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A COMPARISON OF THREE TOPOLOGIES ON ORDERED SETS DRAFT OF FEBRUARY 2, 2007

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Abstract. We introduce two new topologies on ordered sets: the way below topology and weakly way below topology. These are similar in definition to the Scott topology, but are very different if the set is not continuous. The basic properties of these three topologies are compared. We will show that while domain representable spaces must be Baire, this is not the case with the new topologies.

1. Introduction

In 1970 Dana Scott published a model for information systems which used ordered sets with a topology, called the Scott topology, and a relation, called the way below relation, to capture the ideas of completeness and approximation of information. Topologically, these spaces are not very interesting because they are usually no more than T_0. However, the set of maximal elements of such spaces, in its relative Scott topology, can be homeomorphic to very nice topological spaces. In this case, the ordered set is said to model the topological space, or that the topological space is representable. Before proceeding with the background information, we need to review some definitions.

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An order is a relation that is either asymmetric and transitive ($<$) or reflexive, antisymmetric, and transitive ($\leq$). If X is an ordered set and $a \in X$ then $\Uparrow a = \{b \in X : a \leq b\}$ and $\Downarrow a = \{b \in X : b \leq a\}$. Also, $\downarrow a = \{b \in X : b < a\}$ and $\uparrow a = \{b \in X : a < b\}$. If $A \subseteq X$ then $\Uparrow A = \bigcup_{a \in A} \Uparrow a$ and $\Downarrow A = \bigcup_{a \in A} \Downarrow a$. When $A = \Uparrow A$ we say that A is increasing, and when $A = \Downarrow A$ we say that A is decreasing. A subset D of X is directed if and only if for every $a, b \in D$ there is $c \in D$ such that $a, b \leq c$. X is a directed complete ordered set (dcos) if and only if every nonempty directed subset of X has a supremum in X. This property gives the information models of Scott their completeness. A subset U of X is Scott-open if and only if U is increasing and, for every directed $D \subseteq X$, if $\sup D \in U$ then $D \cap U \neq \emptyset$. The Scott topology on X is the collection of all Scott-open subsets of X. Every ordered set admits a Scott topology.

An important relation defined by Scott in connection with the information models is the way below relation. An element a of X is way below an element b (denoted $a \ll b$) if and only if for every directed subset D of X, if $\sup D \geq b$ then $D \cap \Uparrow a \neq \emptyset$. One can think of a as being an essential piece of information that approximates b. For every $a \in X$ let $\twoheaddownarrow a = \{b \in X : b \ll a\}$ and $\twoheaduparrow a = \{b \in X : a \ll b\}$. This ability to approximate is a useful and desirable aspect of the model. To ensure that all elements of the model can be built from approximations, the ordered set is normally assumed to be continuous. An ordered set X is continuous if and only if for every $a \in X$, $\twoheaddownarrow a$ is directed and has supremum a. One important consequence of continuity is that if X is a continuous ordered set then $\{\twoheaduparrow a : a \in X\}$ is a basis for the Scott topology on X. An ordered set that is continuous and directed complete is called a domain. For more information on the basics of the Scott topology, the way below relation, and domains see [1], [7], [10], [12], [13], and [14]

A topological space T is domain representable if and only if there is a domain X such that the set $\max X$ of maximal elements of X, with its relative Scott topology, is homeomorphic to T. It is known that domain representable spaces must be Baire. Here is one way to obtain this result. A closed subset of a topological space T is irreducible if and only if it is not the union of two proper closed subsets. T is sober if and only if every irreducible closed subset of T

is the closure of a single point. Every domain is locally compact and sober ([1]) and every locally compact sober space is Choquet complete ([12]). This means that every domain is Baire, which implies that its set of maximal elements is Baire ([12]). For more information on domain representability see [9], [11], [12], [2], [3], [4], and [5].

We now return to the more general discussion. It is not always that case that the way below relation gives a model the desired approximation properties. It happens that models developed by Coecke and Martin ([6]) for classical and quantum states in physics are not continuous, and the way below relation fails miserably to describe approximation. They therefore made a slight change in the definition of this relation to create a new relation which we will call the *weakly way below relation*.

Definition 1.1. For every a, b in an ordered set X, $a \ll_w b$ if and only if for every directed subset D of X, if $\sup D = b$ then $D \cap \updownarrow a \neq \emptyset$.

If X is continuous, then this relation is the same as the way below relation, and both of them provide a basis for the Scott topology. Without continuity, $\{\twoheaduparrow a : a \in X\}$ and $\{\twoheaduparrow_w a : a \in X\}$ may not generate a topology at all and if they do, the topologies could be very different than the Scott topology and each other. It is the purpose of this paper to compare the basic properties of these three topologies with the specific view of trying to determine whether there are spaces that can be represented by the new topologies that are not domain representable. We will show in section 4 that, in fact, there is a space that is representable by these new topologies that is not Baire. In section 2 we will examine the properties of the way below topology for noncontinuous sets. In section 3 the basic properties of the weakly way below relation and topology are discussed. An example is presented in section 4 which shows that even in the Scott topology, things can be quite different without continuity. Throughout the paper, X and Y will represent ordered sets.

Before we move on to section 2 let us note that a subset C of X is Scott-closed if and only if C is decreasing and closed under the suprema of directed subsets. Also, if X and Y are equipped with their Scott topologies and $f : X \to Y$ then f is continuous if and only if f preserves the suprema of directed sets. This means that f must also be increasing, that is, if $a \leq b$ then $f(a) \leq f(b)$.

2. The Way Below Topology

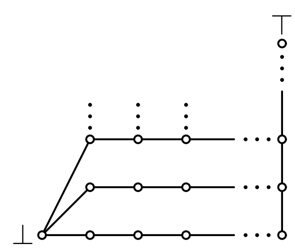
Definition 2.1. If $\{\Uparrow a : a \in X\}$ generates a topology then it will be called the *way below topology* or the *wb topology.*

It follows from the definition of the way below relation that wb-open sets must be increasing. However, they need not capture directed sets whose suprema they contain, which is unexpected. This is seen by the following example.

Example 2.2. Let $X = \{\bot, \top\} \cup [(\omega + 1) \times \omega]$. Define an order on X by setting $p < q$ if and only if one of the following conditions is satisfied.

(1) $p = \bot$ and $q \neq \bot$
(2) $p \neq \top$ and $q = \top$
(3) $p = \langle a, m\rangle$, $q = \langle b, m\rangle$, and $a < b$
(4) $p = \langle a, m\rangle$, $q = \langle \omega, n\rangle$, and $m < n$.

X can be represented by the following figure. In this, and all other figures, circles represents elements of the set, and lines indicate order relation, with larger being up and/or to the right.



In this example, $\Uparrow\bot = X$ and $\Uparrow\top = \emptyset$ while $\Uparrow a = \{\top\}$ for all other $a \in X$. Therefore $\{\omega\} \times \omega$ is a chain, and thus a directed set, whose supremum is an element of $\{\top\}$, but which does not intersect $\{\top\}$.

