x_p(t)$ and is designated as δ -variation. Readers are referred to [29], ([30], pp. 132), [1] and [32] for these variation processes.

For a more general variation process, we define the position of the system in the actual (real) motion by the variables $x_p(t)$ and determine in a varied motion an infinitely closed position, by the function $x_p^*(t + \Delta t)$, where Δt is an infinitesimal change in time and is a differentiable function of time t .

Taking into account only small quantities of first order, we get

$$x_p^*(t + \Delta t) = x_p^*(t) + \dot{x}_p^* \Delta t = x_p(t) + \delta x_p + \dot{x}_p \Delta t, \quad (2.5)$$

where the dot over the letter denotes differentiation with respect to time t . Introducing the symbol Δx_p , we write

$$x_p^*(t + \Delta t) - x_p(t) = \Delta x_p = \delta x_p + \dot{x}_p \Delta t, \quad (2.6)$$

which serves to define the variation Δx_p of the variable x_p where the time varies and the position along the varied path is not simultaneous to the actual path. These variations are called asynchronous (or non-simultaneous) virtual variations of the variable x_p and are indicated by the symbol Δ -variation. Further, the Δ -variation of an arbitrary function $F(x_p, t)$ of the class C^2 in the domain of the variables x_p and time t can be readily obtained as

$$\Delta F = \delta F + \dot{F} \Delta t. \quad (2.7)$$

By the relations (2.2) to (2.5), it can be easily shown that

$$d(\delta x_p)/dt = \delta \dot{x}_p, \quad (d(\Delta x_p)/dt = \Delta \dot{x}_p + \ddot{x}_p \Delta t), \quad (2.8)$$

which leads to the fact that the velocity of the variation and the variation of the velocity are the same (not the same) according to the process of δ -variation (Δ -variation), respectively. It is worth mentioning that the relation (2.7) plays a vital role when the investigations are carried out on the

basis of variational principles and their interpretation varies according to the dynamical system. It is to be noted that the relations (2.8), in fact, represent the commutative (non-commutative) behavior of the operation d/dt and δ (d/dt and Δ), respectively. We assume that the relation $d(\Delta t) = \Delta(dt)$ always holds for the independent variable t , whereas it can be shown that $\Delta\delta x_p = \delta\Delta x_p$ holds for the dependent variables x_p .

In order to extend the existing theory of integral invariants we transform the preceding analysis to the Poincaré formalism that is based on the theory of continuous groups of transformations [21].

Let the group variables x_p be holonomically connected to the Cartesian coordinate u 's by the relations

$$u_p = u_p(x_p, t), \quad p, q = 1, 2, \dots, n, \quad (2.9)$$

and the Poincaré parameters $\eta_p(\omega_p)$ of the actual (virtual) displacement by the relations [7]

$$\eta_p = A_{pq}\dot{x}_q + A_p, \quad (\omega_p = A_{pq}\delta x_q), \quad (2.10)$$

where A_{pq} , A_p are functions of the x 's and the time t and the matrix A_{pq} is non-singular. In this work the following conventions will be observed: (1) the summation convention is employed throughout the work; (2) the indices will have the range of the values $\lambda, \mu, \nu = 0, 1, 2, \dots, n$; $p, q, r, s = 1, 2, \dots, n$; $i, j, k, l = 1, 2, \dots, m < n$; $\alpha, \beta, \gamma, \delta = (m + 1), \dots, n$.

We introduce the following:

Definition 2.1. The change dG of an arbitrary function $G(x_p, t)$, during the actual displacement dx_p in the time dt of the system, is determined by the relations [2]

$$dG = [X_0G + \eta_p X_p G]dt, \quad p = 1, 2, \dots, n, \quad (2.11)$$

where X_0 and X_p , characterizing the infinitesimal displacement, are the operators defined by

$$X_0 = \frac{\partial}{\partial t} + \xi_0^q(x_p) \frac{\partial}{\partial x_q}, \quad X_p = \xi_p^q(x_r) \frac{\partial}{\partial x_q}, \quad (2.12)$$

which form a transitive group of operators if we require that the commutators (see, for example, [30]),

$$[X_0, X_p] = X_0 X_p - X_p X_0, \quad [X_p, X_q] = X_p X_q - X_q X_p,$$

satisfy the relations

$$[X_0, X_p] = C_{0p}^q X_q, \quad [X_p, X_q] = C_{pq}^r X_r, \quad p, q, r = 1, 2, \dots, n. \quad (2.13)$$

Here C_{0p}^q , C_{pq}^r , which depend on the x 's and the time t , are the structure constants corresponding to the operators X_0, X_p .

We now give the following:

Definition 2.2. In a simultaneous virtual displacement $\delta x_1, \delta x_2, \dots, \delta x_n$ of the system, the change δG in an arbitrary function $G(x_p, t)$ is determined by the formula [7]:

$$\delta G = \omega_p X_p G, \quad p = 1, 2, \dots, n. \quad (2.14)$$

Here the quantities ω_p are the parameters corresponding to the synchronous variation δG of the function and are called the Poincaré synchronous virtual displacement parameters or simply the virtual displacement parameters.

In order to include the asynchronous variation ΔG of the function $G(x_p, t)$ during an infinitesimal time Δt , we substitute from (2.11) and (2.14) into the formula (2.7) to obtain

$$\Delta G = (\Delta t) X_0 G + (\omega_p + \eta_p \Delta t) X_p G.$$

If the virtual displacements are characterized by the asynchronous variations Δx_p , then we need to extend definition 2.2. For this purpose, we use the notation $x_0 = t, \dot{\eta}_0 = 1$ and we write the expression (2.12) for the infinitesimal displacement operators in the compact form:

$$X_\mu = \xi_\mu^\nu(x_1, \dots, x_n) \frac{\partial}{\partial x_\nu}, \quad \xi_0^0 = 1, \quad \xi_p^0 = 0, \quad \mu, \nu = 0, 1, 2, \dots, n. \quad (2.15)$$

Further, we set

$$\omega_0 = 0, \quad \Delta x_0 = \Delta t = \Omega_0, \quad \dot{x}_0 = \eta_0 = 1, \quad (2.16)$$

and analogous to the Poincaré parameters ω_p [7], introduce the new parameters corresponding to asynchronous variations by the relation

$$\Omega_\mu = \omega_\mu + \eta_\mu \Omega_0, \quad (2.17)$$

which was first given in [1, 2].

We now present the following new definition:

Definition 2.3. The variation ΔG of an arbitrary function $G(x_p, t)$, during a virtual displacement $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ asynchronous to the time $\Delta t = \Omega_0$, is described by the formula

$$\Delta G = \Omega_\mu X_\mu G, \quad \mu = 0, 1, \dots, n, \quad (2.18)$$

where $\Omega_0, \Omega_1, \dots, \Omega_n$, assumed to be functions of class C^2 , are the parameters of the virtual displacement corresponding to the asynchronous variation. For future reference we shall call them the asynchronous virtual displacement parameters.

