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Entropy in Topological Groups, Part 2

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Entropy in Topological Groups

Dikran Dikranjan

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June 28, 2017
Weiss [1976] Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group $K$. If $\hat{\phi} : \hat{K} \to \hat{K}$ is the Pontryagin dual of $\phi$. Then

$$h_{\text{top}}(\phi) = h_{\text{alg}}(\hat{\phi}).$$

(P)

Peters [1979] proved (P) when $G$ is compact metrizable and $\phi$ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group $G$. Then (P) holds if one of the following condition is fulfilled:

(a) $G$ is totally disconnected (generalizes Weiss);
(b) $G$ is compact (generalizes Peters).

Question

Does (P) hold true for every LCA group $G$?

Yes, for automorphisms (for actions of amenable groups). Virili '13
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Let $X$ be a set and $\lambda : X \to X$ a selfmap. For a finite subset $D$ of $X$ and $n \in \mathbb{N}_+$ the $n$-th $\lambda$-trajectory of $D$ is

$$T_n(\lambda, D) = D \cup \lambda(D) \cup \cdots \cup \lambda^{n-1}(D),$$

while the $\lambda$-trajectory ([positive] orbit) of $D$ under $\lambda$ is

$$T(\lambda, D) = \bigcup_{n \in \mathbb{N}} \lambda^n(D) = \bigcup_{n \in \mathbb{N}_+} T_n(\lambda, D).$$

This is the smallest $\lambda$-invariant subset of $X$ containing $D$. One can define similarly the inverse $n$-th $\lambda$-trajectory of $D$ by

$$T_n^*(\lambda, D) = D \cup \lambda^{-1}(D) \cup \cdots \cup \lambda^{-n+1}(D).$$
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(b) The number $h_c(\lambda) = \sup \{ h_c(\lambda, D) : D \in [X]^<\omega \}$ is the (covariant) combinatorial entropy of $\lambda$.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of $\lambda$ can be defined similarly, by making use of $\mathcal{T}_n^*(\lambda, D)$ in place of $\mathcal{T}_n(\lambda, D)$.

**Example (Generalized shifts)**

Let $K$ be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$.

Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in \text{Sym}(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is $\sigma_\lambda$-invariant in $K^X$ and $h_{alg}(\sigma_\lambda \restriction \bigoplus_X K) = h_c^*(\lambda) \log |K|$.
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Call a compact group *strictly reductive* if it is isomorphic to a cartesian product of simple compact groups.

**Theorem (Countable Layer Theorem, Hofmann-Morris)**

Any compact profinite group $G$ has a canonical countable descending sequence $G = \Omega_0(G) \supseteq \ldots \supseteq \Omega_n(G) \supseteq \ldots$ of closed characteristic subgroups of $G$ such that:

1. $\bigcap_{n=1}^{\infty} \Omega_n(G) = \{e\}$,
2. each layer $L_n = \Omega_{n-1}(G)/\Omega_n(G)$ is a strictly reductive group.

The computation of the topological entropy of an automorphism $f : G \to G$ of a compact profinite group $G$ can be reduced to the case of a strictly reductive compact group $L$. Indeed, $f$ induces an automorphism $f_n : L_n \to L_n$ of the strictly reductive group $L_n$ and $h_{\text{top}}(f) = \sum_{n=1}^{\infty} h_{\text{top}}(f_n)$, as $G = \varprojlim G/\Omega_n(G)$ and the induced automorphism $\overline{f}$ of $G/\Omega_n(G)$ has $h_{\text{top}}(\overline{f}) = \sum_{k=1}^{n} h_{\text{top}}(f_k)$. 
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\begin{theorem}[Countable Layer Theorem, Hofmann-Morris]
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\end{theorem}

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An automorphism \( f \) of a compact group \( L \) induces automorphisms of \( L' \) and \( L/L' \), so by using AT (when \( L' = \overline{L}' \)), one can assume wlog that either \( L = L' \) or \( L \) is abelian when computing \( h_{\text{top}}(f) \).

