On a Construction of Some Class of Metric Spaces

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On a construction of some class of metric spaces

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• Motivations

• Definition

• Examples

• Properties

• Another example

• References
## Definition

**Definition (Aronszajn–Panitchpakdi, 1956)**

We call a metric space hyperconvex, if each family of closed balls $\{\overline{B}(x_i, r_i)\}_{i \in J}$ such that  

$$d(x_i, x_j) \leq r_i + r_j \quad \text{for } i, j \in J$$

has a nonempty intersection.

### Remark

For normed spaces, hyperconvexity means that each pairwise intersecting family of closed balls has a nonempty intersection.
The following spaces are hyperconvex:

- $\mathbb{R}$;
- $\mathbb{R}^n$ with a “maximum” norm;
- $l^\infty$ and $L^\infty(\mathbb{R})$;
- $C_\mathbb{R}(K)$, where $K$ is a compact and extremally disconnected Hausdorff topological space.
The space $\mathbb{R}^2$ with the euclidean norm is not hyperconvex. However, the same space with the “maximum” norm is.
The radial metric
The "river" metric
The construction of certain metrics

Definition (pictorial; Borkowski, DB, Przybycień)
Definition (informal; Aksoy, Maurizi)

Let \((X_c, d_c)\) and \((X_i, d_i)\) for \(i \in I\) be metric spaces. Assume that for every \(i \in I\) the intersection \(X_c \cap X_i\) consists of exactly one point - let us denote it by \(x_i\) - and that for \(i, j \in I\) the intersection \(X_i \cap X_j\) is empty or it contains only the point \(x_i = x_j\).
**Definition (informal; Aksoy, Maurizi)**

Let \((X_c, d_c)\) and \((X_i, d_i)\) for \(i \in I\) be metric spaces. Assume that for every \(i \in I\) the intersection \(X_c \cap X_i\) consists of exactly one point - let us denote it by \(x_i\) - and that for \(i, j \in I\) the intersection \(X_i \cap X_j\) is empty or it contains only the point \(x_i = x_j\). Let us denote \(X := X_c \cup \bigcup_{i \in I} X_i\) and let us define the function \(\varphi : X \times X \to [0, +\infty)\) by formulae

\[
\psi(x, y) := \begin{cases} 
  d_c(x, y), & \text{if } x, y \in X_c \\
  d_i(x, y), & \text{if } x, y \in X_i \\
  d_c(x, x_i) + d_i(x_i, y), & \text{if } x \in X_c \text{ and } y \in X_i \\
  d_i(x, x_i) + d_c(x_i, x_j) + d_j(x_j, y), & \text{if } x \in X_i \text{ and } y \in X_j 
\end{cases}
\]

**Fact**

*The above defined function \(\psi\) is a metric on the set \(X\).*
The definition of the metric $\varphi$

**Definition (Száz)**

Let $(X, d)$ be a metric space, $T : X \to X$ be a mapping, $g$ be a metric on $T(X)$ and let $\sim$ be such an equivalence relation on $X$ that $T$ is constant on its equivalence classes. Let

$\varphi : X \times X \to [0, +\infty)$ be defined by the formula

$$
\varphi(x, y) := \begin{cases} 
    d(x, y), & \text{if } x \sim y, \\
    d(x, T(x)) + g(T(x), T(y)) + d(T(y), y), & \text{otherwise.}
\end{cases}
$$

**Fact**

*It is not difficult to check that the above defined function $\varphi$ is a metric on the space $X$.***
Motivations

Definition

Examples

Properties

Another example

References
# The radial metric and the "river" metric

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $X := \mathbb{R}^2$, $d = g$ be the euclidean metric, $T((x_1, x_2)) := (0, 0)$ and $(x_1, x_2) \sim (y_1, y_2) \iff$ the points $(0, 0), (x_1, x_2), (y_1, y_2)$ are colinear. Then the metric $\varphi$ is the well-known radial metric.</td>
</tr>
</tbody>
</table>
The radial metric and the "river" metric

Example

Let \( X := \mathbb{R}^2 \), \( d = g \) be the euclidean metric, \( T((x_1, x_2)) := (0, 0) \) and \((x_1, x_2) \sim (y_1, y_2) \iff \) the points \((0, 0), (x_1, x_2), (y_1, y_2)\) are colinear.
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Example

Let \( X := \mathbb{R}^2 \), \( d = g \) be the euclidean metric, \( T((x_1, x_2)) := (x_1, 0) \) and \((x_1, x_2) \sim (y_1, y_2) \iff x_1 = y_1. \)
Then the metric \( \varphi \) is the well-known "river" metric.
Motivations

Definition

Examples

Properties

Another example

References
Fact

*If* $d = g$, *the mapping* $T$ *is continuous, the space* $(X, d)$ *is complete and the equivalence classes of the relation “∼” are closed, then the metric* $\varphi$ *is complete.*
Properties

Fact

If \( d = g \), the mapping \( T \) is continuous, the space \((X, d)\) is complete and the equivalence classes of the relation \( "\sim" \) are closed, then the metric \( \varphi \) is complete.

Fact

If the metrics \( d \) and \( g \) are bounded, then the metric \( \varphi \) is also bounded.
Properties

**Fact**

*If* $d = g$, *the mapping* $T$ *is continuous, the space* $(X, d)$ *is complete and the equivalence classes of the relation* "$\sim$" *are closed, then the metric* $\varphi$ *is complete.*

**Fact**

*If the metrics* $d$ *and* $g$ *are bounded, then the metric* $\varphi$ *is also bounded.*

**Fact**

*If the relation* "$\sim$" *is the equality relation, then every cluster point of the space* $(X, \varphi)$ *is a fixed point of the mapping* $T$. 
The "floor" metric

Example

Let $X := \mathbb{R}$, $d = g$ be the euclidean metric, $T(x) := \lfloor x \rfloor$ and $x \sim y \iff x = y$. Then $\varphi$ is a certain metric on the real line which we will denote by $\varphi[\cdot]$. 