Moreover, in view of (2.16-17), the formula (2.11) can be put in a more compact form as

$$dG = [\eta_\mu X_\mu G] dt. \quad (2.19)$$

Since the system is holonomic, by virtue of relations (2.11-13) and the rule $\delta d = d\delta$, the synchronous variations $\delta\eta_p$ of the parameters of real displacement are given by [7]

$$\delta\eta_p = \frac{d\omega_p}{dt} + C_{0q}^p\omega_q + C_{qr}^p\eta_q\omega_r, \quad (2.20)$$

and, by means of the formula (2.7), it follows that

$$\Delta\eta_p = \delta\eta_p + \dot{\eta}_p\Omega_0. \quad (2.21)$$

Thus, using relations (2.17) and (2.20), it can readily be shown that

$$\Delta\eta_p = \dot{\Omega}_p - \eta_p\dot{\Omega}_0 + C_{qr}^p\eta_q\Omega_r + C_{0q}^p\Omega_q, \quad (2.22)$$

which represents the asynchronous variations $\Delta\eta_p$ of the parameters of real displacement in terms of the new parameters Ω 's of possible displacements.

It is known [1] that the process of δ -variation and integration commute with each other for holonomic dynamical systems, but from the preceding analysis it follows that the non-commutativity of the operations ' Δ ' and differentiation ' d ' implies the non-commutativity of the operation Δ and integration. That is,

$$\Delta \int_{t_1}^{t_2} () dt \neq \int_{t_1}^{t_2} \Delta() dt, \quad (2.23)$$

in general, even if the system is holonomic. Precisely, we state the following:

Lemma 2.4. *Let J be a functional defined by the integral*

$$J = \int_0^t f dt, \quad (2.24)$$

where f is an arbitrary function of η_p, x_p and possibly the time t . Then the asynchronous variation ΔJ of the functional J is given by

$$\Delta J = \int_0^t (\Delta f + f\dot{\Omega}_0) dt. \quad (2.25)$$

For the proof of the lemma, readers are referred to [2]. However, one can see that the non-commutativity of the Δ -operation and the integration is obvious due to the presence of the quantity $f\dot{\Omega}_0$ in the expression (2.25).

In a special case when the variation is synchronous, we have $\Delta t = \Omega_0 = 0$. This, together with (2.17) implies that $\Delta \equiv \delta$ and (2.25) includes the well-known result of the commutativity of δ -operation and integration for a holonomic dynamical system. But it is to be noted carefully that for nonholonomic system these results vary in a different way.

3 The two viewpoints

From the analysis discussed in the preceding section, it follows that the commutativity (or non-commutativity) of the operations d with δ (or Δ) is expressed by the transpositional relations (2.20) and (2.22) respectively, which play a crucial role in the dynamics of the nonholonomic systems. In fact, the study of integral invariants for such systems is intimately connected with these relations, since they involve certain variational procedure. In the sequel, we briefly discuss the viewpoints about the validity of these relations which allow us to obtain a generalization of the theory of integral invariants.

We begin by introducing the main ideas of Poincaré's formalism [7, 13]. Let us consider the dynamical system whose configuration is determined by the independent coordinates x_p and moves subject to the nonholonomic constraints expressed by the $(n - m)$ independent equations

$$f_\alpha(\eta_p, x_p, t) = 0, \quad p = 1, 2, \dots, n; \alpha = m + 1, \dots, n, \quad (3.1)$$

where the functions f_α are not necessarily linear in the η_p 's and that the virtual displacement parameters ω_p 's satisfy the Chetaev's relations [7]

$$\frac{\partial f_\alpha}{\partial \eta_p} \omega_p = 0. \quad (3.2)$$

We further assume that the constraint equations (3.1) are expressible in the form

$$f_\alpha(\eta_p, x_p, t) = \eta_\alpha - \phi_\alpha(\eta_i, x_p, t) = 0, \quad (3.3)$$

$$p = 1, 2, \dots, n; i = 1, 2, \dots, m < n; \alpha = m + 1, \dots, n,$$

and the ω 's satisfy the relations [12]

$$\omega_\alpha = \frac{\partial \phi_\alpha}{\partial \eta_i} \omega_i, \quad (3.4)$$

where we designate $\eta_\alpha(\eta_i)$ and $\omega_\alpha(\omega_i)$ as the dependent (independent) Poincaré parameters of real and virtual displacements respectively.

In view of (3.2) or (3.4), we observe that the presence of nonholonomic constraints (3.1) or (3.3) leads to the fact that the ω_p 's are not independent. This reveals that the parameters of virtual displacement are not determined uniquely. Thus, there exists some arbitrariness in the determination of the derivatives $\frac{d\omega_p}{dt}$. Consequently, in using relation (2.20), we can adopt either of the two viewpoints given in the following:

i) According to Hölder, Volterra and Hamel, (2.20) may be used for the δ -variation of all the parameters η_p of real displacement whether the system is holonomic or not. This, in view of the formula (2.14), allows us to express the variations δf_α of f_α for the constraints (3.1) and (3.3) in the form [12]:

$$\delta f_\alpha = \left[X_p f_\alpha + (C_{0p}^r + C_{qp}^r \eta_q) \frac{\partial f_\alpha}{\partial \eta_r} - \frac{d}{dt} \left(\frac{\partial f_\alpha}{\partial \eta_p} \right) \right] \omega_p, \quad (3.5)$$

and

$$\delta f_\alpha = \delta \eta_\alpha - \delta \phi_\alpha = A_i^\alpha \omega_i, \quad (3.6)$$

respectively, by using the conditions (3.2) and (3.4). Here the quantities A_i^α are determined by the relations

$$\begin{aligned} A_i^\alpha &= \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_i} \right) - X_i \phi_\alpha - \frac{\partial \phi_\beta}{\partial \eta_i} X_\beta \phi_\alpha + (C_{0i}^\alpha + C_{qi}^\alpha \eta_q) + (C_{0\beta}^\alpha + C_{q\beta}^\alpha \eta_q) \frac{\partial \phi_\beta}{\partial \eta_i} \\ &\quad - \frac{\partial \phi_\alpha}{\partial \eta_j} \{ (C_{0i}^j + C_{qi}^j \eta_q) + (C_{0\beta}^j + C_{q\beta}^j \eta_q) \frac{\partial \phi_\beta}{\partial \eta_i} \}, \\ &\quad q = 1, 2, \dots, n; i, j = 1, 2, \dots, m < n; \alpha, \beta = m + 1, \dots, n. \end{aligned} \quad (3.7)$$

ii) According to Amaldi, Levi-Civita and Suslov as discussed in [12], the relations (2.20) hold for nonholonomic systems. They may be used only for the variation $\delta \eta_i$ of the independent real parameters η_i together with the assumption that the δ -variation of the f_α vanishes. Precisely,

$$\delta \eta_i = \frac{d\omega_i}{dt} + C_{0q}^i \omega_q + C_{qr}^i \eta_q \omega_r, \quad \delta f_\alpha = 0, \quad (3.8)$$

for the constraints of the type (3.3).