This example also shows that without continuity, another important property of the way below relation is lost.

Definition 2.3. A relation R on a set S is *interpolative* in S if and only if for every $a, b \in S$, if aRb then there is $c \in S$ such that aRc and cRb.

The way below relation is always interpolative in a domain. However, $\ll$ is not interpolative in the last example, which is a noncontinuous dcos. Here $\langle \omega, 0 \rangle \ll \top$ but there is no $p \in X$ such that $\langle \omega, 0 \rangle \ll p \ll \top$. The kind of behavior exhibited in Example 2.2 cannot happen if $\ll$ is interpolative.

Theorem 2.4. *If $\ll$ is interpolative in X and X admits a wb topology then the Scott topology is finer than the wb topology in X.*

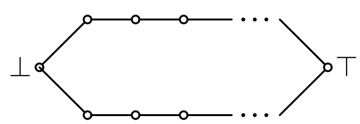
Proof. Let $a \in X$ and let D be a directed subset of X such that $\sup D = b \in \mathop{\Uparrow} a$. There is $c \in X$ such that $a \ll c \ll b$. Let $d \in D$ such that $c \leq d$. Then $a \ll d$, so $D \cap \mathop{\Uparrow} a \neq \emptyset$. Therefore $\mathop{\Uparrow} a$ is Scott-open. $\square$

However, the wb topology and the Scott topology need not be the same, even when $\ll$ is interpolative.

Example 2.5. Let $X = \{\bot, \top\} \cup (\omega \times 2)$. Define an order on X by setting $p < q$ if and only if one of the following conditions is satisfied.

(1) $p = \bot$ and $q \neq \bot$
(2) $p \neq \top$ and $q = \top$
(3) $p = \langle m, i \rangle$ and $q = \langle n, i \rangle$ for some $m < n$ and $i \in 2$

X can be represented by the following figure.



In this example $p \ll q$ if and only if $p = \bot$. Therefore, if $p \ll q$ then $p \ll p \ll q$ and $\ll$ is interpolative. The only nonempty wb-open set is X, but $X - \{\bot\}$ is Scott-open.

A good characterization of wb-open sets is not known when $\ll$ is not interpolative. The next lemma and theorem give a characterization when $\ll$ is interpolative. The proof of the lemma is straightforward.

Lemma 2.6. *If $\ll$ is interpolative in X and X admits a wb topology then $\mathop{\Uparrow}(\mathop{\Uparrow} a) = \mathop{\Uparrow} a$ for all $a \in X$.*

Theorem 2.7. *Let X be an ordered set in which $\ll$ is interpolative and which admits a wb topology. A subset U of X is wb-open if and only if $U = \twoheaduparrow U$.*

Proof. If $U = \twoheaduparrow U$ it is obvious that U is wb-open. So assume that U is a wb-open subset of X. For every $b \in U$ there is $a_b \in X$ such that $b \in \twoheaduparrow a_b \subseteq U$. Then

$$\twoheaduparrow U = \twoheaduparrow \left(\bigcup_{b \in U} \twoheaduparrow a_b \right) = \bigcup_{b \in U} \twoheaduparrow (\twoheaduparrow a_b) = \bigcup_{b \in U} \twoheaduparrow a_b = U$$

□

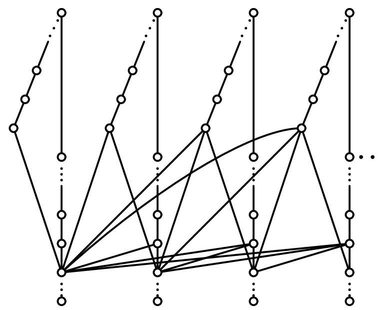
From here we can get local compactness, which is one of the ingredients for the Baire property. This is because $\underline{\twoheaduparrow}\, a = \{a\} \cup \twoheaduparrow a$ is compact.

Theorem 2.8. *If $\ll$ is interpolative in X and X admits a wb topology then X is locally compact in the wb topology.*

Proof. Let U be wb-open in X and let $b \in U$. By Theorem 2.7 there is $a \in U$ such that $b \in \twoheaduparrow a \subseteq \underline{\twoheaduparrow}\, a \subseteq U$. □

It helps to know something about wb-closed sets in order to determine whether such spaces must be sober. Unfortunately, there is not much that can be said about them. They do have to be decreasing, but Example 2.2 shows that if $\ll$ is not interpolative then wb-closed sets need not be closed under suprema of directed sets. If $\ll$ is interpolative, then wb-open sets are Scott-open, so wb-closed sets are Scott-closed. In this case, wb-closed sets are closed under directed suprema. One rather obvious property which should be noted is that the wb-closure of a set A is the set of points p such that $\twoheaddownarrow p \subseteq \twoheaddownarrow A$. One consequence is that, unlike the Scott topology, $\overline{\twoheaddownarrow}\, a$ need not be wb-closed. Another is that we lose sobriety, as is shown by the next example.

Example 2.9. Let $X = \omega \times (\omega \cdot 3 + 1)$. We will order X to match the following diagram.



Define the order on X by setting $p < q$ if and only if one of the following conditions is met.

(1) $p = \langle m, \alpha \rangle$ for some $\alpha \leq \omega$ and $q = \langle m, \beta \rangle$ for some $\beta > \alpha$.
(2) $p = \langle m, \alpha \rangle$ for some $\alpha \leq \omega$, and $q = \langle n, \beta \rangle$ for some $n > m$ and some $\beta > \omega$.
(3) $p = \langle m, \alpha \rangle$ for some $\omega < \alpha \leq \omega \cdot 2$ and $q = \langle m, \beta \rangle$ where either $\alpha < \beta \leq \omega \cdot 2$ or $\beta = \omega \cdot 3$.
(4) $p = \langle m, \alpha \rangle$ for some $\omega \cdot 2 < \alpha < \omega \cdot 3$ and $q = \langle m, \beta \rangle$ for some $\alpha < \beta \leq \omega \cdot 3$.

For every $n \in \omega$ let $X_n = \{n\} \times (\omega \cdot 3 + 1)$. If $m < n$ then $\langle m, \alpha \rangle < \langle n, \beta \rangle$ if and only if $\alpha \leq \omega < \beta$. Therefore, if C is a chain in X then $C \cap X_n \neq \emptyset$ for at most two values of n. Also note that every directed subset of X_n has a supremum in X_n.

Let $D \subseteq X$ be directed. If D is a chain then there is $a \in D$ and $n \in \omega$ such that $D \cap \mapsup a \subseteq X_n$. Therefore D has a supremum. Assume that D is not a chain and that there are $k, m \in \omega$ such that $k < m$, $D \cap X_k \neq \emptyset$, and $D \cap X_m \neq \emptyset$. Let $a \in D \cap X_k$ and $b \in D \cap X_m$. There is $c \in D$ such that $a, b \leq c$. Then $c \in X_n$ for some $n \geq m$ and $c = \langle n, \alpha \rangle$ for some $\alpha > \omega$. The only points of X larger than c lie in X_n. Therefore $D \cap X_n$ is directed and $\sup(D \cap X_n) = \sup D$. Thus X is directed complete.

