A New Class of Dendrites Having Unique Second Symmetric Product

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A new class of dendrites having unique second symmetric product

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A *continuum* is a nonempty compact connected metric space.
The second symmetric product of a continuum $X$, $\mathcal{F}_2(X)$, is the hyperspace of all nonempty subsets of $X$ having at most two elements.

The hyperspace $\mathcal{F}_2(X)$ is a continuum with Hausdorff metric.
A continuum $X$ has unique second symmetric product provided that each continuum $Y$ such that $\mathcal{F}_2(Y)$ is homeomorphic to $\mathcal{F}_2(X)$ must be homeomorphic to $X$. 
Problem. Find condition on a continuum $X$, so that $X$ has unique second symmetric product.
A locally connected continuum contains no simple closed curve is called *dendrite*. 
Each element in following class of continua has unique second symmetric product.

- Dendrites whose set of end points is closed.
- Almost meshed dendrites.
- Meshed dendrites.
Each element in following class of continua has unique second symmetric product.
- Dendrites whose set of end points is closed.
- Almost meshed dendrites.
- Meshed dendrites.
- New class of dendrites.
A connected topological space $X$ is said to be \textit{unicoherent} provided that whenever $A$ and $B$ are connected closed subsets of $X$ such that $X = A \cup B$, then $A \cap B$ is connected.
A point $p$ in a unicoherent topological space $Y$ makes a hole in $Y$, if $Y - \{p\}$ is not unicoherent.
Theorem (Anaya, Maya, Orozco-Zitli (2016))

Let $X$ be a unicoherent locally connected continuum and $p, q \in X$. Then $\{p, q\}$ makes a hole in $\mathcal{F}_2(X)$ if and only if either $p = q$ and $X - \{p\}$ has at least three components or $p \neq q$ and both $X - \{p\}$ and $X - \{q\}$ are not connected.
\[ \mathcal{MH}(X) = \{ A \in \mathcal{F}_2(X) : A \text{ makes a hole in } \mathcal{F}_2(X) \} . \]

**Corollary.**

If \( X \) is a dendrite, then \( \mathcal{MH}(X) \)
\[ \mathcal{MH}(X) = \{ A \in \mathcal{F}_2(X) : A \text{ makes a hole in } \mathcal{F}_2(X) \} \].

**Corollary.**

If \( X \) is a dendrite, then \( \mathcal{MH}(X) = \{ A \in \mathcal{F}_2(X) - \mathcal{F}_1(X) : A \cap E(X) = \emptyset \} \cup \mathcal{F}_1(R(X)) \).
\[ \mathcal{NMH}(X) = \{ A \in \mathcal{F}_2(X) : A \text{ does not make a hole in } \mathcal{F}_2(X) \} \].

**Corollary.**

If \( X \) is a dendrite, then \( \mathcal{NMH}(X) \).
\[ \mathcal{NMH}(X) = \{ A \in \mathcal{F}_2(X) : A \text{ does not make a hole in } \mathcal{F}_2(X) \} \].

**Corollary.**

If \( X \) is a dendrite, then \( \mathcal{NMH}(X) = \mathcal{F}_1(O(X)) \cup \{ A \in \mathcal{F}_2(X) : A \cap E(X) \neq \emptyset \} \).
Theorem

If $X$ and $Y$ are dendrites such that $CR(X) = \cap\{A \in \mathcal{C}(X) : R(X) \subseteq A\}$ and there exists a homeomorphism $h : F_2(X) \rightarrow F_2(Y)$ satisfying that $h(F_1(R_N(X))) = F_1(R_N(Y))$, then $X$ and $Y$ are homeomorphic.
\[ h : \mathcal{F}_2(X) \to \mathcal{F}_2(Y) \text{ such that } h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y)). \]
\[ \varphi : R_N(X) \to R_N(Y) \] such that \( h(\{p\}) = \{\varphi(p)\} \).
\[ h(\mathcal{F}_1(\text{cl}_X R_N(X))) = \mathcal{F}_1(\text{cl}_Y R_N(Y)). \]
\[ \hat{\varphi} : cl_X R_N(X) \to cl_Y R_N(Y) \text{ such that } h(\{p\}) = \{\hat{\varphi}(p)\} \]
$J$ is component of $X - cl_X R_N(X)$ such that $J \cap E(X) = \emptyset$. 
\[ E(\text{cl}_X J) = \{p, q\} \subseteq \text{cl}_X R_N(X) \text{ and} \]
$\mathcal{F}_2(X)$
$\mathcal{F}_2(X)$