The relations $\delta f_\alpha = 0$ enable us to obtain the synchronous virtual variations of the dependent η 's. To distinguish them from the variation $\delta f(\eta_p, x_p, t)$ of an arbitrary function of all the η 's, x 's and t , we employ the symbol δ^* to denote the variation of f when computed in terms of the independent Poincaré parameters by means of (3.3). Thus, from (2.14) and (3.4), it follows that

$$\delta^* f_\alpha = \delta^* \eta_\alpha - \delta^* \phi_\alpha = (A_j^\alpha)^* \omega_j. \quad (3.9)$$

Here $(A_j^\alpha)^*$ are given by

$$\begin{aligned} (A_j^\alpha)^* &= \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) - X_j^* \phi_\alpha + (K_{0j}^\alpha + K_{ij}^\alpha \eta_i + K_{\beta j}^\alpha \phi_\beta) \\ &\quad - (K_{0j}^k + K_{ij}^k \eta_i + K_{\beta j}^k \phi_\beta) \frac{\partial \phi_\alpha}{\partial \eta_k}, \end{aligned} \quad (3.10)$$

where

$$K_{0j}^k = C_{0j}^k + C_{0\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}, \quad K_{0j}^\alpha = C_{0j}^\alpha + C_{0\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}, \quad (3.11)$$

$$K_{qj}^k = C_{qj}^k + C_{q\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}, \quad K_{qj}^\alpha = C_{qj}^\alpha + C_{q\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}, \quad (3.12)$$

and

$$X_j^* = X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta. \quad (3.13)$$

4 Equations of Motion: parametric representation

We start with the general equation of dynamics expressing the d'Alembert-Lagrange-Poincaré principle given by [12]

$$\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \eta_p} \right) - C_{0p}^q \frac{\partial L}{\partial \eta_q} - C_{qp}^r \eta_q \frac{\partial L}{\partial \eta_r} - X_p L \right] \omega_p = 0, \quad (4.1)$$

where $L(\eta_p, x_p, t)$ is the Lagrangian function that describes the dynamical behavior and ω_p determines the virtual displacement of the system. We assume that the system moves subject to the nonlinear nonholonomic constraints (3.3) and adopt the second viewpoint. Since the ω 's are not independent and satisfy the conditions (3.4), the general equation of dynamics (4.1), as it stands, does not yield the equation of motion. Therefore, we write (4.1) in the form

$$\left[\frac{d}{dt} \frac{\partial L}{\partial \eta_j} + \frac{\partial \phi_\beta}{\partial \eta_j} \frac{d}{dt} \frac{\partial L}{\partial \eta_\beta} - (C_{0j}^q + C_{0\beta}^q \frac{\partial \phi_\beta}{\partial \eta_j}) - (C_{qj}^r + C_{q\beta}^r \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q - (X_j L + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta L) \right] \omega_j = 0, \quad (4.2)$$

$$q = 1, 2, \dots, n; i, j = 1, 2, \dots, m < n; \alpha, \beta = m + 1, \dots, n.$$

We have used the relations (3.4) after separating the sum over p from 1 to n into the sums over j from 1 to m and over β from $(m + 1)$ to n .

After expressing the η_α 's in terms of η_j 's we call the resulting function $L^*(\eta_j, x_p, t)$. Thus,

$$L^*(\eta_i, x_p, t) = L(\eta_j, \phi_\alpha(x_p, \eta_j, t), x_p, t), \quad (4.3)$$

as a result of operating by X_p and differentiating partially with respect to η_j , we have

$$X_p L^* = X_p L + \frac{\partial L}{\partial \eta_\alpha} X_p \phi_\alpha, \quad \frac{\partial L}{\partial \eta_j} = \frac{\partial L^*}{\partial \eta_j} - \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j}. \quad (4.4)$$

Substituting from (4.3-4) into (4.2) and performing some mathematical manipulations the general equation of dynamics takes the form

$$\left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) - \left\{ (C_{0j}^k + C_{0\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) + (C_{qj}^k + C_{q\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q \right\} \frac{\partial L^*}{\partial \eta_k} - (X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta) L^* - \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \left\{ \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) - (X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta) \phi_\alpha + (C_{0j}^\alpha + C_{0\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}) + (C_{qj}^\alpha + C_{q\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q - \frac{\partial \phi_\alpha}{\partial \eta_k} \left((C_{0j}^k + C_{0\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) + (C_{qj}^k + C_{q\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q \right) \right\} \right] \omega_j = 0. \quad (4.5)$$

From the independence of the ω_j 's and (3.10-12), the last result leads to

$$\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - X_j^* L^* - (A_j^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* = 0, \quad (4.6)$$

which are the required equations of motion independent of the Lagrange undetermined multipliers for the nonlinear nonholonomic dynamical system. These m equations determine the values of η_j ($j = 1, 2, \dots, m$) which, by virtue of the constraint equations (3.3), allow us to determine the values of the remaining $(n - m)$ η_α 's as functions of the x_p 's and t . The values of x_p are then calculated from the equations

$$\dot{x}_p = X_0 x_p + \eta_q X_q x_p, \quad (4.7)$$

which are obtained from (2.11) by setting $G = x_p$.

In order to obtain the canonical form of equations (4.6), we introduce the generalized momenta y_p by the relations

$$y_p = \frac{\partial L}{\partial \eta_p}, \quad (4.8)$$

and we assume the transformations (4.8) to be invertible so that the η_p 's can be expressed in the form

$$\eta_p = \eta_p(x_q, y_q, t), \quad p, q = 1, 2, \dots, n. \quad (4.9)$$

We note that (4.8) and (4.9) yield, respectively, y 's and η 's as a linear function of η 's and y 's.

Since all the η 's are not independent, it follows that y 's are not independent as well. Therefore, we cannot work in the phase-space of $2n$ variables defined in [2], that is,

Definition 4.1. A set of $2n$ independent quantities (x_p, y_p) is said to form the phase space if the conditions

$$\delta t = 0, \quad \omega_p \neq 0, \quad \delta \eta_p \neq 0, \quad \delta y_p \neq 0, \quad p = 1, 2, \dots, n,$$

hold throughout the motion of the system.

