Disjoint Infinity Borel Functions

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Disjoint ∞-Borel Functions

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Motivating Statement

Consider the following statement:

Each $A \subseteq \mathbb{R}$ of size $2^\omega$ can be surjected onto $\mathbb{R}$ by a Borel function.

- It follows from $ZF + AD$ (as we will see soon).
- It, together with $AC$, implies $\text{add}(\mathcal{M}) < 2^\omega$. That is, there is a size $< 2^\omega$ collection of meager sets of reals whose union is not meager.
- It holds in the *iterated perfect set model* (start with $CH$, then add $\omega_2$ Sacks reals by iterated forcing). Thus, it is consistent with $ZFC$ and $\text{add}(\mathcal{M}) = \omega_1 < \omega_2 = 2^\omega$. It is open whether it is consistent with $ZFC + 2^\omega > \omega_2$. 
Consider the following stronger statement:

Each uncountable $A \subseteq \mathbb{R}$ can be surjected onto $\mathbb{R}$ by a Borel function.

- It implies the statement on the previous slide.
- It is false if we assume ZFC (it is certainly false if we assume $\neg$CH, and if CH holds, then $\text{add}(\mathcal{M}) < 2^{\omega}$ cannot hold).
- It follows from ZF + the statement that every uncountable set of reals has a non-empty perfect subset (which follows from ZF + AD).

Proof: If $A \subseteq \mathbb{R}$ is uncountable and $P \subseteq A$ is a perfect subset, then there is a real which codes the set $P$, and this real can be used to define a continuous function which maps $P \subseteq A$ onto $\mathbb{R}$.

- It also (almost) follows from the statement on the next slide...
Even Stronger Statement: $\Psi$

This will be our focus:

**Definition of $\Psi$**

$\Psi$ is the following statement: for each $a \in \mathbb{R}$ there is a function $f_a : \mathbb{R} \to \mathbb{R}$ such that the following hold:

- The function $(a, x) \mapsto f_a(x)$ is Borel.
- $(\forall g : \mathbb{R} \to \mathbb{R}) \{ a \in \mathbb{R} : f_a \cap g = \emptyset \}$ is countable.
Recall that Uniformization is the fragment of AC which says that given any $R \subseteq \mathbb{R} \times \mathbb{R}$ satisfying $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) (x, y) \in R$, then there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $\text{Graph}(g) \subseteq R$. 

**Proposition**

$ZF + \text{Uniformization} + \Psi$ implies that if $A \subseteq \mathbb{R}$ is uncountable, then it can be surjected onto $\mathbb{R}$ by a Borel function.

Proof: Fix an uncountable set $A \subseteq \mathbb{R}$. For each $x \in \mathbb{R}$, the function $a \mapsto f_a(x)$ is Borel. We claim that for some $x \in \mathbb{R}$, the function $a \mapsto f_a(x)$ surjects $A$ onto $\mathbb{R}$. Suppose this is not the case. For each $x \in \mathbb{R}$, the set $Y_x := \mathbb{R} - \{f_a(x) : a \in A\}$ is non-empty. Apply Uniformization to get $g : \mathbb{R} \to \mathbb{R}$ such that $(\forall x \in \mathbb{R}) g(x) \in Y_x$. Then $g$ is disjoint from $f_a$ for each $a \in A$, which is a contradiction because $g$ can be disjoint from only countably many of the $f_a$ functions.
Proving $\Psi$ from $\text{AD}^+$: Part 1

Without loss of generality, we will use $\omega \omega$ in place of $\mathbb{R}$. $\text{AD}^+$ is a statement that implies both $\text{AD}$ and that every function $g : \mathbb{R} \to \mathbb{R}$ is $\infty$-Borel. Note: $\text{AD}$ implies there is no injection of $\omega_1$ into $\mathbb{R}$.

**Definition**

A set $S \subseteq \omega \omega$ is $\infty$-Borel iff there is a formula $\varphi$ and a set of ordinals $C \subseteq \text{Ord}$ such that $(\forall x \in \omega \omega) x \in S \iff L[C][x] \models \varphi(C, x)$.

**Definition**

A function $g : \omega \omega \to \omega \omega$ is $\infty$-Borel iff there is a set of ordinals $C \subseteq \text{Ord}$ and a formula $\varphi$ such that for all $x \in \mathbb{R}$ and $n, m \in \omega$,

$$g(x)(n) = m \iff L[C][x] \models \varphi(C, x, n, m).$$

Idea: $\infty$-Borel functions are “nice”. If there is a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is universally Baire, then $L(\mathbb{R}, A) \models \text{AD}^+$. Thus, if there is a proper class of Woodin cardinals and $g : \mathbb{R} \to \mathbb{R}$ is universally Baire, then $g$ is $\infty$-Borel.
We will show that $\text{AD}^+$ implies $\Psi$. For each $a \in \omega\omega$, we will define $f_a : \omega\omega \to \omega\omega$ as follows: Fix $a \in \omega\omega$. Pick some $A \subseteq \omega$ such that $A =_T a$, $A$ is infinite, and $A \leq_T B$ whenever $B$ is an infinite subset of $A$. Such a set $A$ is easy to construct. We actually only need $A$ to be $\Delta^1_1$ in every infinite subset of itself.

Let $h : A \to \omega$ be a function such that $(\forall n \in \omega) h^{-1}(n)$ is infinite.

