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Joe Mashburn
University of Dayton, joe.mashburn@udayton.edu

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Joe Mashburn [‡]

Department of Mathematics, University of Dayton, Dayton OH 45469-2316, USA

joe.mashburn@udayton.edu

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In this paper we extend the spectral order of Coecke and Martin to infinite dimensional quantum states. Many properties present in the finite dimensional case are preserved, but some of the most important are lost. The order is constructed and its properties analyzed. Most of the useful measurements of information content are lost. Shannon entropy is defined on only a part of the model, and that part is not a closed subset of the model. The finite parts of the lattices used by Birkhoff and von Neumann as models for classical and quantum logic appear as subsets of the models for infinite classical and quantum states.

1. Introduction

In the ongoing search for interpretations of quantum physics the idea of quantum states as information has gained significant interest. See, for example, (Brukner and Zeilinger 1999; Bub 2005; Clifton, Bub, and Halverson 2003; van Enk 2007; Fuchs 2002; Spekkens 2007). For a different view, see (Hagar and Hemmo 2006). Mathematical models of information which have not received much attention in this endeavor are domains, introduced by Dana Scott in (Scott 1970). A domain is an ordered set on which a special relation, the way-below relation, is defined. The order allows one to say which elements of the domain have a higher information content or a higher degree of certainty than others and the way-below relation allows one to see which elements are approximations of or essential to others. Martin (Martin 2000) introduced a class of functions which serve as measures of information content of the elements of a domain. In 2002 Coecke and Martin (Coecke and Martin 2002) created domain theoretic models for both finite dimensional classical physical states and finite dimensional quantum physical states. They called the order used for the classical states the Bayesian order, and that used for the quantum states the spectral order. Their models are not precisely domains, because the definition of the way-below relation is slightly altered, but they retain most of the desirable characteristics of

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the domain. These models exhibit the properties one would expect in a model of physical states. They have a minimum element corresponding to a state of minimum information in which all outcomes are equally likely. They have a set of maximal elements corresponding to pure states, and every element of the model lies below at least one of these maximal elements. Also, thermodynamic entropy, Shannon entropy, and von Neumann entropy fall into the category of measurements of information content as defined by Martin. Furthermore, the logics of Birkhoff and von Neumann (Birkhoff and von Neumann 1936) for classical and quantum systems are isomorphic to subsets of the models. In (Mashburn 2007b) the Bayesian order was extended to infinite dimensional classical states. Many of the properties of the model for finite dimensional states were retained, but some very important ones were lost. In particular, the model no longer exhibited the continuity property of a domain. In fact, all the ability to approximate or determine which states contained information essential to other states was lost. While thermodynamic entropy was still defined on the model it was no longer a measurement in the sense of Martin. Shannon entropy was no longer defined on the entire model.

In this paper we extend the spectral order to infinite dimensional quantum states. As might be expected, similar properties are lost and kept as were lost and kept in the transition from the finite dimensional Bayesian order to the infinite dimensional Bayesian order. In Section 2 we give some background information for domains and weak domains. In Section 3 we give some background on the finite-dimensional Bayesian and spectral order and in Section 4 we give a brief review of the infinite dimensional Bayesian order. In Section 5 we define the infinite dimensional spectral order and establish some of its basic properties. In Section 6 we see how unitary operators or operators that are almost unitary can be used to manipulate a fixed basis for the Hilbert space to provide structures for comparing density operators via the spectral order. We see in section 7 that the space of infinite dimensional quantum states can be decomposed into order isomorphic pieces in a fashion similar to the decompositions of the space of finite dimensional classical or quantum states. In section 8 we see that the space of infinite dimensional quantum states under the spectral order contains a subset, in fact many of them, which is order isomorphic to an important subset of the space of infinite dimensional classical states under the Bayesian order. We also see that these subsets are retracts of the whole space. Furthermore, we show that the space of infinite dimensional classical states itself is order isomorphic to a subset of the space of infinite dimensional quantum states. This is further evidence that the spectral order is a legitimate extension of the Bayesian order. In section 9 we investigate the domain properties of Ω^ω and show that, like Δ^ω, it fails miserably to be a weak domain. It is a directed complete ordered set, but is not nearly exact. In section 10 we see that reasonable measurements of entropy will preserve the spectral order, another indication that this order does indeed reflect the certainty of states. We also see that they cannot be measurements in the sense of Martin. In section 11 we consider projections of quantum states and show that the spectral order is preserved by projections. In fact, the spectral order can be determined by projections. Finally, in section 12 we show that the lattices used by Birkhoff and von Neumann to provide a structure for quantum logic arise naturally from the quantum states that are irreducible in the spectral order.