A strictly reductive compact group with \( L = L' \) has the form \( \prod_{j \in J} K_j \), where \( K_j = F^l_j \), for some simple finite non-abelian group \( F_j \) and \( l_j \neq \emptyset \neq J \). Then \( f \) induces automorphisms \( f_j \) of \( K_j \) so that \( h_{\text{top}}(f) = \sum_{j \in J} h_{\text{top}}(f_j) \). Each \( f_j \) induces a bijection \( \lambda_j \) of \( l_j \), so that \( \psi_j := \sigma^{-1}_{\lambda_j} \circ f_j \) acts coordinatewise on \( F^l_j \). Thus,

\[
h_{\text{top}}(f_j) = h_{\text{top}}(\sigma_{\lambda_j} \circ \psi_j) = h_{\text{top}}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.
\]

In case \( L \) is abelian, it has the form \( L = \prod_{p \in \pi} K_p \), where \( K_p = \mathbb{Z}_p^{\kappa_p} \) for some set \( \pi \) of primes. Now each \( f_p : K_p \rightarrow K_p \) is conjugated to a direct product of generalized shifts of \( \mathbb{Z}_p^{\kappa_p} \).

Note that in both cases these generalized shifts are just products of periodic automorphisms and Bernoulli automorphisms.
An automorphism $f$ of a compact group $L$ induces automorphisms of $L'$ and $L/L'$, so by using AT (when $L' = \bar{L}'$), one can assume wlog that either $L = L'$ or $L$ is abelian when computing $h_{top}(f)$.

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$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$ 

In case $L$ is abelian, it has the form $L = \prod_{p \in \pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set $\pi$ of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$. Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.
An automorphism $f$ of a compact group $L$ induces automorphisms of $L'$ and $L/L'$, so by using AT (when $L' = \bar{L}'$), one can assume wlog that either $L = L'$ or $L$ is abelian when computing $h_{top}(f)$.

A strictly reductive compact group with $L = L'$ has the form $\prod_{j \in J} K_j$, where $K_j = F_j^{l_j}$, for some simple finite non-abelian group $F_j$ and $l_j \neq \emptyset \neq J$. Then $f$ induces automorphisms $f_j$ of $K_j$ so that $h_{top}(f) = \sum_{j \in J} h_{top}(f_j)$. Each $f_j$ induces a bijection $\lambda_j$ of $l_j$, so that $\psi_j := \sigma^{-1}_{\lambda_j} \circ f_j$ acts coordinatewise on $F_j^{l_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$ 

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Each $f_j$ induces a bijection $\lambda_j$ of $I_j$, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{l_j}$. Thus,
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An automorphism \( f \) of a compact group \( L \) induces automorphisms of \( L' \) and \( L/L' \), so by using AT (when \( L' = \overline{L}' \)), one can assume wlog that either \( L = L' \) or \( L \) is abelian when computing \( h_{\text{top}}(f) \).

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Similarly, one can compute $h_{\text{top}}(f)$ when $G$ is a compact connected group. As mentioned above, we can reduce to the cases when $G$ is abelian or $G' = G$ (note that $G'$ is closed and connected). The abelian case can be reduced, via the Bridge theorem, to the computation of $h_{\text{alg}}(\hat{f})$.

Since $Z(G)$ is characteristic, the computation of $h_{\text{top}}(f)$ can be reduced, due to AT, to the case when $G$ is center-free, as $Z(G/Z(G)) = \{e\}$. In such a case the group $G$ is, again, strictly reductive, i.e., $G = \prod_{i \in I} F_i^j$, where $F_i$ are pairwise non-isomorphic compact connected simple Lie groups with trivial center.

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- either $h_{\text{top}}(f) = 0$ (if all $h_{\lambda_j}(f) = 0$), or
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A *normed semigroup* is a commutative semigroup \((S, +)\) provided with a map \((\text{norm})\) \(v \colon S \to \mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}\) satisfying

\[ v(x + y) \leq v(x) + v(y) \]

for all \(x, y \in S\).