$p$

$q$
$\mathcal{F}_2(X)$

$\{p\}$  $\mathcal{F}_2(\text{cl}_X J)$  $\{q\}$
$\mathcal{J} = F_1(\text{cl}_X J)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{J}) = \{\{p\}, \{q\}\}$ and $\mathcal{J} - E(\mathcal{J}) \subseteq \mathcal{NMH}(X)$. 
$h(\mathcal{J})$ is an arc in $\mathcal{F}_2(Y)$ such that
$E(h(\mathcal{J})) = \{h(\{p\}), h(\{q\})\}$ and
$h(\mathcal{J}) - E(h(\mathcal{J})) \subseteq \mathcal{NMH}(Y)$. 
$a, b \in \text{cl}_Y R_N(Y)$ satisfying $h(\{p\}) = \{a\}$ and $h(\{q\}) = \{b\}$. 
Then the unique component $L$ of $Y - cl_Y R_N(Y)$ such that $E(cl_Y L) = \{a, b\}$ satisfies that $\mathcal{L} = F_1(cl_X L)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{L}) = \{\{a\}, \{b\}\}$ and $\mathcal{L} - E(\mathcal{L}) \subseteq \mathcal{N}\mathcal{M}\mathcal{H}(L)$. 
Then the unique component $L$ of $Y - cl_Y R_N(Y)$ such that $E(cl_Y L) = \{a, b\}$ satisfies that $\mathcal{L} = F_1(cl_X L)$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{L}) = \{\{a\}, \{b\}\}$ and $\mathcal{L} - E(\mathcal{L}) \subseteq \mathcal{NMH}(L)$. So, $h(\mathcal{J}) = \mathcal{L}$. 
$$h(\mathcal{F}_1(CR(X))) = \mathcal{F}_1(CR(Y))).$$
\[ h(\mathcal{F}_1(CR(X))) = \mathcal{F}_1(CR(Y)). \]

\[ \bar{\varphi} : CR(X) \to CR(Y) \text{ such that } h(\{x\}) = \{\bar{\varphi}(x)\}. \]
Theorem (Illanes, 2002)

If $X$ is a dendrite and $Z \in C(X)$ is such that $CR(X) \subseteq Z$ and each component of $X - CR(X)$ intersects $Z$, then $Z$ is homeomorphic to $X$. 
$K$ is component of $X - CR(X)$,
\[ \{p\} = CR(X) \cap (cl_X K) = (cl_X R_N(X)) \cap (cl_X K) \text{ and } K \cap E(X) = \{e\}. \]
$\mathcal{F}_2(X)$
$\mathcal{F}_2(X)$
\[ \mathcal{F}_2(X) \]

\[ \{p\} \]

\[ \mathcal{F}_2(\text{cl}_X K) \]
$\mathcal{K} = \mathcal{F}_1(\text{cl}_X K) \cup \{ A \in \mathcal{F}_2(\text{cl}_X K) : e \in A \}$ is the unique arc in $\mathcal{F}_2(X)$ such that $E(\mathcal{K}) = \{ \{p\}, \{e, p\} \}$, $\mathcal{K} - E(\mathcal{K}) \subseteq \mathcal{NMH}(X)$ and $\mathcal{K} - E(\mathcal{K})$ does not contain ramification points of $\mathcal{NMH}(X)$. 
$h(K)$ is the unique arc in $F_2(Y)$ such that
$E(h(K)) = \{h(\{p\}), h(\{e, p\})\}$, $h(K) - E(h(K)) \subseteq \mathcal{MH}(Y)$ and
$h(K) - E(h(K))$ does not contain ramification points of $\mathcal{MH}(Y)$. 
If $a \in cl_Y R_N(Y)$ such that $h(\{p\}) = \{a\}$, then there exists a component $G$ of $X - CR(X)$ such that $a \in cl_Y G$ and if $v \in (cl_Y G) \cap E(Y)$, then $h(\mathcal{K}) \subseteq \mathcal{F}_1(cl_Y G) \cup \{B \in \mathcal{F}_2(cl_Y G) : v \in B\} = \mathcal{G}$. 