To verify that X admits a wb topology we need to know what $\twoheaduparrow a$ is for all $a \in X$. Let $m \in \omega$.

(1) If $\alpha < \omega$ then $\twoheaduparrow \langle m, \alpha \rangle = \mapsup \langle m, \alpha \rangle$.
(2) $\twoheaduparrow \langle m, \omega \rangle = \uparrow \langle m, \omega \rangle$
(3) If $\omega < \alpha \leq \omega \cdot 2$ then $\twoheaduparrow \langle m, \alpha \rangle = \emptyset$.
(4) If $\omega \cdot 2 < \alpha < \omega \cdot 3$ then $\twoheaduparrow \langle m, \alpha \rangle = \mapsup \langle m, \alpha \rangle$.
(5) $\twoheaduparrow \langle m, \omega \cdot 3 \rangle = \emptyset$

A little checking using the above results will show that if $a, b \in X$ then there is $c \in X$ such that $\twoheaduparrow a \cap \twoheaduparrow b = \twoheaduparrow c$. Therefore X admits a wb topology.

Let $C = \{\langle m, \alpha\rangle : m \in \omega \text{ and } \alpha \leq \omega \cdot 2\}$. C is closed because it is equal to $X - \bigcup_{n\in\omega} \twoheaduparrow\langle n, \omega \cdot 2\rangle$. To see that C is irreducible, consider the following list. Let $n \in \omega$

(1) If $\alpha < \omega$ then $\twoheaddownarrow\langle n, \alpha\rangle = \mapsdown\, \langle n, \alpha\rangle$.
(2) $\twoheaddownarrow\langle n, \omega\rangle = \downarrow \langle n, \alpha\rangle$
(3) If $\omega < \alpha \leq \omega \cdot 2$ then $\twoheaddownarrow\langle n, \alpha\rangle = \bigcup_{m\leq n} \mapsdown\, \langle m, \omega\rangle$.
(4) If $\omega \cdot 2 < \alpha < \omega \cdot 3$ then $\twoheaddownarrow\langle n, \alpha\rangle = \mapsdown\, \langle n, \alpha\rangle$.
(5) $\twoheaddownarrow\langle n, \omega \cdot 3\rangle = \{\langle n, \alpha\rangle : \omega \cdot 2 < \alpha < \omega \cdot 3\} \cup \bigcup_{m\leq n} \mapsdown\, \langle m, \omega\rangle$

Let A and B be closed subsets of C such that $C = A \cup B$. Assume that there is $m \in \omega$ such that $\langle m, \omega \cdot 2\rangle \notin A$. If $\langle n, \omega \cdot 2\rangle \in A$ for some $n > m$ then $\twoheaddownarrow\langle m, \omega \cdot 2\rangle \subseteq \twoheaddownarrow\langle n, \omega \cdot 2\rangle \subseteq A$, so $\langle m, \omega \cdot 2\rangle \in A$, a contradiction. Therefore $\omega \times \{\omega \cdot 2\}$ must be a subset of B. But B is decreasing, so $B = C$. Therefore C is irreducible. But C cannot be the closure of a single point, again because closed sets are decreasing. Thus X is not sober.

Note that $\ll$ is not interpolative in X because $\langle m, \omega\rangle \ll \langle m, \omega \cdot 2\rangle$ but there is no $a \in X$ such that $\langle m, \omega\rangle \ll a \ll \langle m, \omega \cdot 2\rangle$. Finally, X is Baire, because every open dense subset of X must contain all the maximal elements of X, and also the set of maximal elements of X is discrete in its relative wb topology.

Before moving on to the next section, let us take a brief look at continuous functions under the wb topology. The next example shows that the Scott-continuity of a function does not imply that it is wb-continuous.

Example 2.10. Let X be the space of Example 2.5. Give $2 = \{0, 1\}$ its usual order and define $f : X \to 2$ by $f(\bot) = 0$ and $f(a) = 1$ for all $a \neq \bot$. Then f is Scott-continuous, but is not wb-continuous because $\{1\}$ is wb-open in 2, but $f^{-1}[\{1\}] = X - \{\bot\}$ is not wb-open in X.

Because the wb topology on the space X used in the last example is the indiscrete topology, we can easily define a wb-continuous function from X into itself which does not preserve order or the suprema of a directed set.

3. The Weakly Way Below Relation and Topology

The following properties follow easily from the definition of $\ll_w$ and are all surely known to everyone who has seen this relation.

Theorem 3.1. *Let $a, b, c \in X$.*

(1) $a \ll b \implies a \ll_w b$
(2) $a \ll_w b \implies a \leq b$
(3) $a \leq b \ll_w c \implies a \ll_w c$
(4) $a \ll_w b \ll_w c \implies a \ll_w c$

There is one property enjoyed by $\ll$ which $\ll_w$ does not have. If $a \ll b \leq c$ then $a \ll c$. This is not necessarily true for $\ll_w$, which has a big impact on the topology generated by $\ll_w$.

Definition 3.2. For every $a \in X$ let $\downarrow_w a = \{b \in X : b \ll_w a\}$ and $\uparrow_w a = \{b \in X : a \ll_w b\}$.

We do not know that $\uparrow_w a$ is increasing. In fact, if $\uparrow_w a$ is increasing for every $a \in X$ then $\ll_w = \ll$, as Coecke and Martin noted in [6]. In this paper, Martin has defined a weaker version of the increasing property which is useful.

Definition 3.3. A relation R is *weakly increasing* in an ordered set X if and only if for every $a, b, c \in X$ if $aRb \leq c$ and there is $d \in X$ such that cRd then aRc.

The following property, defined by Keye Martin in [6], is the weakly way below relation's version of continuity.

Definition 3.4. X is *exact* if and only if for every $a \in X$, $\downarrow_w a$ is directed and $\sup \downarrow_w a = a$.

Definition 3.5. A *weak domain* is an exact dcos in which $\ll_w$ is weakly increasing.

Exactness does not have much effect on $\ll$. All the examples we have given so far have been exact. Two important properties follow from exactness. The first involves interpolation, and the second is the existence of a topology.

Theorem 3.6. *If X is a weak domain then $\ll_w$ is interpolative in X.*

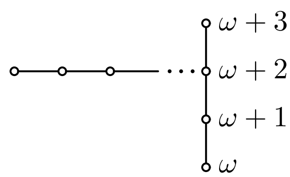
This theorem is due to Martin. We will include a proof because the author has not found one in the literature, and because it follows the proof that $\ll$ is interpolative in continuous dcos, which is also hard to find.