Fact

The metric $\varphi[\cdot]$ is neither translation invariant nor homogeneous. It is not metrically convex, however it is complete.
The "floor" metric

**Example**

Let $X := \mathbb{R}$, $d = g$ be the euclidean metric, $T(x) := \lfloor x \rfloor$ and $x \sim y \iff x = y$. Then $\varphi$ is a certain metric on the real line which we will denote by $\varphi_{\lfloor \cdot \rfloor}$.

**Fact**

*The metric $\varphi_{\lfloor \cdot \rfloor}$ is neither translation invariant nor homogeneous. It is not metrically convex, however it is complete.*
Shape of balls

Fact

Let $x \in \mathbb{Z}$ and $r > 0$. Then

$$B_{\varphi_{\lfloor \cdot \rfloor}}(x, r) = [x - \lfloor r \rfloor, x - \lfloor r \rfloor + r - \lfloor r \rfloor] \cup [x - \lfloor r \rfloor + 1, x + r).$$
Shape of balls

Fact

Let $x \in \mathbb{Z}$ and $r > 0$. Then

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Fact

Let $x \in \mathbb{R}$ and $r > 0$. Then

$$B_{\varphi_{\lfloor \cdot \rfloor}}(x, r) = B_{\varphi_{\lfloor \cdot \rfloor}}([x], r - (x - \lfloor x \rfloor)) \cup \{x\}.$$
**Shape of balls**

**Fact**

Let $x \in \mathbb{Z}$ and $r > 0$. Then

$$B_{\lfloor \cdot \rfloor}(x, r) = \left[ x - \lfloor r \rfloor, x - \lfloor r \rfloor + r - \lfloor r \rfloor \right] \cup \left[ x - \lfloor r \rfloor + 1, x + r \right).$$

**Fact**

Let $x \in \mathbb{R}$ and $r > 0$. Then

$$B_{\lfloor \cdot \rfloor}(x, r) = B_{\lfloor \cdot \rfloor}(\lfloor x \rfloor, r - (x - \lfloor x \rfloor)) \cup \{x\}.$$

**Remark**

Analogous formulae hold for closed balls.
Shape of balls – examples

Examples

\[ B_{\varphi_{[\cdot]}} (1, \frac{3}{2}) = \left[ 0, \frac{1}{2} \right) \cup \left[ 1, \frac{5}{2} \right) \]
Shape of balls – examples

Examples

\[ B_{\varphi} \left( 1, \frac{3}{2} \right) = \left[ 0, \frac{1}{2} \right) \cup \left[ 1, \frac{5}{2} \right) \]

\[ B_{\varphi} \left( \frac{3}{4}, 1 \right) = \left[ 0, \frac{1}{4} \right) \cup \left\{ \frac{3}{4} \right\} \]
Form of open sets

Fact

A subset of the space \((\mathbb{R}, \varphi_{\lfloor . \rfloor})\) is \(\varphi\)-open if and only if it is a union of pairwise disjoint maximal intervals with the property that if the right endpoint of any of these intervals in an integer, then this endpoint does not belong to that interval.
Form of open sets

**Fact**

A subset of the space \((\mathbb{R}, \varphi_{[\cdot, \cdot])}\) is \(\varphi\)-open if and only if it is a union of pairwise disjoint maximal intervals with the property that if the right endpoint of any of these intervals is an integer, then this endpoint does not belong to that interval.

**Corollary**

A ball in the space \((\mathbb{R}, \varphi_{[\cdot, \cdot])}\) need not be a connected set. For example \(B(1, \frac{3}{2}) = [0, \frac{1}{2}) \cup [1, \frac{5}{2})\) is the union of two open sets.
Measure of noncompactness

Fact

Let $A \subset [k, k + 1)$ for some $k \in \mathbb{Z}$. The Kuratowski measure of noncompactness of the set $A$ (in view of the metric $\varphi_{[\cdot]}$) expresses by the formula

$$\alpha(A) = 2 \max(\{x - k | x \text{ is a cluster point of the set } A\} \cup \{0\}).$$
Measure of noncompactness

Fact

Let $A \subset [k, k + 1)$ for some $k \in \mathbb{Z}$. The Kuratowski measure of noncompactness of the set $A$ (in view of the metric $\varphi_{[\cdot]}$) expresses by the formula

$$\alpha(A) = 2 \max\left(\{x - k \mid x \text{ is a cluster point of the set } A\} \cup \{0\}\right).$$

Remark

In the above formula a cluster point is considered in view of the euclidean metric.
Measure of noncompactness

**Corollary**

Let $A$ be a bounded subset of $\mathbb{R}$. The Kuratowski measure of noncompactness of the set $A$ (in view of the metric $\varphi_{[\cdot]}$) expresses by the formula

$$\alpha(A) = 2 \max_{k \in \mathbb{Z}} \max \left\{ \left. x - k \right| x \text{ is a cluster point of the set } A \cap [k, k+1) \right\} \cup \{0\}$$
Motivations
Definition
Examples
Properties
Another example
References
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A common generalization of the postman, radial, and river metrics

Thank you for your attention.