\[ \]
$h(\mathcal{F}_1(\text{cl}_X K)) \cap \mathcal{F}_1(\text{cl}_Y G)$ is an arc contained in $\mathcal{F}_1(\text{cl}_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(\text{cl}_X K)) \cap \mathcal{F}_1(\text{cl}_Y G))$. 
\( h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G) \) is an arc contained in \( \mathcal{F}_1(cl_X G) \) such that \( \{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)) \). Let \( Y_K \in \mathcal{C}(cl_X G) \) such that \( \mathcal{F}_1(Y_K) = h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y K) \).
$h(\mathcal{F}_1(\text{cl}_X K)) \cap \mathcal{F}_1(\text{cl}_Y G)$ is an arc contained in $\mathcal{F}_1(\text{cl}_X G)$ such that $\{a\} \in E(h(\mathcal{F}_1(\text{cl}_X K)) \cap \mathcal{F}_1(\text{cl}_Y G))$. Let $Y_K \in \mathcal{C}(\text{cl}_X G)$ such that $\mathcal{F}_1(Y_K) = h(\mathcal{F}_1(\text{cl}_X K)) \cap \mathcal{F}_1(\text{cl}_Y K)$. $Y_Z = CR(Y) \cup \bigcup \{Y_K : K \text{ is component of } X - CR(X)\}$. 
\[ h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G) \] is an arc contained in \( \mathcal{F}_1(cl_X G) \) such that \( \{a\} \in E(h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y G)) \). Let \( Y_K \in \mathcal{C}(cl_X G) \) such that \( \mathcal{F}_1(Y_K) = h(\mathcal{F}_1(cl_X K)) \cap \mathcal{F}_1(cl_Y K) \). \( Y_Z = CR(Y) \cup \bigcup \{Y_K : K \text{ is component of } X - CR(X)\} \).

Thus, \( Y_Z \) and \( X \) are homeomorphic.
$F$ is a component of $Y - CR(Y)$ such that $a \in cl_Y F$. 
\[ X_F \in \mathcal{C}(cl_X I) \text{ such that } \mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I). \]
$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.

$X_Z = CR(X) \cup \bigcup \{X_F : F \text{ is a component of } Y - CR(Y)\}$. 
$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.

$X_Z = CR(X) \cup \bigcup \{X_F : F \text{ is a component of } Y - CR(Y)\}$.

$X_Z$ is homeomorphic to $Y$ and $X$. 
$X_F \in \mathcal{C}(cl_X I)$ such that $\mathcal{F}_1(Y_F) = h^{-1}(\mathcal{F}_1(cl_Y F)) \cap \mathcal{F}_1(cl_X I)$.

$X_Z = CR(X) \cup \bigcup \{X_F : F \text{ is a component of } Y - CR(Y)\}$.

$X_Z$ is homeomorphic to $Y$ and $X$. 
Problem. \( h(\mathcal{F}_1(R_N)) = \mathcal{F}_1(R_N(Y)) \).
The *multicoherence degree* of a connected topological space $Y$, $r(Y)$, is defined by

$$\sup \left\{ b_0(L \cap K) : \begin{array}{l}
L \text{ and } K \text{ are connected closed subset of } Y \\
\text{and } Y = L \cup K
\end{array} \right\} - 1.$$
The *multicoherence degree* of a connected topological space $Y$, $r(Y)$, is defined by

$$
\sup \left\{ b_0(L \cap K) : \begin{array}{l}
L \text{ and } K \text{ are connected closed subset of } Y \\
\text{and } Y = L \cup K
\end{array} \right\} - 1.
$$

