Hausdorff Dimension of Kuperberg Minimal Sets

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Hausdorff Dimension of Kuperberg Minimal Sets

Daniel Ingebritson

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Kuperberg’s flow is the flow of a $C^\infty$ aperiodic vector field on a three-manifold called a plug.

This flow preserves a unique minimal set with a fractal structure.

Problem: What is the Hausdorff dimension of this minimal set?
History: Seifert’s conjecture

- Seifert 1950: Does every nonsingular vector field on the three-sphere $S^3$ have a periodic orbit?
- Wilson 1966: On any three-manifold there exists a nonsingular $C^\infty$ vector field with only 2 periodic orbits.
  - These orbits are contained inside a plug.
  - The plug is inserted to break periodic orbits outside the plug.
- Schweitzer 1974: There exists a $C^1$ plug with no periodic orbits.
- Kuperberg 1994: Constructed a $C^\infty$ aperiodic plug.
  - Modified Wilson’s construction using self-insertion.
The Wilson Plug

Wilson’s plug is a three-manifold with boundary, supporting a smooth vector field $\mathcal{W}$ defining a flow $\phi_t$ with three orbit types:

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- Large radius:

- Radius close to 2:

- Radius $= 2$: 
Wilson’s minimal set

Two periodic orbits
Kuperberg’s plug

The Kuperberg plug is the Wilson plug with self-insertions. The resulting plug $K$ inherits a vector field $\mathcal{K}$ with flow $\psi_t$. 
Dynamics of the Kuperberg flow

Kuperberg (1994)

The $C^\infty$ vector field $\mathcal{K}$ has no closed orbits.
The Kuperberg minimal set

- Ghys (1995): conjectured that $K$ has a unique minimal set $\mathcal{M}$ with topological dimension 2.

- Hurder and Rechtman (2016):
  - $\mathcal{M} \subset K$ is nontrivial.
  - $\mathcal{M}$ is a surface lamination with radial Cantor transversal of Lebesgue measure zero.

- To estimate the Hausdorff dimension of $\mathcal{M}$ we model the transverse Cantor set as the attractor of a conformal graph-directed pseudo-Markov system.
A collection $S = \{\phi_i : X \rightarrow X\}_{i \in I}$ of injective contractions of a compact metric space $X$ is an IFS. For $\omega \in I^n$, denote

$$\phi_\omega = \omega_1 \circ \cdots \circ \omega_n$$

Then $J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X)$ is the limit set of $S$. 

Nesting condition: $\phi_{\omega,i}(X) \subset \phi_\omega(X)$. 

If $X \subset \mathbb{R}$ and $\phi_i$ are $C^{1+\alpha}$, then $S$ has bounded distortion.
Iterated Function Systems

A collection $S = \{\phi_i : X \to X\}_{i \in I}$ of injective contractions of a compact metric space $X$ is an IFS. For $\omega \in I^n$, denote

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Then $J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X)$ is the limit set of $S$.

- $J$ is invariant under $S$.
- If $S$ satisfies the open set condition and the bounded distortion property, then $J$ is a Cantor set.
- Nesting condition: $\phi_{\omega,i}(X) \subset \phi_\omega(X)$.
- If $X \subset \mathbb{R}$ and $\phi_i$ are $C^{1+\alpha}$, then $S$ has bounded distortion.
Topological Pressure of an IFS

Each IFS $S = \{\phi_i : X \to X\}_{i \in I}$ has an associated topological pressure:

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|\phi'_{\omega}\|^t$$
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- $P : [0, \infty) \to \mathbb{R}$ is continuous, convex, and strictly decreasing.
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Theorem (Bowen 1979)

Let $J$ be the limit set of an IFS, and $s = \dim_H(J)$. Then $s$ is the unique solution of $P(s) = 0$. 
Dimension theory of limit sets

Thermodynamic formalism:

- Mathematical formulation of equilibrium statistical mechanics developed by Sinai and Ruelle.
- Idea: study space of probability measures on phase space, with good ergodic properties.
- Hausdorff dimension estimates involve probability measures (Frostman’s lemma).
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Generalizations of Bowen’s theorem:

- Mauldin and Urbański (1996) extended to graph-directed constructions on infinite alphabet.
- Barreira, Pesin and Weiss (1996) developed a non-additive thermodynamic formalism.
There exists a curve $\gamma$ transverse to the flow so that

$$\mathcal{M} = \bigcup_{-\infty < t < \infty} \psi_t(\gamma).$$

We can decompose $\mathcal{M}$ by level of insertion:

$$\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$$

Each $\mathcal{M}_n$ forms a propeller winding around the plug.
Level-one propeller $\mathcal{M}_1$

- Curves are generated the holonomy pseudogroup of the Wilson flow.
- Propeller $\mathcal{M}_1$ bounds a family of closed regions $R_i$.
- The pseudogroup contracts $R_i$ in the radial direction.
- Infinite returns of the propeller implies symbolic dynamics on an infinite alphabet.
Level-two propeller $P_2$

- Curves are generated by composition of Wilson pseudogroup with one insertion.
- Propeller $\mathcal{M}_2$ bounds a family of closed regions $R_{i,j}$.
- Nesting property: $R_{i,j} \subset R_j$
Pressure for pseudogroups

\[ P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in A_n \subset I^n} \|\phi'_\omega\|^t \]
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- Need to determine the admissible words \( A_n \subset I^n \).
- \( A_n \) is defined by the symbolic dynamics of the pseudogroup.
Pressure for pseudogroups

\[ P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in A_n \subset I^n} \| \phi'_\omega \|^t \]

- Need to determine the admissible words \( A_n \subset I^n \).
- \( A_n \) is defined by the symbolic dynamics of the pseudogroup.
- Necessary to determine growth of derivatives \( \phi'_\omega \) of the holonomy maps.
- Bounds on \( \phi'_\omega \) lead to generalized pressure functions.
- To estimate \( P(t) = 0 \), we find zeros of generalized pressure functions.
Dimension estimates on $\mathcal{M}$

**Theorem (I.)**

Let $C \subset [0, 1]$ be the transverse Cantor set of $\mathcal{M}$.

- There exists a conformal graph-directed pseudo-Markov function system on $[0, 1]$ with limit set $C$.
- $s = \dim_H(C)$ is the unique root of a dynamically defined pressure function.
- $0.544 \leq \dim_H(C) \leq 0.863$.

**Corollary:** $2.544 \leq \dim_H(\mathcal{M}) \leq 2.863$. 

**Question:** Can we use similar thermodynamic formalism to find dimension estimates for other codimension-one attractors?
Dimension estimates on $\mathcal{M}$

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