Proof. Let $a, c \in X$ such that $a \ll_w c$. We will show that $\twoheaddownarrow_w(\twoheaddownarrow_w c)$ is directed and has supremum c. Let $p, q \in \twoheaddownarrow_w(\twoheaddownarrow_w c)$. There are $r, s \ll_w c$ such that $p \ll_w r$ and $q \ll_w s$. But $\twoheaddownarrow_w c$ is directed, so there is $t \in \twoheaddownarrow_w c$ such that $r, s \leq t$. Then $p \ll_w r \leq t \ll_w c$ and $q \ll_w s \leq t \ll_w c$. Since $\ll_w$ is weakly increasing, we have $p, q \ll_w t$. But $\twoheaddownarrow_w t$ is directed so there is $u \in \twoheaddownarrow_w t$ such that $p, q \leq u$. Obviously $u \in \twoheaddownarrow_w(\twoheaddownarrow_w c)$, so $\twoheaddownarrow_w(\twoheaddownarrow_w c)$ is directed.

Now X is a dcos, so $\twoheaddownarrow_w(\twoheaddownarrow_w c)$ has a supremum s. Since $\twoheaddownarrow_w(\twoheaddownarrow_w c) \subseteq \twoheaddownarrow_w c$ we know that c is an upper bound of $\twoheaddownarrow_w(\twoheaddownarrow_w c)$ and that $s \leq c$. Also, if $b \in \twoheaddownarrow_w c$ then s is an upper bound of $\twoheaddownarrow_w b$. Therefore $b \leq s$. It follows that s is an upper bound of $\twoheaddownarrow_w c$ and that $c \leq s$. Since $\twoheaddownarrow_w(\twoheaddownarrow_w c)$ is directed and has supremum c, there must be $x \in \twoheaddownarrow_w(\twoheaddownarrow_w c)$ such that $a \leq x$. Let $b \in \twoheaddownarrow_w c$ such that $x \ll_w b$. Then $a \ll_w b \ll_w c$. $\square$

The following example shows that X can be interpolative without $\ll_w$ being weakly increasing.

Example 3.7. Let $X = \omega + 4$ and define an order X by giving $\omega \cup \{\omega + 2, \omega + 3\}$ its usual order, and declaring $\omega < \omega + 1 < \omega + 2$. X can be represented by the following figure.



Here $\omega \ll_w \omega + 1 \leq \omega + 2 \ll_w \omega + 3$, but $\omega \not\ll_w \omega + 2$. It is easy to check that $\ll_w$ is interpolative.

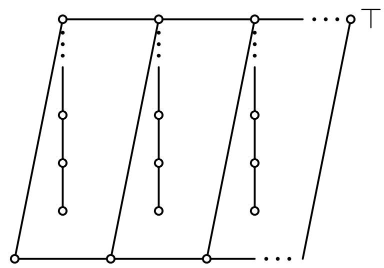
Theorem 3.8. *If X is exact then $\{\twoheaduparrow_w a : a \in X\}$ is a basis for a topology on X.*

Proof. If $b \in X$ then $\twoheaddownarrow_w b$ is a directed set with supremum b so there is $a \in X$ such that $b \in \twoheaduparrow_w a$.

Let $a, b, c \in X$ with $c \in (\twoheaduparrow_w a) \cap (\twoheaduparrow_w b)$. Then $a, b \in \twoheaddownarrow_w c$, which is directed, and there is $d \ll_w c$ such that $a, b \leq d$. If $d \ll_w e$ then $a \ll_w e$ and $b \ll_w e$, so $c \in \twoheaduparrow_w d \subseteq (\twoheaduparrow_w a) \cap (\twoheaduparrow_w b)$. $\square$

It is possible for $\{\twoheaduparrow_w a : a \in X\}$ to generate a topology on X even when X is not exact. Whenever $\{\twoheaduparrow_w a : a \in X\}$ does generate a topology, we will call it the *weakly way below topology* or *wwb topology*. Obviously wwb-open sets do not have to be increasing. Example 2.2 shows that wwb-open sets need not capture directed sets whose suprema they contain. The next example also shows this, but it has more properties.

Example 3.9. Let $X = (\omega \times 2) \cup (\omega \times \{1\} \times \omega) \cup \{\top\}$. We will define an order on X to match the following diagram.



For every $p, q \in X$ set $p < q$ if and only if one of the following conditions is met.

(1) $p \neq \top$ and $q = \top$
(2) $p \neq q$, $p = \langle m, i \rangle$, $q = \langle n, j \rangle$, $m \leq n$, and $i \leq j$
(3) $p = \langle m, 1, k \rangle$, $q = \langle n, 1 \rangle$, and $m \leq n$.

For every $n \in \omega$, $\{n\} \times \{1\} \times \omega$ is a directed set whose supremum is $\langle n, 1 \rangle$. Also, $\omega \times \{0\}$ and $\omega \times \{1\}$ are both directed sets whose supremum is $\top$.

(1) $\twoheaddownarrow_w \top = \omega \times \{0\}$ and $\twoheaduparrow_w \top = \emptyset$
(2) For every $n \in \omega$, $\twoheaddownarrow_w \langle n, 1 \rangle = \{n\} \times \{1\} \times \omega$ and $\twoheaduparrow_w \langle n, 1 \rangle = \emptyset$.
(3) For every $m, k \in \omega$, $\twoheaddownarrow_w \langle m, 1, k \rangle = \mapsdown \langle m, 1, k \rangle$ and $\twoheaduparrow_w \langle m, 1, k \rangle = \{\langle m, 1, n \rangle : k \leq n\} \cup \{\langle m, 1 \rangle\}$.
(4) For every $m \in \omega$, $\twoheaddownarrow_w \langle m, 0 \rangle = \mapsdown \langle m, 0 \rangle$ and $\twoheaduparrow_w \langle m, 0 \rangle = \{\langle n, 0 \rangle : m \leq n\} \cup \{\top\}$.

X is a weak domain. But $U = (\omega \times \{0\}) \cup \{\top\}$ is a wwb-open subset of X, $\sup(\omega \times \{1\}) = \top$, and $(\omega \times \{1\}) \cap U = \emptyset$.

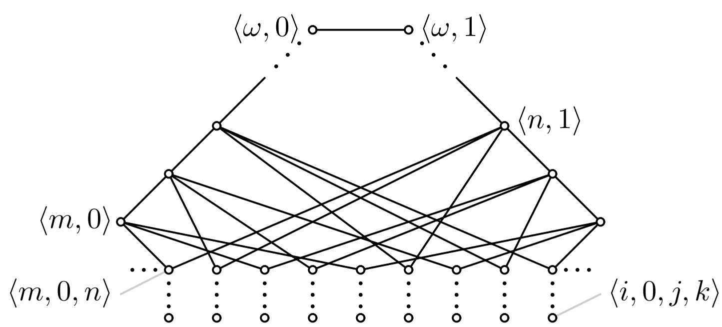
The next theorem shows that in exact sets the relation between the Scott topology and the wwb topology is reversed from that of the Scott topology and the wb topology.