The category \(\mathcal{G}\) of normed semigroups has as morphisms all *contractive* semigroup homomorphism \(f : (S, v) \to (S_1, v_1)\) (i.e., \(\phi(x + y) = \phi(x) + \phi(y)\) and \(v_1(\phi(x)) \leq v(x)\) hold for every \(x, y \in S\)).

For \((S, v) \in \mathcal{G}\) we say that the norm is *\(s\)-monotone*, if

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Definition

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For \((S, \nu) \in \mathcal{G}, x \in S\) and \(n \in \mathbb{N}_+\) consider the \textit{n-th trajectory of} \(x\) \textit{under} \(\phi\)

\[T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x)\] and \(c_n(\phi, x) = \nu(T_n(\phi, x))\). Then \((c_n(\phi, x))\) is subadditive and \(c_n \leq n \cdot \nu(x)\), so the growth of the function \(n \mapsto c_n(\phi, x)\) is at most linear.

**Theorem**

\(\text{Let } \phi : S \to S \text{ be an endomorphism in } \mathcal{G}. \text{ Then for every } x \in S \text{ the limit } h_{\mathcal{G}}(\phi, x) := \lim_n \frac{c_n(\phi, x)}{n} \text{ exists and satisfies } h_{\mathcal{G}}(\phi, x) \leq \nu(x).\)

The existence of the limit is ensured by Fekete Lemma.

**Definition**

\(\text{Let } \phi : S \to S \text{ be an endomorphism in } \mathcal{G}. \text{ The } \textbf{semigroup entropy}\ \text{of} \ \phi \text{ is}\)

\[h_{\mathcal{G}}(\phi) = \sup_{x \in S} h_{\mathcal{G}}(\phi, x).\]
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Let \(\phi : S \to S\) be an endomorphism in \(\mathcal{G}\). Then for every \(x \in S\) the limit \(h_S(\phi, x) := \lim_n \frac{c_n(\phi, x)}{n}\) exists and satisfies \(h_S(\phi, x) \leq \nu(x)\).

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Lemma (\(h_{\mathcal{G}}\) is monotone under taking quotients)

If \(\phi : S \to S\) and \(\psi : T \to T\) are endomorphisms in \(\mathcal{G}\) and \(\alpha : T \to S\) is a surjective homomorphism between normed semigroups such that \(\alpha \circ \psi = \phi \circ \alpha\), then \(h_{\mathcal{G}}(\phi) \leq h_{\mathcal{G}}(\psi)\).

Corollary (\(h_{\mathcal{G}}\) is invariant under conjugation)

If \(\phi : S \to S\) is an endomorphism in \(\mathcal{G}\) and \(\alpha : T \to S\) is an isomorphism in \(\mathcal{G}\), then \(h_{\mathcal{G}}(\phi) = h_{\mathcal{G}}(\alpha \circ \phi \circ \alpha^{-1})\).

Lemma (\(h_{\mathcal{G}}\) is invariant under inversion)

If \(\phi : S \to S\) is an isomorphism in \(\mathcal{G}\), then \(h_{\mathcal{G}}(\phi^{-1}) = h_{\mathcal{G}}(\phi)\).

Lemma (Logarithmic Law)

Let \((S, v)\) be a normed semigroup and \(\phi : S \to S\) an endomorphism. Then \(h_{\mathcal{G}}(\phi^k) \leq k \cdot h_{\mathcal{G}}(\phi)\) for every \(k \in \mathbb{N}\). Furthermore equality holds if \(v\) is \(s\)-monotone.
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### Lemma (\(h_{\mathcal{G}}\) is invariant under inversion)

If \(\phi: S \to S\) is an isomorphism in \(\mathcal{G}\), then \(h_{\mathcal{G}}(\phi^{-1}) = h_{\mathcal{G}}(\phi)\).