To overcome this problem we introduce the new independent variables y_j^* that are defined by means of equations of transformation

$$y_j^* = \frac{\partial L^*}{\partial \eta_j}, \quad (4.10)$$

which are assumed to be invertible and allow to express the η_j 's as functions of y_j^* , x_p and time t in the form

$$\eta_j = \eta_j(x_p, y_j^*, t), \quad j = 1, 2, \dots, m < n; p = 1, 2, \dots, n. \quad (4.11)$$

Note that the constraint equations (3.3) are expressed in terms of the independent y_j^* 's by means of the equations (4.11).

We now give the following:

Definition 4.2. The $2m$ quantities $(x_1, \dots, x_m; y_1^*, \dots, y_m^*)$ are said to form a $2m$ -dimensional reduced phase space provided that they are independent and satisfy the conditions

$$\delta t = 0, \quad \omega_j \neq 0, \quad \delta \eta_j \neq 0, \quad \delta y_j^* \neq 0, \quad (4.12)$$

throughout the motion of the dynamical system under the nonlinear non-holonomic constraints (3.3) with (3.4).

Let us consider the motion of the system in this $2m$ -dimensional reduced phase space and introduce the Hamiltonian function $H^*(x_p, y_j^*, t)$ corresponding to Lagrangian L^*

$$H^*(x_p, y_j^*, t) = \eta_j y_j^* - L^*(x_p, \eta_j, t). \quad (4.13)$$

Performing the δ -variation according to (2.14), in conjunction with (4.10), we have

$$\delta H^*(x_p, y_j^*, t) = \delta[\eta_j y_j^* - L^*(x_p, \eta_j, t)]. \quad (4.14)$$

On one hand, using the second viewpoint and (4.10), we have

$$\delta^* H^*(x_p, y_j^*, t) = \eta_j \delta^* y_j^* - \omega_p X_p L^*.$$

Breaking the sum over the index p from 1 to n into the sums over the indices j from 1 to m and α from $(m+1)$ to n and using conditions (3.4) and (3.13), the last result assumes the form

$$\delta^* H^*(x_p, y_j^*, t) = \eta_j \delta^* y_j^* - \omega_j X_j^* L^*,$$

which, in view of equation (4.6), becomes

$$\delta[\eta_j y_j^* - L^*(x_p, \eta_j, t)] = \eta_j \delta y_j^* - \omega_j [\dot{y}_j^* - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - (A_j^\alpha)^* (\frac{\partial L}{\partial \eta_\alpha})^*]. \quad (4.15)$$

On the other hand, we can find

$$\delta H^*(x_p, y_j^*, t) = \frac{\partial H^*}{\partial y_j^*} \delta y_j^* + \omega_p X_p H^*,$$

which is equivalent to

$$\delta H^*(x_p, y_j^*, t) = \frac{\partial H^*}{\partial y_j^*} \delta y_j^* + \omega_j X_p^* H^*, \quad (4.16)$$

where we have used (3.4) and (3.12).

From (4.15) and (4.16), it follows that

$$(\frac{\partial H^*}{\partial y_j^*} - \eta_j) \delta y_j^* + [X_j^* H^* + \dot{y}_j^* - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - (A_j^\alpha)^* (\frac{\partial L}{\partial \eta_\alpha})^*] \omega_j = 0.$$

Since all the δy_j^* and ω_j are independent, the coefficient of each of the δy_j^* and ω_j vanishes and we obtain

$$\eta_j = \frac{\partial H^*}{\partial y_j^*}, \quad \dot{y}_j^* = -X_j^* H^* + (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - (A_j^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^*, \quad (4.17)$$

$$j, k = 1, 2, \dots, m; \alpha = m + 1, \dots, n; q = 1, 2, \dots, n.$$

The equations (4.17) together with (4.7) are the required Poincaré-Hamilton (PH) equations of motion for the nonlinear nonholonomic dynamical system. They together with the constraint equations (3.3) determine the $(n + m)$ quantities $x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_m$. In fact, we can find η_j ($j = 1, 2, \dots, m$) as functions of x_p, y_j and t and then substituting the values of η_j into the equations (3.3), determine the remaining η_α ($\alpha = m + 1, \dots, n$). In this way all the η_j 's are determined and then, by using (4.7), all the x_p 's can be obtained as functions of the time t .

5 Poincaré-Cartan Integral Invariant of Nonholonomic Dynamical System

We now turn to the investigation of the Poincaré-Cartan integral invariants for the conservative nonlinear nonholonomic system whose motion is determined by the Poincaré-Hamilton (PH) equations (4.17) with (4.7). In order to achieve our goal we need to compute, subject to the constraints (3.3) with (3.4), the asynchronous variation of the action integral defined by

$$S = \int_{t_1}^{t_2} L dt, \quad (5.1)$$

where $L dt$ expresses the small element of action with L , describing the dynamical behavior of the system, as the Lagrangian function of all the x 's, η 's and possibly the time t .

Let us now consider the motion of the system along the real (actual) trajectory between the two positions P_1 and P_2 corresponding to the initial and final instants t_1 and t_2 , respectively. We assume that the varied path is determined by means of Δ -variation in which not only the coordinates but also the time varies at the initial and final moments of the motion. Since the Lagrangian is a function of class C^2 with respect to all of its arguments, using Lemma 2.1, we perform the Δ -variation of (4.1) according to (2.25) to get

$$\Delta S = \int_{t_1}^{t_2} (\Delta L + L \dot{\Omega}_0) dt,$$

which, by virtue of (A.4) from the APPENDIX, assumes the form

$$\begin{aligned} \Delta S &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \dot{\eta}_p \Omega_0 + \eta_\mu \Omega_0 X_\mu L - \Omega_0 \dot{L} \right) dt \\ &+ \int_{t_1}^{t_2} (L \Omega_0) \cdot dt - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - X_j^* L^* \right. \\ &\left. - (A_j^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \right] \omega_j dt, \end{aligned} \quad (5.2)$$

where the superscript “*” shows that the quantities are expressed in terms of the independent Poincaré parameters of real displacement.