Our notation is the usual mathematical (set-theoretic) notation. The set ω of natural numbers is the set of all nonnegative integers and we will think of every $n \in \omega$ as the set of all smaller elements of ω. So $0 = \emptyset$ and $n = \{0, 1, \ldots, n-1\}$ when $n > 0$. A finite sequence is a function defined on a natural number and a (infinite) sequence is a function defined on ω. If f is a function and A is a set then $f[A] = \{f(x) : x \in A\}$. A (partial) order is a relation that is reflexive, antisymmetric, and transitive. If X is an ordered set, we will use X^* to denote the set X with the reverse of its usual order.

2. A Brief Review of Domains and Weak Domains

Throughout this section, X is an ordered set. When an ordered set is used as a model for an information system it is standard to interpret $a < b$ to mean that b contains more information than does a. During the rest of this section, X will be an ordered set with order $<$. A nonempty subset D of X is said to be directed if and only if for every $a, b \in D$ there is $c \in D$ such that $a \leq c$ and $b \leq c$. X is said to be directed-complete if every directed set has a supremum.

The basic relation, besides the order, of a domain is the way-below relation.

Definition 2.1. For $a, b \in X$, $a \ll b$ if and only if for every directed subset D of X with $\sup D \geq b$, there is $c \in D$ such that $a \leq c$.

For every element a of X let $\twoheaddownarrow a = \{b \in X : b \ll a\}$ and $\twoheaduparrow a = \{b \in X : a \ll b\}$.

Definition 2.2. The set X is said to be *continuous* if and only if for every $a \in X$, $\twoheaddownarrow a$ is directed and $\sup \twoheaddownarrow a = a$.

A domain is a continuous directed-complete ordered set, although some refer to it as a continuous domain.

A subset U of X is said to be Scott open if and only if for every directed subset D of X, if $\sup D \in U$ then $D \cap U \neq \emptyset$. The Scott topology is then the collection of all Scott open subsets of X. Every ordered set admits a Scott topology, but domains have a special relation with this topology because $\{\twoheaduparrow a : a \in X\}$ is a base for the Scott topology in a domain.

But the ordered sets used by Coecke and Martin as their models for physical states are not domains. They are not continuous. This is overcome in (Coecke and Martin 2002) by changing the definition of the way-below relation, although they still refer to the new relation as way-below. To distinguish between the two, we will call the new relation the weakly way below relation.

Definition 2.3. For all $a, b \in X$, a is *weakly way-below* b, denoted $a \ll_w b$, if and only if for every directed subset D of X, if $\sup D = b$ then there is $c \in D$ with $a \leq c$.

Let $\twoheaddownarrow_w a = \{b \in X : b \ll_w a\}$ and $\twoheaduparrow_w a = \{b \in X : a \ll_w b\}$.

Definition 2.4. X is *exact* if and only if for every $a \in X$, $\twoheaddownarrow_w a$ is directed and $\sup \twoheaddownarrow_w a = a$.

One further property is required of the ordered set which is automatically present in domains. We require that if $a \ll_w b \leq c$ and $\Uparrow_w c \neq \emptyset$ then $a \ll_w c$. A weak domain is an exact directed-complete ordered set which satisfies this last property. The models of Coecke and Martin are weak domains. Weak domains have, of course, a Scott topology, but $\{\Uparrow_w a : a \in X\}$ is no longer a basis for the Scott topology unless the weak domain is actually a domain. The set $\{\Uparrow_w a : a \in X\}$ is a basis for a different topology whose relation to the Scott topology is not completely understood.

3. A Brief Review of the Models of Finite Dimensional Classical and Quantum States

We first give an overview of the Bayesian order and the model for finite dimensional classical states. The classical states are what one has when an observable is measured in a quantum system. So a classical state should give us probabilities of various possible outcomes. For every $n \in \omega$ with $n \geq 2$

$$\Delta^n = \left\{ f \in {}^n\mathbb{R} : \forall k \in n (f(k) \geq 0) \text{ and } \sum_{k \in n} f(k) = 1 \right\}$$

For every $f, g \in \Delta^n$ set $f \leq g$ if and only if there is a permutation $\sigma : n \to n$ such that $f \circ \sigma$ and $g \circ \sigma$ are both decreasing and $(f \circ \sigma)(k)(g \circ \sigma)(k+1) \leq (f \circ \sigma)(k+1)(g \circ \sigma)(k)$ for all $k < n$. This relation results in a legitimate order on Δ^n for each n. The least element of Δ^n is the sequence in which each coordinate has the same value, so one cannot say that any outcome is more likely than another. The maximal elements are the elements which assign a probability of 1 to one of the outcomes and a probability of 0 to all other outcomes.

Each Δ^n is a weak domain or, in the terminology of Coecke and Martin, an exact domain under the Bayesian order. This allows the Bayesian order to distinguish between partial and total elements of Δ^n. Intuitively, a partial state is one which provides only partial, not total, information about each outcome.

We next give a brief description of Martin's measurements of information content. For a more thorough description see (Coecke and Martin 2002) or (Martin 2000).

Definition 3.1. Let D be a domain and let $x \in D$. A Scott continuous function $\mu : D \to [0, \infty)^*$ is said to *measure the content of* x if and only if for every Scott open subset U of D, if $x \in U$ then there is $\epsilon > 0$ such that $\{y \in D : y \leq x \text{ and } |\mu(x) - \mu(y)| < \epsilon\} \subseteq U$.