### Lemma (Logarithmic Law)

Let \((S, \nu)\) be a normed semigroup and \(\phi: S \to S\) an endomorphism. Then \(h_{\mathcal{G}}(\phi^k) \leq k \cdot h_{\mathcal{G}}(\phi)\) for every \(k \in \mathbb{N}\). Furthermore equality holds if \(\nu\) is \(s\)-monotone.
Lemma \((h_{G} is monotone under taking quotients)\)

If \(\phi : S \to S\) and \(\psi : T \to T\) are endomorphisms in \(G\) and \(\alpha : T \to S\) is a surjective homomorphism between normed semigroups such that \(\alpha \circ \psi = \phi \circ \alpha\), then \(h_{G}(\phi) \leq h_{G}(\psi)\).

Corollary \((h_{G} is invariant under conjugation)\)

If \(\phi : S \to S\) is an endomorphism in \(G\) and \(\alpha : T \to S\) is an isomorphism in \(G\), then \(h_{G}(\phi) = h_{G}(\alpha \circ \phi \circ \alpha^{-1})\).

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Lemma \((h_\mathbb{G} \text{ is monotone under taking quotients})\)

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Corollary \((h_\mathbb{G} \text{ is invariant under conjugation})\)

If \(\phi : S \rightarrow S\) is an endomorphism in \(\mathbb{G}\) and \(\alpha : T \rightarrow S\) is an isomorphism in \(\mathbb{G}\), then \(h_\mathbb{G}(\phi) = h_\mathbb{G}(\alpha \circ \phi \circ \alpha^{-1})\).

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Lemma \((\text{Logarithmic Law})\)

Let \((S, \nu)\) be a normed semigroup and \(\phi : S \rightarrow S\) an endomorphism. Then \(h_\mathbb{G}(\phi^k) \leq k \cdot h_\mathbb{G}(\phi)\) for every \(k \in \mathbb{N}\). Furthermore equality holds if \(\nu\) is \(s\)-monotone.
Lemma (*h_S* is monotone under taking quotients)

If φ : S → S and ψ : T → T are endomorphisms in S and α : T → S is a surjective homomorphism between normed semigroups such that α ◦ ψ = φ ◦ α, then h_S(φ) ≤ h_S(ψ).

Corollary (*h_S* is invariant under conjugation)

If φ : S → S is an endomorphism in S and α : T → S is an isomorphism in S, then h_S(φ) = h_S(α ◦ φ ◦ α^{-1}).

Lemma (*h_S* is invariant under inversion)

If φ : S → S is an isomorphism in S, then h_S(φ^{-1}) = h_S(φ).

Lemma (Logarithmic Law)

Let (S, υ) be a normed semigroup and φ : S → S an endomorphism. Then h_S(φ^k) ≤ k · h_S(φ) for every k ∈ ℤ. Furthermore equality holds if υ is s-monotone.
**Lemma (\(h_{\mathcal{G}}\) is monotone under taking quotients)**

If \(\phi : S \to S\) and \(\psi : T \to T\) are endomorphisms in \(\mathcal{G}\) and \(\alpha : T \to S\) is a surjective homomorphism between normed semigroups such that \(\alpha \circ \psi = \phi \circ \alpha\), then \(h_{\mathcal{G}}(\phi) \leq h_{\mathcal{G}}(\psi)\).

**Corollary (\(h_{\mathcal{G}}\) is invariant under conjugation)**

If \(\phi : S \to S\) is an endomorphism in \(\mathcal{G}\) and \(\alpha : T \to S\) is an isomorphism in \(\mathcal{G}\), then \(h_{\mathcal{G}}(\phi) = h_{\mathcal{G}}(\alpha \circ \phi \circ \alpha^{-1})\).

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**Lemma (Logarithmic Law)**

Let \((S, \nu)\) be a normed semigroup and \(\phi : S \to S\) an endomorphism. Then \(h_{\mathcal{G}}(\phi^k) \leq k \cdot h_{\mathcal{G}}(\phi)\) for every \(k \in \mathbb{N}\). Furthermore equality holds if \(\nu\) is \(s\)-monotone.
For a topological space $X$ the family $\text{cov}(X)$ of all open covers of $X$ is a commutative monoid $(\text{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.