$r(Y) = 0$ if and only if $Y$ is *unicoherent*. 
Theorem

If $X$ is a dendrite and $p \in R_N(X)$ is such that $\text{ord}(p, X) = n$, then

$$r(\mathcal{F}_2(X) - \{\{p\}\}) = \frac{(n - 1)(n - 2)}{2}.$$
Theorem

If \( X \) is a dendrite and \( p, q \in R_N(X) \cup O(X) \) are such that \( p \neq q \), \( \text{ord}(p, X) = n \) and \( \text{ord}(q, X) = m \), then

\[
r(F_2(X) - \{\{p, q\}\}) = (n - 1)(m - 1).
\]
For a dendrite $X$, set
\[ \Omega_X = \{ \text{ord}(p, X) : p \in R_N(X) \} \]
For a dendrite $X$, set
$\Omega_X = \{ \text{ord}(p, X) : p \in R_N(X) \}$

**Lemma**

If $X$ and $Y$ are dendrites such that there exists an homeomorphism $h : F_2(X) \to F_2(Y)$ and $\Omega_X \subseteq \{5, 6, \ldots\}$, then $\Omega_Y \subseteq \{5, 6, \ldots\}$. 
Theorem

Let $X$ and $Y$ be dendrites. If $|\Omega_X| = 1$, $\Omega_X \subseteq \{5, 6, \ldots\}$ and $h : F_2(X) \to F_2(Y)$ is a homeomorphism, then $h(F_1(R_N(X))) = F_1(R_N(Y))$. 
Theorem

Let $X$ and $Y$ be dendrites. If $h : \mathcal{F}_2 \to \mathcal{F}_2(Y)$ be a homeomorphism, $\Omega_X \subseteq \{5, 6 \ldots\}$ and

1. $\Omega_X \cap \left\{ \frac{(j-1)(j-2)}{2} + 1 : j \geq 5 \right\} = \emptyset,$

2. $\{(n - 1)(m - 1) : n, m \in \Omega_X\} \cap \left\{ \frac{(j-1)(j-2)}{2} : j \geq 5 \right\} = \emptyset,$

then $h(\mathcal{F}_1(R_N(X))) = \mathcal{F}_1(R_N(Y)).$
$F_2(X) \rightarrow F_2(Y)$ is a homeomorphism.
$h : \mathcal{F}_2(X) \rightarrow \mathcal{F}_2(Y)$ is a homeomorphism
\( a \in \mathbb{R}^N(Y) \Rightarrow \{a\} \in MH(Y) \).

\[ \text{ord}(a, Y) = m \geq 5 \Rightarrow r(F_2(Y) - \{\{a\}\}) = (m - 2)(m - 1)^2. \]

\( w, z \in X \) such that \( h(\{w, z\}) = \{a\} \Rightarrow \{w, z\} \in MH(X) \).
\(a \in R_N(Y)\)
\[a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{M}\mathcal{H}(Y).\]
\(a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{MH}(Y)\).

\(\text{ord}(a, Y) = m \geq 5 \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2}\).
\( a \in R_N(Y) \Rightarrow \{a\} \in \mathcal{MH}(Y). \)

\( \text{ord}(a, Y) = m \geq 5 \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2}. \)

\( w, z \in X \text{ such that } h(\{w, z\}) = \{a\} \Rightarrow \{w, z\} \in \mathcal{MH}(X) \)
\[ w, z \in O(X), w \neq z \Rightarrow r(F_2(X) - \{w, z\}) = 1. \]

\[ (m - 1)(m - 2)^2 = 1 \Rightarrow m = 3. \]
$w, z \in O(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = 1.$
\[ w, z \in O(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = 1. \]
\[ \Rightarrow \frac{(m-1)(m-2)}{2} = 1 \Rightarrow m = 3. \]
\[ w \in \mathbb{R}^N(X), \ z \in O(X) \Rightarrow r(F_2(X) - \{w, z\}) = \text{ord}(w, X) - 1. \]

\[ (m - 1)(m - 2)^2 + 1 = \text{ord}(w, X) \in \Omega_X. \]