We will now define $f_a : \omega\omega \to \omega\omega$. Fix $x = \langle x_0, x_1, ... \rangle \in \omega\omega$. Let $i_0 < i_1 < ...$ be the sequence of indices listing which numbers $x_i$ are in $A$. That is, each $x_{i_k} \in A$, but no other $x_i$ is in $A$. Define

$$f_a(x) := \langle h(x_{i_0}), h(x_{i_1}), ... \rangle$$

If there are only finitely many $x_i$ in $A$, define $f_a(x)$ to be anything.
Main Theorem (ZF)

Assume there is no injection of $\omega_1$ into $\omega\omega$. Let $g : \omega\omega \to \omega\omega$ be $\infty$-Borel, as witnessed by the set of ordinals $C \subseteq \text{Ord}$. For each $a \in \omega\omega$,

$$f_a \cap g = \emptyset \Rightarrow a \in L[C].$$

Since $L[C]$ has only countably many reals in it (because $\omega_1^{L[C]}$ injects into $\omega\omega \cap L[C]$), this theorem gives us that AD$^+$ implies $\Psi$.

To prove the theorem, fix $a \in \omega\omega$ and assume $a \notin L[C]$. Let $A \subseteq \omega$ be the set associated with $a$ such that $a =^T A$ and $A$ is computable from every infinite subset of itself. We will construct an $x \in \omega\omega$, by forcing over $L[C]$, such that $f_a(x) = g(x)$. 
Let \( \mathbb{H} \) be the poset of all trees \( T \subseteq \omega^\omega \) with cofinite splitting beyond the stem. We have \( T_2 \leq T_1 \) iff \( T_2 \subseteq T_1 \). Define the stronger ordering \( \leq^A \) by \( T_2 \leq^A T_1 \) iff \( T_2 \leq T_1 \) and

\[
(\forall t \in \text{Stem}(T_2) - \text{Stem}(T_1))\ t(|t| - 1) \notin A.
\]

That is, \( T_2 \leq^A T_1 \) means that \( T_2 \leq T_1 \) and the part of the stem of \( T_2 \) not in the stem of \( T_1 \) does not hit \( A \). Idea: If \( x \in \omega^\omega \) is the generic real being added by \( \mathbb{H} \) (\( x = \bigcap \) the generic filter), then if \( T_2 \leq T_1 \), then \( T_2 \) does not decide any more of \( f_a(x) \) than \( T_1 \) already does. The main lemma says we can hit dense subsets of \( \mathbb{H} \) without deciding more of \( f_a(x) \):

**Main Lemma**

Let \( M \) be an inner model such that \( A \notin M \). Let \( U \in M \) be such that \( U \subseteq \mathbb{H}^M \) is dense\(^M \) in \( \mathbb{H}^M \). Fix \( T_1 \in \mathbb{H}^M \). Then there is \( T_2 \leq^A T_1 \) such that \( T_2 \in U \).
Step 0: Let $\langle U_n : n < \omega \rangle$ be an enumeration of the dense $L[C]$ subsets of $\mathbb{H}^{L[C]}$ in $L[C]$. Let $\dot{x}$ be the canonical name for the generic real $x \in \omega$. Let $T_0 = 1$.

Step 1:
- Let $T'_0 \leq^A T_0$ be in $U_0$.
- Let $T''_0 \leq^A T'_0$ and $m_0 \in \omega$ be such that $T''_0 \models g(\dot{x})(0) = \dot{m}_0$.
- Let $T_1 \leq T''_0$ have stem 1 longer than $T''_0$ such that $T_1$ ensures that $f(x)(0) = m_0$.

Step 2:
- Let $T'_1 \leq^A T_1$ be in $U_1$.
- Let $T''_1 \leq^A T'_1$ and $m_1 \in \omega$ be such that $T''_1 \models g(\dot{x})(1) = \dot{m}_1$.
- Let $T_2 \leq T''_1$ have stem 1 longer than $T''_1$ such that $T_2$ ensures that $f(x)(1) = m_1$.

... 

We have $\forall i \in \omega \: f_a(x)(i) = m_i = g(x)(i)$. Thus, $f_a(x) = g(x)$.

This completes the proof.
Proving $\Psi$ from $\text{AD}^+$: Picture Summary

\[ f_\alpha(x) = g(x) \]
Corollary of the theorem

The following follows from the theorem:

**Corollary**

Assume there is a proper class of Woodin cardinals. Let \( g : \mathbb{R} \to \mathbb{R} \) be universally Baire. Then \( g \) is disjoint from at most countably many of the functions \( f_a : \mathbb{R} \to \mathbb{R} \).
Earlier we showed that ZFC implies $\neg \Psi$. We ask whether the weaker statement $\Psi^-$ is consistent with ZFC:

**Definition of $\Psi^-$**

$\Psi^-$ is the following statement: for each $a \in \mathbb{R}$ there is a function $f^a : \mathbb{R} \to \mathbb{R}$ such that the following hold:

- The function $(a, x) \mapsto f^a(x)$ is Borel.
- $\left( \forall g : \mathbb{R} \to \mathbb{R} \right) \left\{ a \in \mathbb{R} : f^a \cap g = \emptyset \right\}$ has size $< 2^\omega$.

Note: In a model of ZFC + $\Psi^-$, it must be that CH fails.
Thank You!