One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\text{cov}(C)$ ($\mathcal{V}$ refines $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\text{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)

For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \text{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$ 

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This defines a contravariant functor $\text{cov}$ from the category of all topological spaces to the category of commutative semigroups.
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One has a natural a preorder $U \prec V$ on $\text{cov}(C)$ ($V$ refines $U$, i.e, if for every $V \in V$ there exists $U \in U$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $U \lor U \neq U$ (so $\text{cov}(C)$ is not a semilattice), yet $U \lor U \sim U$ (where $U \sim V$ means $U \prec V$ abd $V \prec U$).

For a continuous map $\phi : X \to Y$ and $U \in \text{cov}(Y)$ let

$$\phi^{-1}(U) = \{\phi^{-1}(U) : U \in U\}.$$

The assignment $U \mapsto \phi^{-1}(U)$ gives a semigroup homomorphism $\text{cov}(\phi) : \text{cov}(Y) \to \text{cov}(X)$ (as $\phi^{-1}(U \lor V) = \phi^{-1}(U) \lor \phi^{-1}(V)$). This defines a contravariant functor $\text{cov}$ from the category of all topological spaces to the category of commutative semigroups.
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To get a norm on the semigroup $\text{cov}(X)$ we restrict this functor to the subcategory $\text{CTop}$ of compact spaces. For $X \in \text{CTop}$, $\mathcal{U} \in \text{cov}(X)$ let $\nu(\mathcal{U}) = N(\mathcal{U})$.

**Lemma**

For a compact space $X$, $(\text{cov}(X), \lor, \nu)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $\nu(\phi^{-1}(\mathcal{W})) \leq \nu(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.

By the lemma $\text{cov}(\phi) : \text{cov}(Y) \to \text{cov}(X)$ is a morphism in $\mathcal{G}$, so that the assignement $X \mapsto \text{cov}(X)$ defines a contravariant functor $\text{cov} : \text{CTop} \to \mathcal{G}$, that sends embeddings in $\text{CTop}$ to surjective morphisms in $\mathcal{G}$ and sends surjective maps in $\text{CTop}$ to embeddings in $\mathcal{G}$. 
To get a norm on the semigroup $\text{cov}(X)$ we restrict this functor to the subcategory $\text{CTop}$ of compact spaces. For $X \in \text{CTop}$, $\mathcal{U} \in \text{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

**Lemma**

For a compact space $X$, $(\text{cov}(X), \lor, v)$ is a normed semigroup. For every continuous map $\phi : X \rightarrow Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.

By the lemma $\text{cov}(\phi) : \text{cov}(Y) \rightarrow \text{cov}(X)$ is a morphism in $\mathcal{S}$, so that the assignment $X \mapsto \text{cov}(X)$ defines a contravariant functor

$$\text{cov} : \text{CTop} \rightarrow \mathcal{S},$$

that sends embeddings in $\text{CTop}$ to surjective morphisms in $\mathcal{S}$ and sends surjective maps in $\text{CTop}$ to embeddings in $\mathcal{S}$. 
To get a norm on the semigroup \( \text{cov}(X) \) we restrict this functor to the subcategory \( \text{CTop} \) of compact spaces. For \( X \in \text{CTop} \), \( U \in \text{cov}(X) \) let \( v(U) = N(U) \).

**Lemma**

For a compact space \( X \), \( (\text{cov}(X), \lor, v) \) is an normed semigroup. For every continuous map \( \phi : X \to Y \) of compact spaces the inequality \( v(\phi^{-1}(W)) \leq v(W) \) holds for every \( W \in \text{cov}(Y) \).