Since the Poincaré equations (4.6) hold along the real (actual) trajectory, using (2.11), (2.16) and (4.3), the last integral on the right hand side of (5.2) vanishes, and we are left with

$$\begin{aligned} \Delta S &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j + L^* \Omega_0 \right) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \dot{\eta}_p \Omega_0 + \eta_\mu \Omega_0 X_\mu L - \frac{\partial L}{\partial \eta_p} \dot{\eta}_p \Omega_0 \right. \\ &\left. - \eta_\mu \Omega_0 X_\mu L \right) dt. \end{aligned}$$

And, in view of (4.10), simplifies to

$$\Delta S = (y_j^* \omega_j + L^* \Omega_0) \Big|_{t_1}^{t_2}. \quad (5.3)$$

Using the relations (2.17) and (4.13), the last result becomes

$$\Delta S = (y_j^* \Omega_j + H^* \Omega_0) \Big|_{t_1}^{t_2}, \quad (5.4)$$

where the Hamiltonian function H^* is computed along the actual trajectory of the system. Denoting it by $-y_0^*$, the asynchronous variation ΔS of S can be put in a more concise form

$$\Delta S = y_\lambda^* \Omega_\lambda \Big|_{t_1}^{t_2}, \quad \lambda = 0, 1, 2, \dots, m. \quad (5.5)$$

If we take $G = x_\lambda$ in formula (2.18), we find that

$$\Delta x_\lambda = \Omega_\mu X_\mu x_\lambda, \quad \lambda, \mu = 0, 1, 2, \dots, m, \quad (5.6)$$

and, by using (2.12) it can readily be shown that Ω_μ are linear combinations of the quantities Δx_λ in the form

$$\Omega_\mu = \zeta_\lambda^\mu \Delta x_\lambda, \quad (5.7)$$

where $\|\zeta_\mu^\lambda(x_\nu)\|$ is the inverse of the matrix $\|\xi_\lambda^\mu(x_\nu)\|$. Then the variational equation (5.5) becomes

$$\Delta S = (y_\lambda^* \xi_\mu^\lambda \Delta x_\mu) \Big|_{t_1}^{t_2} = (y_\mu^* \Delta x_\mu) \Big|_{t_1}^{t_2}. \quad (5.8)$$

Equation (5.8) yields in concise form the variation of the functional S in the space of $(2m+1)$ variables $(x_1, x_2, \dots, x_m; y_1^*, y_2^*, \dots, y_m^*; t)$ called the ‘‘Extended Reduced Phase-Space’’ and may be applied to any functional of this type.

Let us now consider the motion of the system in this space. We note the fact that corresponding to the different initial values of the x ’s and y ’s we obtain a set of initial points at the initial time t_1 and through each point we can draw the appropriate real paths satisfying (4.17) and giving rise to a set of terminal points at the other end of each real path at the same terminal time t_2 . If the initial set of points forms a closed curve C_1 , correspondingly, the terminal set of points will also form a closed curve C_2 giving a tube of real trajectories.

Thus integration of (5.5) along a closed curve C (that is, the locus at any time t of the points which are initially on C_1) that passes from C_1 to C_2 , yields

$$0 = \oint_C \Delta S = \oint_C (y_\lambda^* \Omega_\lambda |_{t_1}^{t_2}) = \oint_C (y_\lambda^{*(2)} \Omega_\lambda^{(2)} - y_\lambda^{*(1)} \Omega_\lambda^{(1)}),$$

where $y_\lambda^{*(1)}, y_\lambda^{*(2)}, \Omega_\lambda^{(1)}, \Omega_\lambda^{(2)}$ are the values of the y^* ’s and the Ω ’s at the time t_1 and t_2 respectively. In view of the continuity of the motion, the last result may be written as

$$\oint_{C_1} y_\lambda^* \Omega_\lambda = \oint_{C_2} y_\lambda^* \Omega_\lambda,$$

which leads to

$$I = \oint_C (y_j^* \Omega_j - H^* \Omega_0),$$

taken along a closed contour C remain invariant during an arbitrary displacement with deformation of the system. Thus, we have the following:

Theorem 5.1. *The line integral*

$$I = \oint_C (y_j^* \Omega_j - H^* \Omega_0), \quad (5.9)$$

along an arbitrary closed curve C remains invariant with arbitrary deformation of this curve along the tube of real trajectories of a conservative nonlinear nonholonomic dynamical system whose motion is governed by the equations (4.17) provided that the relations (3.8) hold.

We remark that the integral (5.9) is a generalized form of the relative integral invariant of Poincaré-Cartan given in [3]. It is shown [2, 3] that these integrals are important in the study of analytical dynamics of holonomic system, since their invariance yields their relationship with the Poincaré-Hamiltonian system. In what follows, we shall show that this relationship can also be established for nonlinear nonholonomic dynamical systems as well.

6 Poincaré-Hamiltonian Systems and the Poincaré-Cartan Integral Invariant.

In the preceding section we have established the integral invariant of a nonlinear nonholonomic dynamical system. Now we shall discuss the consequences of the invariant property of the integral (5.9). Precisely, we prove the converse of Theorem 5.1 by making use of the property of asynchronous variation as discussed in Section 2.

We start with the assumption that the generalized Poincaré-Cartan integral (5.9) is invariant with respect to the tube of real trajectories of the system whose motion is governed by the set of equations of the form

$$\eta_j = \psi_j(x_p, y_k^*, t), \quad (6.1)$$

$$\dot{y}_j^* = \phi_j(x_p, y_k^*, t), \quad j, k = 1, 2, \dots, m; p = 1, 2, \dots, n, \quad (6.2)$$

where ψ_j and ϕ_j are arbitrary functions to be determined.

The invariance of the integral (5.9) along the tube of real trajectories, in accordance with (2.19), (6.1) and (6.2) implies that

$$dI = 0.$$

Thus, we have

$$0 = d \oint (y_j^* \Omega_j - H^* \Omega_0) = \oint_C (dy_j^* \Omega_j + y_j^* d\Omega_j - dH^* \Omega_0 - H^* d\Omega_0). \quad (6.3)$$

Recalling that relation $d(\Delta t) = \Delta(dt)$ it follows that

$$H^* d\Omega_0 = H^* d(\Delta t) = H^* \Delta(dt) = \Delta(H^* dt) - \Delta H^* dt.$$

Combining this with (2.22), and the identity

$$y_j^* \eta_j d\Omega_0 = y_j^* \eta_j d(\Delta t) = y_j^* \eta_j \Delta(dt) = \Delta(y_j^* \eta_j dt) - \Delta(y_j^* \eta_j) dt,$$

we obtain

$$0 = d \oint \left[dy_j^* \Omega_j + y_j^* \eta_j d\Omega_0 + \left(y_j^* \Delta \eta_j - y_j^* C_{qr}^j \eta_q \Omega_r - y_j^* C_{0q}^j \Omega_q \right) dt - dH^* \Omega_0 - \Delta(H^* dt) + \Delta H^* dt \right],$$

which takes the form

$$0 = d \oint \Delta \left[(y_j^* \eta_j - H^*) dt \right] + \oint \left[dy_j^* \Omega_j + y_j^* \Delta \eta_j dt - \Delta(y_j^* \eta_j) dt - y_j^* C_{qr}^j \eta_q \Omega_r dt - y_j^* C_{0q}^j \Omega_q dt - dH^* \Omega_0 + \Delta H^* dt \right].$$

