Some Applications of the Point-Open Subbase Game

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The point-open subbase game

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Definition

Given a subbase $S$ of a space $X$, the game $PO(S, X)$ is defined for two players $P$ and $O$ who respectively pick, at the $n$-th move, a point $x_n \in X$ and a set $U_n \in S$ such that $x_n \in U_n$. The game stops after the moves $\{x_n, U_n : n \in \omega\}$ have been made and the player $P$ wins if $\bigcup_{n \in \omega} U_n = X$; otherwise $O$ is the winner.
A strategy for Player $P$ in the point-open game $PO(X)$ on a space $X$ is a function $\sigma$ with values in $X$ defined on the initial segments of $PO(X)$ called $\sigma$-admissible; it is inductively defined as follows. The empty segment is $\sigma$-admissible; if $n > 0$, then a segment $\{x_0, U_0, \ldots, x_n, U_n\}$ is $\sigma$-admissible if $\{x_0, U_0, \ldots, x_{n-1}, U_{n-1}\}$ is $\sigma$-admissible and $x_n = \sigma(x_0, U_0, \ldots, x_{n-1}, U_{n-1})$.

The definition of a strategy $s$ for Player $O$ is analogous for $s$-admissible segments $\{x_0, U_0, \ldots, x_{n-1}, U_{n-1}, x_n\}$. A play $P = \{x_n, U_n : n \in \omega\}$ is called $\sigma$-admissible for a strategy $\sigma$ of Player $P$ if every initial segment of $P$ is $\sigma$-admissible; in this case we will also say that $P$ applies the strategy $\sigma$. 
An $s$-admissible play for a strategy $s$ of Player $O$ is defined analogously. A strategy $\sigma$ of Player $P$ is winning on $X$ if $P$ wins in any $\sigma$-admissible play.

Analogously, a strategy $s$ of Player $O$ is winning on $X$ if $O$ is the winner in any $s$-admissible play.

A game $PO(X)$ or $PO(S, X)$ is undetermined on a space $X$ if neither of the players $P$ and $O$ has a winning strategy in the respective game on $X$. If a game is considered on a space $X$ and $A$ is one of the players, then $X$ is called $A$-favorable if $A$ has a winning strategy on $X$. 
The Player $P$

**Theorem**
If $X$ is a space and $S$ is a subbase in $X$, then the games $PO(X)$ and $PO(S, X)$ are equivalent for $P$, i.e., Player $P$ has a winning strategy in the game $PO(X)$ if and only if $P$ has a winning strategy in the game $PO(S, X)$.

**Corollary**
If $PO(X)$ is undetermined on a space $X$, then so is $PO(S, X)$ for any subbase $S$ of the space $X$. 
Telgarsky constructed a Lindelöf $P$-space $X$ on which $PO(X)$ is undetermined. By the previous Corollary, on the same space $X$ the game $PO(S, X)$ is undetermined for any subbase $S$. It is also worth mentioning that Under Martin’s Axiom, if $M \subset \mathbb{R}$ and $\omega < |M| < \mathfrak{c}$, then the game $PO(M)$ is undetermined on $M$; this was proved by Galvin. Applying our Corollary once again we conclude that $PO(S, M)$ is undetermined on $M$ for any subbase $S$ of the space $M$. 
A complete characterization was given by Pawlikowski for the game $PO(X)$ to be undetermined on a space $X$ of countable pseudocharacter. In particular, the game $PO(M)$ is undetermined on a set $M \subset \mathbb{R}$ if and only if $|M| > \omega$ and $M$ is a $C''$-set, i.e., for every sequence $\{U_n : n \in \omega\}$ of open covers of $M$, there exists a sequence $\{U_n : n \in \omega\} \subset \tau(X)$ such that $U_n \in U_n$ for each $n \in \omega$ and $\bigcup_{n \in \omega} U_n = M$. Therefore the game $PO(S, M)$ is undetermined on a set $M \subset \mathbb{R}$ for every subbase $S$ of $M$ if $M$ is a $C''$-set. We will see later that the above implication cannot be reversed.
Theorem

Assume that a space $X$ has a pseudocompact crowded subspace. Then Player $O$ has a winning strategy in $PO(S, X)$ for any subbase $S$ in the space $X$.

Corollary

If $X$ is a compact space and $S$ is a subbase of $X$, then the following conditions are equivalent:

1. $X$ is scattered;
2. Player $P$ has a winning strategy in the game $PO(S, X)$;
3. Player $O$ has no winning strategy in the game $PO(S, X)$.
Theorem
Suppose that \(X \subset [0, 1]\) is a Bernstein set. Consider the families \(S_0 = \{[0, x] \cap X : x \in X\}\) and \(S_1 = \{[x, 1] \cap X : x \in X\}\); then \(S = S_0 \cup S_1\) is a subbase for the discrete topology on \(X\) and neither of the players has a winning strategy in the game \(PO(S, X)\).

Corollary
There exists a space \(X \subset [0, 1]\) such that \(PO(X)\) is determined on \(X\) but \(PO(S, X)\) is undetermined for some subbase \(S\) of \(X\). In particular, Pawlikowski’s characterization does not hold for the game \(PO(S, X)\).
Recall that a cardinal $\kappa$ is called measurable if there exists a free $\sigma$-complete ultrafilter on $\kappa$.

**Theorem**

If $\kappa$ is a measurable cardinal and $X$ is a discrete space of cardinality $\kappa$, then Player $O$ has a winning strategy in the game $PO(S, X)$ for any subbase $S$ of the space $X$. 
Open problems

1. Suppose that $X$ is a discrete space such that Player $O$ has a winning strategy in the game $PO(S, X)$ for every subbase $S$ in $X$. Must the cardinality of $X$ be measurable?

2. Suppose that $X$ is a discrete space of cardinality $2^c$. Does there exist a subbase $S$ in $X$ for which Player $O$ has no winning strategy in the game $PO(S, X)$?

3. Does there exist a pseudocompact space $X$ such that the games $PO(X)$ and $PO(S, X)$ are not equivalent for Player $O$ for some subbase $S$ in the space $X$?