By the lemma \( \text{cov}(\phi) : \text{cov}(Y) \to \text{cov}(X) \) is a morphism in \( \mathcal{G} \), so that the assignement \( X \mapsto \text{cov}(X) \) defines a contravariant functor \( \text{cov} : \text{CTop} \to \mathcal{G} \),

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**Lemma**

For a compact space $X$, $(\text{cov}(X), \lor, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.

By the lemma $\text{cov}(\phi) : \text{cov}(Y) \to \text{cov}(X)$ is a morphism in $\mathcal{G}$, so that the assignment $X \mapsto \text{cov}(X)$ defines a contravariant functor $\text{cov} : \text{CTop} \to \mathcal{G}$, that sends embeddings in $\text{CTop}$ to surjective morphisms in $\mathcal{G}$ and sends surjective maps in $\text{CTop}$ to embeddings in $\mathcal{G}$. 
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**Lemma**

*For a compact space $X$, $(\text{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \rightarrow Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.*

By the lemma $\text{cov}(\phi) : \text{cov}(Y) \rightarrow \text{cov}(X)$ is a morphism in $\mathcal{G}$, so that the assignement $X \mapsto \text{cov}(X)$ defines a contravariant functor $\text{cov} : \text{CTop} \rightarrow \mathcal{G}$, that sends embeddings in $\text{CTop}$ to surjective morphisms in $\mathcal{G}$ and sends surjective maps in $\text{CTop}$ to embeddings in $\mathcal{G}$. 
To get a norm on the semigroup $\text{cov}(X)$ we restrict this functor to the subcategory $\text{CTop}$ of \textit{compact spaces}. For $X \in \text{CTop}$, $\mathcal{U} \in \text{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

\textbf{Lemma}

\textit{For a compact space $X$, $(\text{cov}(X), \lor, v)$ is an normed semigroup. For every continuous map $\phi : X \rightarrow Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.}

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**Lemma**

For a compact space $X$, $(\text{cov}(X), \lor, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \text{cov}(Y)$.

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that sends embeddings in $\mathbf{CTop}$ to surjective morphisms in $\mathcal{G}$ and sends surjective maps in $\mathbf{CTop}$ to embeddings in $\mathcal{G}$.
Let $F : \mathcal{X} \to \mathcal{G}$ a functor. Define the entropy function $h_F$ in the category $\mathcal{X}$ by

$$h_F(\phi) = h_{\mathcal{G}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor $F$ preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \text{End}_{\mathcal{X}}(X)$, the entropy $h_F$ will satisfy:

**[Invariance under conjugation]** If $\alpha : Y \to X$ is an isomorphism, then $h_F(\phi) = h_F(\alpha^{-1} \circ \phi \circ \alpha)$.

**[Invariance under inversion]** $h_F(\phi^{-1}) = h_F(\phi)$, if $\phi$ is an isomorphism.

**[Logaritmic law]** If the norm of the semigroup $F(X)$ is $s$-monotone, then $h_F(\phi^k) = k \cdot h_F(\phi)$, for all $k \in \mathbb{N}$. 
Let $F : \mathcal{X} \to \mathcal{S}$ a functor. Define the entropy function $h_F$ in the category $\mathcal{X}$ by

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$$h_F(\phi) = h_{\mathcal{S}}(F\phi),$$

for an endomorphism $\phi : X \rightarrow X$ in $\mathcal{X}$. The functor $F$ preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \text{End}_\mathcal{X}(X)$, the entropy $h_F$ will satisfy:

**[Invariance under conjugation]** If $\alpha : Y \rightarrow X$ is an isomorphism, then $h_F(\phi) = h_F(\alpha^{-1} \circ \phi \circ \alpha)$.

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Obtaining the topological entropy $h_{\text{top}}$ as $h_{\text{cov}}$

For the contravariant functor $\text{cov} : \text{CTop} \to S$ the entropy $h_{\text{cov}} : \text{CTop} \to \mathbb{R}_+$ coincides with the topological entropy $h_{\text{top}}$ defined by Adler et al.