Since the motion is represented by the closed curve C which is completely arbitrary, the first integral on the right hand side vanishes. Thus we have

$$0 = d \oint \left[dy_j^* \Omega_j + y_j^* \Delta \eta_j dt - \Delta(y_j^* \eta_j) dt - y_j^* \eta_q C_{qr}^j \Omega_r dt - y_j^* C_{0q}^j \Omega_q dt - dH^* \Omega_0 + \Delta H^* dt \right].$$

Interchanging the indices r and q and separating the sum over the index $q = 1, 2, \dots, n$ into the sums over the indices $i = 1, \dots, m$ and $\alpha = m + 1, \dots, n$, we have

$$0 = d \oint \left[dy_j^* \Omega_j + y_j^* \Delta \eta_j dt - \Delta(y_j^* \eta_j) dt - y_j^* \left(C_{ri}^j \eta_r + C_{0i}^j \right) \Omega_i dt - y_j^* \left(C_{r\alpha}^j \eta_r + C_{0\alpha}^j \right) \Omega_\alpha dt - dH^* \Omega_0 + \Delta H^* dt \right].$$

Using the formulas (2.18-19) and performing some algebra we get

$$0 = \oint \left[\left(\dot{y}_i^* - C_{0i}^j y_j^* - C_{ri}^j \eta_r y_j^* \right) \Omega_i dt + y_j^* \Delta \eta_j dt - \Delta y_j^* \eta_j dt - y_j^* \Delta \eta_j dt - y_j^* \left(C_{0\alpha}^j + C_{r\alpha}^j \eta_r \right) (\omega_\alpha + \eta_\alpha \Omega_0) dt - \eta_0 X_0 \Omega_0 dt - \eta_j X_j H^* \Omega_0 dt - \eta_\alpha \Omega_0 X_\alpha H^* dt - \frac{\partial H^*}{\partial y_j^*} dy_j^* \Omega_0 + \Omega_0 X_0 H^* dt + \Omega_j X_j H^* dt + \Omega_\alpha X_\alpha H^* dt + \frac{\partial H^*}{\partial y_j^*} \Delta y_j^* dt \right].$$

Taking into account (2.16-17) and (3.4), we obtain

$$0 = \oint \left[\left\{ \dot{y}_i^* - C_{0i}^j y_j^* - C_{ri}^j \eta_r y_j^* - y_j^* (C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} + X_i H^* + \frac{\partial \phi_\alpha}{\partial \eta_i} X_\alpha H^* \right\} \Omega_i dt - \Delta y_j^* \left(\eta_j - \frac{\partial H^*}{\partial y_j^*} \right) dt + \left\{ y_j^* (C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} - X_i H^* - \frac{\partial \phi_\alpha}{\partial \eta_i} X_\alpha H^* \right\} \eta_i \Omega_0 dt + \left\{ -y_j^* (C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \eta_\alpha \Omega_0 dt - \eta_\alpha X_\alpha H^* \Omega_0 dt - \frac{\partial H^*}{\partial y_j^*} dy_j^* \Omega_0 + \eta_\alpha \Omega_0 X_\alpha H^* dt \right\} \right],$$

which, by using (3.11-13), reduces to

$$0 = \oint \left[\left\{ \dot{y}_i^* - y_j^* (K_{0i}^j + K_{ri}^j \eta_r) + X_i^* H^* \right\} \Omega_i dt - \Delta y_j^* \left(\eta_j - \frac{\partial H^*}{\partial y_j^*} \right) dt + \left\{ y_j^* (C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} \eta_i - X_i^* H^* \eta_i \right\} \Omega_0 dt - \left\{ y_j^* (C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \eta_\alpha - \frac{\partial H^*}{\partial y_j^*} y_j^* \right\} \Omega_0 dt \right]. \quad (6.4)$$

As required by the present investigation, we adopt the second viewpoint for which $\delta^* f_\alpha = 0$ for nonlinear nonholonomic dynamical system and multiply (3.9) with $(\frac{\partial L}{\partial \eta_\alpha})^*$ we get

$$(A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \omega_i = 0, \quad i = 1, \dots, m; \alpha = m + 1, \dots, n. \quad (6.5)$$

Since the relations (2.17) hold for all the ω_i 's, we integrate (6.5) along the closed curve C and incorporate the result into (6.4), we obtain

$$\begin{aligned} 0 = & \oint \left[\left\{ y_i^* - y_j^*(K_{0i}^j + K_{ri}^j \eta_r) + X_i^* H^* - (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \right\} \Omega_i dt \right. \\ & - \Delta y_j^* \left(\eta_j - \frac{\partial H^*}{\partial y_j^*} \right) dt + \left\{ y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} \eta_i - X_i^* H^* \eta_i \right. \\ & \left. \left. + \eta_i \Omega_0 (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* - y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \eta_\alpha - \frac{\partial H^*}{\partial y_j^*} y_j^* \right\} \Omega_0 dt \right], \end{aligned}$$

which, in view of the equations (6.1) and (6.2), takes the form

$$\begin{aligned} 0 = & \oint \left[\left\{ \phi_i - y_j^*(K_{0i}^j + K_{ri}^j \eta_r) + X_i^* H^* - (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \right\} \Omega_i \right. \\ & - \Delta y_j^* \left(\psi_j - \frac{\partial H^*}{\partial y_j^*} \right) + \left\{ y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} \eta_i - X_i^* H^* \eta_i \right. \\ & \left. \left. + \eta_i \Omega_0 (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* - y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \eta_\alpha - \frac{\partial H^*}{\partial y_j^*} y_j^* \right\} \Omega_0 \right] dt. \end{aligned}$$

In order that the equations (6.1) and (6.2) must be satisfied for the asynchronous variation in which the quantities Ω_0, Ω_i and Δy_j^* are arbitrary, the coefficient of each of these must vanish.