Theorem 3.10. *If X is exact then the wwb topology on X is finer than the Scott topology.*

Proof. Let $U \subseteq X$ be Scott-open and let $b \in U$. Now $\Downarrow_w b$ is directed and $b = \sup \Downarrow_w b$, so there is $a \in (\Downarrow_w b) \cap U$. Since U is increasing we have $b \in \Uparrow_w a \subseteq \uparrow a \subseteq U$. $\square$

So if X is exact and $\ll$ is interpolative in X then the wwb topology is finer that the wb topology. In the next example, the Scott and wb topologies are equal, while the wwb topology is strictly finer.

Example 3.11. Set $X = [(\omega+1)\times 2]\cup(\omega\times\{0\}\times\omega)\cup(\omega\times\{0\}\times\omega\times\omega)$. We will define an order on X to match the following diagram.



Define the order on X by the following rules. Here $m, j, k \in \omega$ and $i \in 2$.

(1) $p < \langle \omega, 1\rangle$ for all $p \in X - \{\langle \omega, 1\rangle\}$.
(2) $p < \langle \omega, 0\rangle$ for all $p \in X - [(\omega+1)\times\{1\}]$.
(3) $\langle m, i\rangle < \langle n, i\rangle$ for all $n \geq m$.
(4) $\langle m, 0, j\rangle < \langle n, 1\rangle$ for all $n \geq j$.
(5) $\langle m, 0, j\rangle < \langle n, 0\rangle$ for all $n \geq m$.
(6) $\langle m, 0, j, k\rangle < \langle n, 1\rangle$ for all $n \geq j$.
(7) $\langle m, 0, j, k\rangle < \langle n, 0\rangle$ for all $n \geq m$.
(8) $\langle m, 0, j, k\rangle < \langle m, 0, j\rangle$.
(9) $\langle m, 0, j, k\rangle < \langle m, 0, j, n\rangle$ for all $n \geq k$.

For $i = 0, 1$, $\omega \times \{i\}$ is a chain whose supremum is $\langle \omega, i\rangle$. The set $\omega \times \{0\} \times \omega$ is an antichain. If $p \in \omega \times \{0\}$ and $q \in \omega \times \{1\}$ then there is $r \in \omega \times \{0\} \times \omega$ such that $r < p, q$. For every $r \in \omega \times \{0\} \times \omega$ there are $p \in \omega \times \{0\}$ and $q \in \omega \times \{1\}$ such that $r < p, q$. For every $m, j \in \omega$ the set $\{m\} \times \{0\} \times \{j\} \times \omega$ is a chain whose supremum is $\langle m, 0, j\rangle$. X is a directed complete ordered set.

Let $m, j, k \in \omega$ and $i \in 2$.

(1) $\twoheaduparrow\langle \omega, i\rangle = \twoheaduparrow\langle m, 0\rangle = \emptyset$
(2) $\twoheaduparrow\langle m, 1\rangle = \mapsup \langle m, 1\rangle$
(3) $\twoheaduparrow\langle m, 0, j\rangle = \uparrow \langle m, 0, j\rangle = (\mapsup \langle m, 0\rangle) \cup (\mapsup \langle j, 1\rangle)$
(4) $\twoheaduparrow\langle m, 0, j, k\rangle = \mapsup \langle m, 0, j, k\rangle$

A little checking will show that X admits a wb topology and that $\ll$ is interpolative in X. So the Scott topology is finer than the wb topology. Let $U \subseteq X$ be Scott-open.

If $\langle \omega, 1\rangle \in U$ then there is $m \in \omega$ such that $\langle m, 1\rangle \in U$. But $\twoheaduparrow\langle m, 1\rangle = \mapsup \langle m, 1\rangle$ so $\langle \omega, 1\rangle \in \twoheaduparrow\langle m, 1\rangle \subseteq U$.

If $\langle \omega, 0\rangle \in U$ then $\langle \omega, 1\rangle \in U$. There are $j, m \in \omega$ such that $\langle j, 1\rangle \in U$ and $\langle m, 0\rangle \in U$. But $\twoheaduparrow\langle m, 0, j\rangle = (\mapsup \langle m, 0\rangle) \cup (\mapsup \langle j, 1\rangle)$ so $\langle \omega, 0\rangle \in \twoheaduparrow\langle m, 0, j\rangle \subseteq U$.

If $\langle m, 1\rangle \in U$ then $\langle m, 1\rangle \in \twoheaduparrow\langle m, 1\rangle \subseteq U$.

If $\langle m, 0\rangle \in U$ then $\mapsup \langle m, 0\rangle \subseteq U$. In particular, $\langle \omega, 1\rangle \in U$. Thus there is $j \in \omega$ such that $\langle j, 1\rangle \in U$ and therefore $\mapsup \langle j, 1\rangle \subseteq U$. So $\langle m, 0\rangle \in \twoheaduparrow\langle m, 0, j\rangle \subseteq U$.

If $\langle m, 0, j\rangle \in U$ then $\mapsup \langle m, 0, j\rangle \subseteq U$. Also, there is $k \in \omega$ such that $\langle m, 0, j, k\rangle \in U$. But then $\mapsup \langle m, 0, j, k\rangle \subseteq U$, so $\langle m, 0, j\rangle \in \twoheaduparrow\langle m, 0, j, k\rangle \subseteq U$.

If $\langle m, 0, j, k\rangle \in U$ then $\mapsup \langle m, 0, j, k\rangle \subseteq U$ and $\langle m, 0, j, k\rangle \in \twoheaduparrow\langle m, 0, j, k\rangle \subseteq U$. Therefore the Scott and wb topologies are the same.

We next turn our attention to the weakly way below relation. Again let $m, j, k \in \omega$.

(1) $\twoheaddownarrow_w\langle \omega, 1\rangle = (\omega \times \{1\}) \cup (\omega \times \{0\} \times \omega) \cup (\omega \times \{0\} \times \omega \times \omega)$
(2) $\twoheaddownarrow_w\langle \omega, 0\rangle = \downarrow \langle \omega, 0\rangle$
(3) $\twoheaddownarrow_w\langle m, 1\rangle = \mapsdown \langle m, 1\rangle$
(4) $\twoheaddownarrow_w\langle m, 0\rangle = \mapsdown \langle m, 0\rangle$
(5) $\twoheaddownarrow_w\langle m, 0, j\rangle = \downarrow \langle m, 0, j\rangle$
(6) $\twoheaddownarrow_w\langle m, 0, j, k\rangle = \mapsdown \langle m, 0, j, k\rangle$

It is easy to see from this list that X is exact and so it admits a wwb topology. A calculation of $\twoheaduparrow_w a$ for all $a \in X$ will show that $\twoheaduparrow_w a$ is increasing for all $a \in X - (\omega \times \{0\})$. Now $\uparrow \langle m, 0\rangle - \twoheaduparrow_w \langle m, 0\rangle = \{\langle \omega, 1\rangle\}$ and $\twoheaduparrow_w \langle \omega, 1\rangle = \emptyset$. It follows that $\ll_w$ is weakly increasing in X. Also, $\twoheaduparrow_w \langle 0, 0\rangle = (\omega + 1) \times \{0\}$ which is not increasing, and therefore not Scott-open.