Since the functor $\text{cov}$, 
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Entropy in Topological Groups

The general scheme for obtaining the entropies and their properties

The category $\text{MesSp}$ of probability measure spaces

For a measure space $(X, \mathcal{B}, \mu)$ let $\mathcal{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of $X$. For $\xi, \eta \in \mathcal{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathcal{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \ldots, A_k\} \in \mathcal{P}(X)$ of $X$ define the entropy of $\xi$ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \quad (\text{Shannon entropy})$$

This is a monotone norm making $\mathcal{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^{k} \in \mathcal{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^{k}$. Since $T$ is measure preserving, one has $T^{-1}(\xi) \in \mathcal{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all $i$. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathcal{P}(X)$ defines a contravariant functor

$$\mathcal{P} : \text{MesSp} \longrightarrow \mathcal{L}.$$
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Obtaining the measure entropy \( h_{mes} \) as \( h_P \)

For the contravariant functor \( \mathcal{P} : \text{MesSp} \to \mathcal{L} \) the entropy
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This is why, similarly to \( h_{top} \), also the measure-theoretic entropy
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Example (measure entropy vs topological entropy)

Let \( X \) be a compact topological group, let \( \mu \) be its Haar measure
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The general scheme for obtaining the entropies and their properties

The category $\text{MesSp}$ of probability measure spaces

Obtaining the measure entropy $h_{mes}$ as $h_\Psi$

For the contravariant functor $\Psi : \text{MesSp} \to \mathcal{L}$ the entropy $h_\Psi = h_\mathcal{O} \circ \Psi : \text{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy $h_m$ defined by Kolmogorov and Sinai in ergodic theory in the fifties.

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Obtaining the measure entropy $h_{mes}$ as $h_\mathcal{P}$

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Example (Adler, Konrad and McAndrew's algebraic entropy $\text{ent}$)

Let $G$ be an Abelian group and let $(\mathcal{F}(G), +)$ be the semilattice of all finite subgroups of $G$. Letting $\nu(F) = \log |F|$ for $F \in \mathcal{F}(G)$, makes $\mathcal{F}(G)$ a normed semilattice with a monotone norm. For every homomorphism $\phi : G \to H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathcal{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$
\mathcal{F} : \mathbf{AbGrp} \longrightarrow \mathcal{S}.
$$

The entropy $h_\mathcal{F} = h_\mathcal{S} \circ \mathcal{F}$ coincides with the algebraic entropy $\text{ent}$ defined by Adler, Konrad and McAndrew. So $\text{ent}$ satisfies the invariance under conjugation and inversions as well as the logarithmic law. Since $\mathcal{F}$ sends monomorphisms to embeddings, $\text{ent}$ is also monotone w.r.t. taking invariant subgroups (not w.r.t. taking factors).
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For every homomorphism $\phi : G \rightarrow H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \rightarrow \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathcal{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

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Let \( G \) be an Abelian group and let \((\mathcal{F}(G), +)\) be the semilattice of all finite subgroups of \( G \). Letting \( \nu(F) = \log |F| \) for \( F \in \mathcal{F}(G) \), makes \( \mathcal{F}(G) \) a normed semilattice with a monotone norm.

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Example (The algebraic entropy $h_{\text{alg}}$)

For $G \in \text{AbGrp}$ let $\mathcal{H}(G)$ be the family of all finite non-empty subsets of $G$. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \rightarrow H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \rightarrow \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.

Letting $\nu(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignments $G \mapsto (\mathcal{H}(G), \nu)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \text{AbGrp} \longrightarrow \mathcal{S}.$$ 

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm $\nu$ is $s$-monotone. The entropy $h_\mathcal{H} = h_\mathcal{S} \circ \mathcal{H}$ coincides with the algebraic entropy $h_{\text{alg}}$. So $h_{\text{alg}}$ is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logarithmic law.
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For $G \in \textbf{AbGrp}$ let $\mathcal{H}(G)$ be the family of all finite non-empty subsets of $G$. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \rightarrow H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \rightarrow \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.