(2) \[ \Omega_X \cap \{ (j - 1)(j - 2)^2 + 1 : j \geq 5 \} = \emptyset. \]
$w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = \text{ord}(w, X) - 1.$
\( w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = \text{ord}(w, X) - 1. \)

\( \Rightarrow \frac{(m-1)(m-2)}{2} + 1 = \text{ord}(w, X) \in \Omega_X. \)
\[ w \in R_N(X), z \in O(X) \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = \text{ord}(w, X) - 1. \]
\[ \Rightarrow \frac{(m-1)(m-2)}{2} + 1 = \text{ord}(w, X) \in \Omega_X. \]
(2) \[ \Omega_X \cap \left\{ \frac{(j-1)(j-2)}{2} + 1 : j \geq 5 \right\} = \emptyset. \]
\[
\forall w, z \in \mathbb{R}^N (X) \land w \neq z \Rightarrow r (F^2 (X) - \{w, z\}) = (\text{ord}(w, X) - 1)(\text{ord}(z, X) - 1)
\]

\[
\Rightarrow r (F^2 (Y) - \{a\}) = (m - 2)(m - 1)
\]

\[
2 \in \{(n - 1)(m - 1) : n, m \in \Omega_X\}
\]

\[
\{(n - 1)(m - 1) : n, m \in \Omega_X\} \cap \{(j - 1)(j - 2) : j \geq 5\} = \emptyset
\]
\[ w, z \in R_N(X), w \neq z \]
\[ w, z \in R_N(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = (\text{ord}(w, X) - 1)(\text{ord}(z, X) - 1) \]
\( w, z \in R_N(X), w \neq z \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = (\text{ord}(w, X) - 1)(\text{ord}(z, X) - 1) \)

\( \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2} \in \{(n-1)(m-1) : n, m \in \Omega_X\} \).
\( w, z \in R_N(X), w \neq z \)
\[ \Rightarrow r(\mathcal{F}_2(X) - \{\{w, z\}\}) = (\text{ord}(w, X) - 1)(\text{ord}(z, X) - 1) \]
\[ \Rightarrow r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-2)(m-1)}{2} \in \{(n-1)(m-1) : n, m \in \Omega_X\}. \]

(1) \( \{(n-1)(m-1) : n, m \in \Omega_X\} \cap \left\{ \frac{(j-1)(j-2)}{2} : j \geq 5 \right\} = \emptyset. \)
$w = z \in R_N(X)$
\( \text{ord}(z, X) = n \Rightarrow r(\mathcal{F}_2(X) - \{\{z\}\}) = \frac{(n-1)(n-2)}{2} \)
\[ ord(z, X) = n \Rightarrow r(\mathcal{F}_2(X) - \{\{z\}\}) = \frac{(n-1)(n-2)}{2} \]
\[ r(\mathcal{F}_2(Y) - \{\{a\}\}) = \frac{(m-1)(m-2)}{2} \]
ord(\(z, X\)) = n \Rightarrow r(\(\mathcal{F}_2(X) - \{\{z\}\}\)) = \frac{(n-1)(n-2)}{2}

r(\(\mathcal{F}_2(Y) - \{\{a\}\}\)) = \frac{(m-1)(m-2)}{2}

\Rightarrow \frac{(n-1)(n-2)}{2} = \frac{(m-1)(m-2)}{2} \Rightarrow m = n
\[ \mathcal{F}_1(R_N(Y)) \subseteq h(\mathcal{F}_1(R_N(X))) \]
Theorem

Let \( X \) be a dendrite. If \( CR(X) = \cap \{ Z \in C(X) : R_N(X) \subseteq Z \} \), \( \Omega_X \subseteq \{5, 6, \ldots\} \) and either

1. \[ \Omega_X \cap \left\{ \frac{(j-1)(j-2)}{2} + 1 : j \geq 5 \right\} = \emptyset, \]

2. \[ \{(n - 1)(m - 1) : n, m \in \Omega_X\} \cap \left\{ \frac{(j-1)(j-2)}{2} : j \geq 5 \right\} = \emptyset, \]

then \( X \) has unique second symmetric product.
Thank you!