We can give a characterization of wwb-open sets that is the same as our earlier characterization of wb-open sets. The proofs are the same as before.

Lemma 3.12. *If $\ll_w$ is interpolative in X and X admits a wwb topology then $\twoheaduparrow_w(\twoheaduparrow_w a) = \twoheaduparrow_w a$ for all $a \in X$.*

Theorem 3.13. *Let X be an ordered set in which $\ll_w$ is interpolative and which admits a wwb topology. A subset U of X is wwb-open if and only if $U = \twoheaduparrow_w U$.*

Theorem 3.14. *If $\ll_w$ is interpolative in X and X admits a wwb topology then X is locally compact in the wwb topology.*

When it comes to characterizing wwb-closed sets, we can do a bit better than we did with wb-closed sets, but not much. That is because we have the exactness property. We will denote the closure of A in the wwb-topology by $Cl_{wwb}A$. As in the case with the way below topology, $Cl_{wwb}A = \{p \in X : \twoheaddownarrow_w p \subseteq \twoheaddownarrow_w A\}$.

Definition 3.15. The directed closure of a subset A of X is the set $DC(A)$ of all elements of X which are the suprema of a directed subset of A.

Theorem 3.16. *If X is an exact ordered set and $A \subseteq X$ then $Cl_{wwb}A = DC(\twoheaddownarrow_w A)$.*

Proof. If $p \in Cl_{wwb}A$ then $\twoheaddownarrow_w p \subseteq \twoheaddownarrow_w A$. But $\twoheaddownarrow_w p$ is a directed set with supremum p, so $p \in DC(\twoheaddownarrow_w A)$. If $p \in DC(\twoheaddownarrow_w A)$ then there is a directed subset D of $\twoheaddownarrow_w A$ such that $p = \sup D$. Since X is exact there must be some $a \in X$ such that $a \ll_w p$. Let $b \in D$ such that $a \leq b$. There is $c \in A$ such that $b \ll_w c$. Therefore $a \ll_w c$. So $\twoheaddownarrow_w p \subseteq \twoheaddownarrow_w A$ and $p \in Cl_{wwb}A$. $\square$

That weak domains need not be sober or Baire is seen from the following example.

Example 3.17. Let $X = (\omega \times \{0\}) \cup [(\omega + 1) \times \{1\}]$ with its usual product order. Then $\Uparrow_w \langle \omega, 1 \rangle = \emptyset$ and $\Downarrow_w \langle \omega, 1 \rangle = \omega \times \{0\}$. Also, for every $m \in \omega$, $\Uparrow_w \langle m, 0 \rangle = [(\omega - m) \times 2] \cup \{\langle \omega, 1 \rangle\}$, $\Uparrow_w \langle m, 1 \rangle = (\omega - m) \times 2$, $\Downarrow_w \langle m, 0 \rangle = \downarrow \langle m, 0 \rangle$, and $\Downarrow_w \langle m, 1 \rangle = \downarrow \langle m, 1 \rangle$. X is a weak domain.

If A and B are closed subsets of X whose union is X, then one of these two sets must contain a cofinal subset of $\omega \times \{1\}$. Whichever one it is will be equal to X. Therefore X is an irreducible closed subset of itself, but is not the closure of a point.

For every $m \in \omega$, $\Uparrow_w \langle m, 1 \rangle$ is an open dense subset of X, and $\bigcap_{m \in \omega} \Uparrow_w \langle m, 1 \rangle = \emptyset$. Therefore X is not Baire.

Let us again say a brief word about continuous functions before we end this section. Just as was the case for the wb topology, the continuity of a function in the wwb topology is unaffected by the preservation properties of the function. It is easy to find examples of functions that are Scott-continuous, but not wwb-continuous, or that are wwb-continuous, but not Scott-continuous.

4. Representability

As we mentioned in the introduction, every domain representable topological space must be Baire. It is obvious that the completeness property plays an important role, here, but the importance of the continuity condition is not so obvious. Is it possible to weaken or even eliminate this property and still retain Baire? In this section we will see that the answer is no. First, a little vocabulary.

Definition 4.1. A topological space T is *weak domain representable* if and only if there is a weak domain X such that $\max X$, in its relative wwb topology, is homeomorphic to T.

The first theorem concerns spaces that are not represented by domains or weak domains, but rather those that can be represented by noncontinuous directed complete ordered sets in their Scott topology.

Theorem 4.2. *If T is a first countable topological space then there is a weak domain X such that* $\max X$*, with its relative Scott topology, is homeomorphic to T.*

Proof. For every $a \in T$ let $\{B(a,n) : n \in \omega\}$ be a nested neighborhood base for a in T. Set $X = T \times (\omega+1)$ and define an order on X by setting $\langle a,m\rangle < \langle b,n\rangle$ if and only if either $a = b$ and $m < n$ or $n = \omega$ and $b \in B(a,m)$. For every $a \in T$, $\{a\} \times \omega$ is a chain whose supremum is $\langle a,\omega\rangle$. If $a,b \in T$ with $a \neq b$ and $m,n \in \omega$ then $\langle a,m\rangle$ and $\langle b,n\rangle$ are incomparable. This means that if D is a directed subset of T and there are $a,b \in T$ such that $a \neq b$ and D intersects both $\{a\} \times \omega$ and $\{b\} \times \omega$ then D must contain an element of $T \times (\omega+1)$. Also, $\max X = T \times (\omega+1)$.

For every $a \in T$ and $n \in \omega$, $\downarrow_w\langle a,n\rangle = \overline{\downarrow}\,\langle a,n\rangle$ and $\downarrow_w\langle a,\omega\rangle = \{a\}\times\omega$. Therefore X is exact. It is easy to check that $\ll_w$ is weakly increasing in X.

Let D be a directed subset of X. If there is $a \in T$ such that $\langle a,\omega\rangle \in D$ then $\langle a,\omega\rangle = \sup D$. So assume that $D \cap (T \times \{\omega\} = \emptyset$. There must be $a \in T$ such that $D \subseteq \{a\} \times \omega$. But then D has a supremum. Therefore D is directed complete.

Define $f : T \to \max X$ by $f(a) = \langle a,\omega\rangle$. This function is obviously one-to-one and onto. We will show that it is a homeomorphism. Let U be an open subset of T and set $V = f[U]$. For every $a \in U$ let $n_a \in \omega$ such that $B(a,n_a) \subseteq U$. Set $W = \bigcup_{a\in U} \underline{\uparrow}\,\langle a,n_a\rangle$. Then W is increasing. Let D be a directed subset of X such that $\sup D \in W$. If $\sup D \notin D$ then there is $a \in T$ such that $\sup D = \langle a,\omega\rangle$ and thus $D \cap (\{a\}\times\omega)$ is cofinal in $\{a\}\times\omega$. Therefore there is $n > n_a$ such that $\langle a,n\rangle \in D$. Since $n > n_a$ we know that $\langle a,n\rangle \in W$. So W is Scott-open.