This implies that

$$\phi_i = -X_i^* H^* + (K_{0i}^j + K_{ri}^j \eta_r) y_j^* + (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^*, \quad \psi_i = \frac{\partial H^*}{\partial y_i^*},$$

$$j = 1, 2, \dots, m; \alpha = m + 1, \dots, n; r = 1, 2, \dots, n,$$

which yield the desired values of the functions ϕ_i and ψ_i and hence leads to the Poincaré-Hamilton equations of motion. Also we have

$$\begin{aligned} y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \frac{\partial \phi_\alpha}{\partial \eta_i} \eta_i - X_i^* H^* \eta_i + \eta_i \Omega_0 (A_i^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \\ - y_j^*(C_{0\alpha}^j + C_{r\alpha}^j \eta_r) \eta_\alpha - \frac{\partial H^*}{\partial y_j^*} y_j^* = 0. \end{aligned}$$

Thus the preceding analysis can be summarized in the following:

Theorem 6.1. *If the line integral (5.9) is invariant under a deformation of an arbitrary curve C along the tube of real trajectories, then the motion of the conservative nonlinear nonholonomic dynamical system is determined by the Poincaré-Hamilton (PH) equations (4.17) together with (4.7), provided that the relations (3.8) hold.*

It is remarkable to note that Theorems 5.1 and 6.1 furnish the necessary and sufficient condition which allow us to connect the theory of integral invariants with the theory of Poincaré-Hamiltonian systems. Thus, these integrals, as discussed in [2, 3, 11, 17, 19, 31], are very important in analytical dynamics in the sense that they provide another foundation for not only holonomic or linear nonholonomic systems but also furnishes a foundation for nonlinear nonholonomic dynamical systems whose motion is determined by the PH-system of equations (4.17) with (4.7).

7 A Generalization of the Poincaré Linear Integral Invariant

We now turn to a generalization of a theorem analogous to the theorem of Poincaré [3, 19] as a special case of the results established in the last section. To achieve this aim, we proceed with the assumption that the variations under consideration are synchronous. This implies that the quantity $\Omega_0 = \Delta t \equiv 0$. From (2.17) and (2.21), it follows that $\Omega_p = \omega_p$ and $\Delta\eta_p = \delta\eta_p$. Thus $\Delta \equiv d$ holds and therefore $d\delta = \delta d$ holds.

Under such conditions, we may restate Theorem 5.1 as follows:

Theorem 7.1. *The line integral*

$$I_1 = \oint_C y_j^* \omega_j, \quad (7.1)$$

along any closed curve C that consists of the simultaneous states of the system does not change with arbitrary deformation of this curve along the tube of real trajectories of the conservative nonlinear nonholonomic dynamical system which is described by the PH system of equations (4.17) with (4.7).

The proof of the theorem adopts a procedure similar to that discussed in the preceding sections but without requiring Lemma 2.4. The invariance of the integral (7.1) can be obtained by considering the simultaneous states of the dynamical system during its motion in the ‘‘Reduced Phase-space’’ of $2m$ variables (x_j, y_j^*) and using the fact that, contrary to the relation (2.23), the d -operation and integration commute with each other. Provided we have synchronous variation, we may assert the converse of Theorem 7.1 in the following:

Theorem 7.2. *If the line integral I_1 , given by (7.1), remains invariant under an arbitrary deformation along the tube of real trajectories of any closed*

curve C consisting of the simultaneous states of a conservative nonlinear non-holonomic dynamical system, then the motion of the system is determined by the PH system of equations (4.17) with (4.7).

We remark that the results above show that the theory of integral invariants forms another basis for both the Hamiltonian dynamics of holonomic systems, linear nonholonomic systems and for nonlinear nonholonomic dynamical systems as well.

In order to demonstrate the utility of the four theorems, we derive special cases of our general results that are analogous to well known results.

- (i) Suppose that all the x 's are the Lagrangian coordinates and the η 's are the generalized velocities \dot{x} 's. In this case, the relations (2.17) reduce to the result given in [29] and [31] by Vujanovic and Whittaker, respectively; the operators X_0 and X_p 's becomes $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_p}$'s. Consequently all the C_{0q}^p 's and C_{qp}^r 's vanish. Theorem 7.1 and 7.2 furnish the results analogous to those obtained in [6, 11, 19, 31] by Cartan, Gantmacher, Pars and Whittaker, respectively while Theorems 7.1 and 7.2 subsume the results that are discussed in [20] and [21] by Poincaré and in [19] by Pars. It is to be noted that our results are analogous to these but the content is quite different, since our system is nonlinear nonholonomic.
- (ii) If the group variables are the quasi-variables (nonholonomic coordinates) π 's then the η 's becomes $\dot{\pi}$'s and the relations (2.10) express these as non-integrable linear combinations of the quasi-velocities and all the C_{qp}^r 's reduce to Hamel-Boltzmann's three indexed symbols γ_{pq}^r . In this case our theorems subsume the results analogous to those obtained in [8] by Djukic.

APPENDIX

Asynchronous Variation of Functional (5.1)

Performing the Δ -variation of (5.1) according to the Lemma 2.4, we have

$$\begin{aligned} \Delta S &= \int_{t_1}^{t_2} (\Delta L + L\dot{\Omega}_0)dt, \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \Delta \eta_p + \Omega_\mu X_\mu L + (L\Omega_0) - \dot{L}\Omega_0 \right) dt, \end{aligned}$$

where we have used Definition 2.3 and the identity $(L\Omega_0\dot{)} = L\dot{\Omega}_0 + \dot{L}\Omega_0$. Taking into account (2.16), (2.17) and (2.21), the variation ΔS of S becomes

$$\Delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \delta \eta_p + \omega_p X_p L \right) dt + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \dot{\eta}_p \Omega_0 + \eta_\mu \Omega_0 X_\mu L - \Omega_0 \dot{L} \right) dt + \int_{t_1}^{t_2} (L\Omega_0\dot{)} dt. \quad (\text{A-1})$$

Let

$$I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p} \delta \eta_p + \omega_p X_p L \right) dt.$$

Separating the sum over the index p from 1 to n into the sums over j from 1 to m and over α from $(m+1)$ to n and using the equation of motion (4.1) to get

$$I = \int_{t_1}^{t_2} \left\{ \left(\frac{\partial L^*}{\partial \eta_j} - \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} \right) \delta \eta_j + \frac{\partial L}{\partial \eta_\alpha} \delta \eta_\alpha + \omega_p X_p L \right\} dt.$$

Here the “*” over the quantities show that they are expressed in terms of the independent parameters η_j of real displacement.