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In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces or measure spaces was introduced by Ornstein and Weiss [1987]. Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \curvearrowright X$ of a locally compact semigroup $S$ on a metric space $X$, depending on a countable system $\alpha$ of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of $S$ satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n - 1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen’s topological entropy $h_U(f)$. 
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Let $S$ be a semigroup and let $\mathcal{P}_{\text{fin}}(S)$ be the family of its non-empty subsets; $S$ is right amenable, if for every $K \in \mathcal{P}_{\text{fin}}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{\text{fin}}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.

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**Theorem (Ceccherini-Silberstein, Coornaert and Krieger 2014)**

Let $S$ be a cancellative left amenable monoid and let $f : \mathcal{P}(S) \to \mathbb{R}$ be a subadditive, right-subinvariant map. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left-Følner net $(F_i)_{i \in I}$ of $S$,

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Let $X$ be a compact topological space, let $S$ be a cancellative left-amenable monoid and consider the left action $S \curvearrowright X$ by continuous maps. For $\mathcal{U} \in \text{cov}(X)$ and for every $F \in \mathcal{P}_{\text{fin}}(S)$, let

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is non-decreasing, subadditive and right-subinvariant. The above theorem gives the following:

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Let $S$ be a cancellative left-amenable semigroup acting $S \curvearrowright X$ on a compact space $X$. For $\mathcal{U} \in \text{cov}(X)$, the **topological entropy** of $\gamma$ with respect to $\mathcal{U}$ is

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Let $S$ be a cancellative right-amenable semigroup acting $S \overset{\alpha}{\curvearrowright} A$ on an abelian group $A$ by endomorphisms. For $X \in \mathcal{P}_{\text{fin}}(A)$ and for every $F \in \mathcal{P}_{\text{fin}}(S)$, let

$$T_F(\alpha, X) = \sum_{s \in F} \alpha(s)(X) = \sum_{s \in F} s \cdot X \in \mathcal{P}_{\text{fin}}(A)$$

be the $\alpha$-trajectory of $X$ with respect to $F$.

The function

$$f_X : \mathcal{P}_{\text{fin}}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

is subadditive, left-subinvariant, so the limit

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These entropies share many of the properties of the algebraic entropies $h_{\text{alg}}$ and $\text{ent}$ defined for single endomorphisms. Moreover, if $f \in \text{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

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Let $S$ be a cancellative right-amenable semigroup, $A$ an abelian group and consider $S \xrightarrow{\alpha} A$. If $A$ is a direct limit of $\alpha$-invariant subgroups $\{A_i \mid i \in I\}$, then $h_{\text{alg}}(\alpha) = \sup_{i \in I} h_{\text{alg}}(\alpha_{A_i})$.

Theorem (Logarithmic Law)

Let $G$ be an amenable group, $A$ an abelian group and $G \xrightarrow{\alpha} A$. If $H$ is a subgroup of $G$ of finite index $[G : H] = k \in \mathbb{N}$, then

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Theorem (Fornasiero, Giordano Bruno, DD - 2017)

Let $A$ be a torsion abelian group, $S$ be a right-amenable monoid, $\alpha$ be a left action of $S$ on $A$, and $B$ be an $\alpha$-invariant subgroup of $A$. Then

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Entropy in Topological Groups
Entropy of semigroup actions

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A left action $S \bowtie K$ of a cancellative left-amenable semigroup $S$ on a compact abelian group $K$ induces a right dual action $\hat{K} \bowtie \hat{S}$ on the discrete group $\hat{K}$, defined by

$$\hat{\gamma}(s) = \hat{\gamma}(s) : \hat{K} \to \hat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\hat{\gamma}^{\text{op}}$ is the left action of $S^{\text{op}}$ associated to $\hat{\gamma}$):

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For a left action $S \bowtie K$ of a cancellative left-amenable semigroup $S$ on a compact totally disconnected abelian group $K$

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