Now let W be a Scott-open subset of X and set $V = W \cap \max X$. Let $U = f^{-1}[V]$. If $a \in U$ then $\langle a,\omega\rangle \in V$ and there is $n \in \omega$ such that $\langle a,n\rangle \in W$. But W is increasing, so $B(a,n) \subseteq U$. Therefore U is open in T. □

Note that no separation properties are assumed. In general, if we were using the wwb topology rather than the Scott topology, then $\max X$ would have to be T_1 since X is exact. In this example, $\max X$ is discrete in the wb or wwb topology. This construction will work with any topological space T having the following property. For every element a of T there is a limit ordinal λ_a and a collection $\{B(a,\alpha) : \alpha \in \lambda_a\}$ of subsets of T such that a is in the interior of $B(a,\alpha)$ for all $\alpha \in \lambda_a$, $B(a,\beta) \subseteq B(a,\alpha)$ if and only if $\alpha < \beta$, and if $\beta \in \lambda_a$ is a limit ordinal then $B(a,\beta) = \bigcap_{\alpha\in\beta} B(a,\alpha)$. In fact, if one is willing to surrender the completeness property as well as continuity, this construction works with every topological space.

The maximal elements of an ordered set X see no difference in the way below and weakly way below relations. For this reason it is reasonable to expect that domain representable and weak domain representable spaces will be the same, or at least that they are both Baire. The following example shows that this is not the case. We begin with the rational numbers. A common technique used to generate domains is to combine the singletons of a space with a collection of compact neighborhoods and order by reverse inclusion. The singletons provide suprema for directed sets of the neighborhoods. But if we use the rational numbers then there will be directed sets of these neighborhoods which should be converging to irrational numbers, so the ordered set will not be directed complete. To overcome this shortcoming, we will include the irrationals, as well, but will create an order in such a way that the rational numbers are an open dense subset in the set of maximal elements.

Example 4.3. Let $X = \mathbb{R} \times (\omega + 1)$. In the following discussion, we will set $2^{-\omega} = 0$. Define an order on X by setting $\langle a, m\rangle < \langle b, n\rangle$ if and only if at least one of the following conditions is satisfied.

(1) $m < n = \omega$ and $b \in [a - 2^{-m}, a + 2^{-m}]$
(2) $b \in \mathbb{Q}$ and $[b - 2^{-n}, b + 2^{-n}] \subsetneq [a - 2^{-m}, a + 2^{-m}]$
(3) $a = b \in \mathbb{P}$, and $m < n$

If $\langle a, m\rangle < \langle b, n\rangle$ then $m < n$. For every $n \in \omega + 1$, $\mathbb{R} \times \{n\}$ is an antichain. Also, $\mathbb{R} \times \{\omega\}$ is the set of maximal elements of X. For every $a \in \mathbb{R}$, $\{a\} \times \omega$ is a chain whose supremum is $\langle a, \omega\rangle$. Note that in properties 1 and 3 we have $[b-2^{-n}, b+2^{-n}] \subseteq [a-2^{-m}, a+2^{-m}]$.

We will show that X is directed complete. Let D be a nonempty directed subset of X. If there is $a \in \mathbb{R}$ such that $\langle a, \omega\rangle \in D$ then $\langle a, \omega\rangle$ is the maximum element of D. So we may assume that $D \subseteq \mathbb{R} \times \omega$.

Next assume that there is $n \in \omega$ such that $D \subseteq \mathbb{R} \times n$. Let $m = \max\{n \in \omega : D \subseteq \mathbb{R} \times (n+1)\}$. Then $D \cap (\mathbb{R} \cap \{m\}) \neq \emptyset$. Since $\mathbb{R} \times \{m\}$ is an antichain there is $a \in \mathbb{R}$ such that $C \cap (\mathbb{R} \times \{m\}) = \{\langle a, m\rangle\}$. So $\langle a, m\rangle$ is the maximum element of D.

Finally, assume that for every $m \in \omega$ there is $n \in \omega$ such that $m < n$ and $D \cap (\mathbb{R} \times \{n\}) \neq \emptyset$. Let $\langle a, j\rangle, \langle b, k\rangle \in D$. There is

$\langle c, m\rangle \in D$ such that $\langle a, j\rangle \le \langle c, m\rangle$ and $\langle b, k\rangle \le \langle c, m\rangle$. Then $[c - 2^{-m}, c+2^{-m}] \subseteq [a-2^{-j}, a+2^{-j}] \cap [b-2^{-k}, b+2^{-k}]$. Therefore $\{[a-2^{-j}, a+2^{-j}] : \langle a, j\rangle \in D\}$ has the finite intersection property. Since the intervals associated with the elements in D become arbitrarily small there is $s \in \mathbb{R}$ such that $\{s\} = \cap\{[a - 2^{-j}, a + 2^{-j}] : \langle a, j\rangle \in D\}$. Therefore $\langle s, \omega\rangle = \sup D$ and X is directed complete.

Before we move on to the exactness of X we need to establish the following property of directed sets in X. Let D be a directed subset of X such that for every $m \in \omega$ there is $n \in \omega$ such that $m < n$ and $D \cap (\mathbb{R} \times \{n\}) \ne \emptyset$. If there is $j \in \omega$ such that $D \cap (\mathbb{R} \times \{k\}) \subseteq D \cap (\mathbb{P} \times \{k\})$ for all $k > j$ then there is $c \in \mathbb{P}$ such that $D \cap (\mathbb{R} \times \{k\}) \subseteq \{\langle c, k\rangle\}$ for all $k > j$ and therefore $\langle c, \omega\rangle = \sup D$. That is, if the directed set D picks up only irrational numbers from some level on then it must converge to an irrational. Let $k, m > j$ and $\langle a, k\rangle, \langle b, m\rangle \in D$. There is $\langle c, n\rangle \in D$ such that $\langle a, k\rangle \le \langle c, n\rangle$ and $\langle b, m\rangle \le \langle c, n\rangle$. Now $k \le n$ so $j < n$ and therefore $c \in \mathbb{P}$. It follows that $a = c = b$. Thus $D \cap (\mathbb{P} \times \{n \in \omega : n > j\}) \subseteq \{c\} \times \omega$ and $\sup D = \langle c, \omega\rangle$.

In order to show that X is exact, we need to when one element is weakly way below another. We will show that $\langle a, j\rangle \ll_w \langle b, k\rangle$ if and only if one of the following properties is met.