Following the second viewpoint and using relation (2.20), we have

$$I = \int_{t_1}^{t_2} \frac{\partial L^*}{\partial \eta_j} (\dot{\omega}_j + C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) dt - \int_{t_1}^{t_2} \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} (\dot{\omega}_j + C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) dt + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \eta_\alpha} (\dot{\omega}_\alpha + C_{0q}^\alpha \omega_q + C_{qr}^\alpha \eta_q \omega_r) + \omega_p X_p L \right\} dt.$$

Integrating by parts, the first term of each integral on the right hand side of the last result, we get

$$\begin{aligned} I = & \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) \omega_j dt + \int_{t_1}^{t_2} \frac{\partial L^*}{\partial \eta_j} (C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) dt \\ & - \left(\frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \omega_j \frac{d}{dt} \left(\frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} \right) dt \\ & - \int_{t_1}^{t_2} \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} (C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) dt \\ & + \left(\frac{\partial L}{\partial \eta_\alpha} \omega_\alpha \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \eta_\alpha} \right) \omega_\alpha dt \\ & + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_\alpha} (C_{0q}^\alpha \omega_q + C_{qr}^\alpha \eta_q \omega_r) + \omega_p X_p L \right) dt, \end{aligned}$$

which by use of (3.4), reduces to

$$\begin{aligned}
I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) \omega_j - \frac{\partial L^*}{\partial \eta_j} (C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) \right. \\
&\quad - \omega_j \frac{\partial L}{\partial \eta_\alpha} \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) - \omega_j \frac{d}{dt} \left(\frac{\partial L}{\partial \eta_\alpha} \right) \frac{\partial \phi_\alpha}{\partial \eta_j} + \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} (C_{0q}^j \omega_q + C_{qr}^j \eta_q \omega_r) \\
&\quad \left. + \frac{d}{dt} \left(\frac{\partial L}{\partial \eta_\alpha} \right) \frac{\partial \phi_\alpha}{\partial \eta_j} \omega_j - \frac{\partial L}{\partial \eta_\alpha} (C_{0q}^\alpha \omega_q + C_{qr}^\alpha \eta_q \omega_r) - \omega_p X_p L \right\} dt.
\end{aligned}$$

Again breaking the sum over the indices q and r from 1 to n into the sums over the indices j, k from 1 to m and over the indices α, β from $(m+1)$ to n , the last expression takes the form

$$\begin{aligned}
I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) \omega_j - \frac{\partial L^*}{\partial \eta_j} (C_{0k}^j \omega_k + C_{0\alpha}^j \omega_\alpha + C_{qk}^j \eta_q \omega_k \right. \\
&\quad + C_{q\alpha}^j \eta_q \omega_\alpha) - \omega_j \frac{\partial L}{\partial \eta_\alpha} \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) + \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} (C_{0k}^j \omega_k + C_{0\beta}^j \omega_\beta + C_{qk}^j \eta_q \omega_k \\
&\quad + C_{q\beta}^j \eta_q \omega_\beta) - \frac{\partial L}{\partial \eta_\alpha} (C_{0j}^\alpha \omega_j + C_{0\beta}^\alpha \omega_\beta + C_{qj}^\alpha \eta_q \omega_j + C_{q\beta}^\alpha \eta_q \omega_\beta) \\
&\quad \left. - \omega_p X_p L \right\} dt.
\end{aligned}$$

Interchanging the indices j and k in the second and fourth terms of the integrand, using (3.4) and rearranging the terms, we find that

$$\begin{aligned}
I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) \omega_j - \frac{\partial L^*}{\partial \eta_k} \left\{ (C_{0j}^k + C_{0\alpha}^k \frac{\partial \phi_\alpha}{\partial \eta_j}) + \eta_q (C_{qj}^k \right. \right. \\
&\quad \left. \left. + C_{q\alpha}^k \frac{\partial \phi_\alpha}{\partial \eta_j}) \right\} \omega_j - \omega_j \frac{\partial L}{\partial \eta_\alpha} \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) + \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_k} \left\{ (C_{0j}^k + C_{0\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) + (C_{qj}^k \right. \right. \\
&\quad \left. \left. + C_{q\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q \right\} \omega_j - \frac{\partial L}{\partial \eta_\alpha} \left\{ (C_{0j}^\alpha + C_{0\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}) + (C_{qj}^\alpha + C_{q\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}) \eta_q \right\} \omega_j \right. \\
&\quad \left. - \omega_p X_p L \right] dt. \quad (\text{A-2})
\end{aligned}$$

Taking into account equations (3.11) and (3.12), the last result reduces to

$$\begin{aligned}
I &= \frac{\partial L^*}{\partial \eta_j} \omega_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) \omega_j - \frac{\partial L^*}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta_q) \omega_j \right. \\
&\quad \left. - \omega_j \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \left\{ \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) - \frac{\partial \phi_\alpha}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta_q) + (K_{0j}^\alpha + K_{qj}^\alpha \eta_q) \right\} \omega_j \right. \\
&\quad \left. - \omega_p X_p L \right] dt, \quad (\text{A-3})
\end{aligned}$$

where the “*” over the quantities $\left(\frac{\partial L}{\partial \eta_\alpha} \right)^*$ indicates that they are expressed in terms of the independent parameters η_j .

Let us consider the term $\omega_p X_p L$ which, in view of (4.4), can be written as

$$\begin{aligned}\omega_p X_p L &= \omega_p X_p L^* - \omega_p \frac{\partial L}{\partial \eta_\alpha} X_p \phi_\alpha \\ &= \omega_j X_j L^* + \omega_\alpha X_\alpha L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} X_j \phi_\alpha - \omega_\beta \frac{\partial L}{\partial \eta_\alpha} X_\beta \phi_\alpha \\ &= \omega_j X_j L^* + \frac{\partial \phi_\alpha}{\partial \eta_j} \omega_j X_\alpha L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} X_j \phi_\alpha - \frac{\partial \phi_\beta}{\partial \eta_j} \omega_j \frac{\partial L}{\partial \eta_\alpha} X_\beta \phi_\alpha,\end{aligned}$$

where to obtain this result we have separated the sum over the index $p = 1, 2, \dots, n$ into the sums over $j = 1, 2, \dots, m$ and $\alpha = m+1, \dots, n$ and also used the relation (4.3). Simplifying the last expression, we get

$$\omega_p X_p L = \omega_j \left(X_j + \frac{\partial \phi_\alpha}{\partial \eta_j} X_\alpha \right) L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} \left(X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta \right) \phi_\alpha,$$

which together with (3.13), becomes

$$\omega_p X_p L = \omega_j X_j^* L^* - \omega_j \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* X_j^* \phi_\alpha, \quad (\text{A-4})$$

where we have expressed all the quantities in terms of the independent parameters η_j 's of real displacement. This allows us to write (A.1) as

$$\begin{aligned}I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) - \frac{\partial L^*}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta_q) - X_j^* L^* \right. \\ &\quad \left. - \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \left\{ \frac{d}{dt} \left(\frac{\partial \phi_\alpha}{\partial \eta_j} \right) - \frac{\partial \phi_\alpha}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta_q) + (K_{0j}^\alpha + K_{qj}^\alpha \eta_q) - X_j^* \phi_\alpha \right\} \right] \omega_j dt.\end{aligned}$$

$$\begin{aligned}I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L^*}{\partial \eta_j} \right) - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - X_j^* L^* \right. \\ &\quad \left. - (A_j^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha} \right)^* \right] \omega_j dt, \quad (\text{A-5})\end{aligned}$$

which expresses the asynchronous variation of the action integral (5.1).

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