1. $k \in \omega$ and $\langle a, j\rangle \le \langle b, k\rangle$
2. $j < k = \omega$ and either
   - (a) $b \in \mathbb{Q} \cup (a - 2^{-j}, a + 2^{-j})$ or
   - (b) $a = b \in \mathbb{P}$

Assume that $\langle a, j\rangle \ll_w \langle b, k\rangle$. It follows automatically that $\langle a, j\rangle \le \langle b, k\rangle$, so we may assume that $k = \omega$. In that case, $b \in [a - 2^{-j}, a + 2^{-j}]$. Also, no maximal element of X is weakly way below itself, so $j < k$. First assume that $b \in \mathbb{Q}$. If $a \in \mathbb{P}$ then $b \in \mathbb{Q} \cup (a - 2^{-j}, a + 2^{-j})$, so we may assume that $a \in \mathbb{Q}$. If $b \in \{a - 2^{-j}, a + 2^{-j}\}$ then $\{\langle b, n\rangle : n \in \omega\}$ is a chain in X whose supremum is $\langle b, \omega\rangle$ but none of whose elements is larger than $\langle a, j\rangle$. This contradicts $\langle a, j\rangle \ll_w \langle b, \omega\rangle$, so $b \in \mathbb{Q} \cup (a - 2^{-j}, a + 2^{-j})$.

Now assume that $b \in \mathbb{P}$. If $a \in \mathbb{Q}$ then $\{\langle b, n\rangle : n \in \omega\}$ is a chain in X whose supremum is $\langle b, \omega\rangle$ but none of whose elements is larger than $\langle a, j\rangle$. This contradicts $\langle a, j\rangle \ll_w \langle b, \omega\rangle$, so we may assume that $a \in \mathbb{P}$. Then $a = b$.

To establish the implication in the other direction, let $\langle a, j\rangle, \langle b, k\rangle \in X$. Assume that $k \in \omega$ and that $\langle a, j\rangle \leq \langle b, k\rangle$. Let D be a directed subset of X with $\sup D = \langle b, k\rangle$. Then $\langle b, k\rangle$ is the maximum element of D and $\langle a, j\rangle \ll_w \langle b, k\rangle$.

Now assume that $j < k = \omega$. Let $b \in \mathbb{Q} \cup (a - 2^{-j}, a + 2^{-j})$. Then $\langle a, j\rangle \leq \langle b, \omega\rangle$. Let D be a directed subset of X such that $\sup D = \langle b, \omega\rangle$. We may assume that $\langle b, \omega\rangle \notin D$. For every $m \in \omega$ there is $n \in \omega$ such that $m < n$ and $D \cap (\mathbb{R} \times \{n\}) \neq \emptyset$. If there is $m \in \omega$ such that $D \cap (\mathbb{R} \times \{n\}) \subseteq D \cap (\mathbb{P} \times \{n\})$ for all $n > m$ then we have shown that there is $c \in \mathbb{P}$ such that $\langle c, \omega\rangle = \sup D$. This is impossible, since $b \in \mathbb{Q}$. Thus for every $m \in \omega$ there is $n \in \omega$ such that $m < n$ and $D \cap (\mathbb{Q} \times \{n\}) \neq \emptyset$. Choose $m \in \omega$ such that 2^{-m} is less than half the distance from b to $\{a - 2^{-j}, a + 2^{-j}\}$. There is $n > m$ and $c \in \mathbb{Q}$ such that $\langle c, n\rangle \in D$. Then $b \in [c - 2^{-n}, c + 2^{-n}]$ so $[c - 2^{-n}, c + 2^{-n}] \subsetneq [a - 2^{-j}, a + 2^{-j}]$ and $\langle a, j\rangle < \langle c, n\rangle$.

Finally, let $a = b \in \mathbb{P}$. We will show that $\langle b, j\rangle \ll_w \langle b, \omega\rangle$. Let D be a directed subset of X with $\sup D = \langle b, \omega\rangle$. We may assume that $\langle b, \omega\rangle \notin D$. So for every $m \in \omega$ there is $n \in \omega$ such that $m < n$ and $D \cap (\mathbb{R} \times \{n\}) \neq \emptyset$. First consider the case when there is $n > j$ such that $D \cap (\mathbb{Q} \times \{n\}) \neq \emptyset$ and let $\langle c, n\rangle \in D \cap (\mathbb{Q} \times \{n\})$. Now $b \in [c - 2^{-n}, c + 2^{-n}]$ so $[c - 2^{-n}, c + 2^{-n}] \subseteq [b - 2^{-j}, b + 2^{-j}]$ and $\langle b, j\rangle < \langle c, n\rangle$. Next assume that $D \cap (\mathbb{R} \times \{n\}) \subseteq \mathbb{P} \times \{n\})$ for all $n > j$. This means that there is $c \in \mathbb{P}$ such that $D \cap (\mathbb{R} \times \{n\}) \subseteq \mathbb{P} \times \{n\})$ for all $n > j$ and $\sup D = \langle c, b\rangle$. It follows that $c = b$ and there is $n > j$ such that $\langle b, j\rangle < \langle b, n\rangle \in D$. This completes the proof of our claim.

Let $\langle b, k\rangle \in X$. If $k \in \omega$ then $\mathop{\downarrow\!\!\downarrow}\langle b, k\rangle = \downarrow \langle b, k\rangle$ which is directed and has supremum $\langle b, k\rangle$. Assume that $k = \omega$. If $b \in \mathbb{Q}$ then $\mathop{\downarrow\!\!\downarrow}\langle b, \omega\rangle = \{\langle a, j\rangle \in X : b \in (a - 2^{-j}, a + 2^{-j})\}$, which is directed and has supremum $\langle b, \omega\rangle$. If $b \in \mathbb{P}$ then $\mathop{\downarrow\!\!\downarrow}\langle b, \omega\rangle = \{b\} \times \omega$, which is directed and has supremum $\langle b, \omega\rangle$. Therefore X is exact. Our characterization of $\ll_w$ on X also shows that $\ll_w$ is weakly increasing on X. Thus X is a weak domain.

For every $\langle a, n\rangle \in X$ with $n \in \omega$ we have $\mathop{\uparrow\!\!\uparrow}_w\langle a, n\rangle \cap \max X = ([(a - 2^{-n}, a + 2^{-n}) \cap \mathbb{Q}] \times \{\omega\}) \cup \{\langle a, \omega\rangle\}$. Therefore $\max X$ is homeomorphic to $\mathbb{R}$ with the topology generated by assigning to each element a of $\mathbb{R}$ a neighborhood base consisting of all sets of the form $\{a\} \cup (U \cap \mathbb{Q})$, where U is a neighborhood of a in the usual topology of $\mathbb{R}$. All cofinite subsetes of $\mathbb{Q}$ are dense open subsets of this space, so it is not Baire. Note that $\max X$ is Hausdorff, but not regular.

5. Questions

(1) Which topological spaces can be represented by a dcos in its Scott topology?
(2) Which topological spaces are weak domain representable? In particular, is $\mathbb{Q}$ weak domain representable?
(3) Example 4.3 suggests that there may by a "completion" process for exact ordered sets. If so, what is it, and to which sets does it apply?
(4) Is there a Scott-like characterization of wb-open or wwb-open sets?
(5) Is there a Scott-like characterization of wb-continuous or wwb-continuous functions?
(6) Are the wb or wwb topologies locally compact when the defining relations are not interpolative?

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