Definition 3.2. Let X be a subset of a domain D. A function is said to *measure* X if it measures the content of each element of X.

Definition 3.3. A *measurement* of a domain D is a function μ which measures $\ker \mu = \{x \in D : \mu(x) = 0\}$.

These functions measure the content of the elements of D which are supposed to have the maximum information content. Among the measurements of Δ^n are $\mu(f) = 1 - f^+$

where f^+ is the maximum value of f, a thermodynamic measure of entropy $\mu(f) = -\ln f^+$, and Shannon entropy $\mu(f) = -\sum_{j=1}^{n} f(j)\ln f(j)$.

Of course, the reason for developing an order-theoretic model for the classical states is so that it can be used to develop an order-theoretic model for quantum states. In their development of the spectral order on quantum states, Coecke and Martin use density operators to represent the states. They use Ω^n to represent the set of all density operators on the n-dimensional Hilbert space H. Their observables are self-adjoint linear operators. One can choose a sequence of n orthogonal unit eigenvectors of an observable, which will provide a basis by which to compare two states. An observable e is said to label a state r if and only if e and r commute, or r diagonalizes along the sequence of eigenvectors provided by e. These eigenvectors are used to produce a sequence of eigenvalues of r. The spectrum of r has now become an element of Δ^n. This listing of the eigenvalues of r is denoted $\mathrm{spec}(r|e)$. The spectral order on Ω^n is then defined as follows. For $r, s \in \Omega^n$ set $r \sqsubseteq s$ if and only if there is a labeling e which is admitted by both r and s and $\mathrm{spec}(r|e) \leq \mathrm{spec}(s|e)$ in Δ^n.

Ω^n has many of the same structural properties as Δ^n. It is shown by Coecke and Martin to be an effective qualitative model for finite-dimensional quantum states.

4. A Brief Review of the Infinite Dimensional Bayesian Order

In this section we highlight the main characteristics of the infinite dimensional Bayesian order that we will use to define and study the properties of the infinite dimensional spectral order. See (Mashburn 2007b) for details.

Definition 4.1. $\Delta^\omega = \{f \in {}^\omega\mathbb{R} : \forall n \in \omega(f(n) \geq 0)$ and $\sum_{n\in\omega} f(n) = 1\}$.

These functions represent the classical physical states with $f(n)$ being the probability that one obtains outcome n.

Definition 4.2. For every $f, g \in \Delta^\omega$ set $f \leq g$ if and only if there is a one-to-one function $\sigma : \omega \to \omega$ such that the following three properties are satisfied.

1 $f \circ \sigma$ and $g \circ \sigma$ are both decreasing.
2 If $f(n) \neq 0$ or $g(n) \neq 0$ then there is $m \in \omega$ such that $\sigma(m) = n$.
3 $(f \circ \sigma)(n)(g \circ \sigma)(n+1) \leq (f \circ \sigma)(n+1)(g \circ \sigma)(n)$.

Note that σ is no longer a permutation, but merely a one-to-one function.

Theorem 4.1. Let $f, g \in \Delta^\omega$. If $f \leq g$ and there is $n \in \omega$ such that $f(n) = 0$ then $g(n) = 0$. If $\{m \in \omega : g(m) > 0\}$ is infinite and $g(n) = 0$ then $f(n) = 0$. Furthermore, if there are $m, n \in \omega$ such that $g(m) = g(n) > 0$ then $f(m) = f(n)$.

This is the infinite dimensional version of the property called degeneracy by Coecke and Martin. But note that if f has an infinite number of positive coordinates and $f \leq g$ then either g is positive on all of those same coordinates or g is zero on all but a finite number of them. We cannot change only a finite number of them to zero.

Theorem 4.2. Let $f, g \in \Delta^\omega$. If $f \leq g$ and $t \in (0, 1)$ then $f \leq (1-t)f + tg \leq g$.

For every $m \in \omega$ let $e_m(n) = 1$ when $n = m$ and $e_m(n) = 0$ when $n \neq m$.

Theorem 4.3. For every $f \in \Delta^\omega$ and every $m \in \omega$, $f \leq e_m$ if and only if $f(m) = \max\{f(n) : n \in \omega\}$.

The following theorem shows that the infinite dimensional Bayesian order also satisfies one of the basic properties needed to be a domain or weak domain: it is directed-complete.

Theorem 4.4. Δ^ω is a dcpo and every directed subset D of Δ^ω contains an increasing sequence $\langle f_n : n \in \omega \rangle$ with $\sup_{n\in\omega} f_n = \sup D$.

But the infinite dimensional Bayesian order fails miserably to be even a weak domain. In fact, one cannot find a pair of elements of Δ^ω which are related by the weakly way below relation. See Theorem 32 of (Mashburn 2007b). So some of the most desirable aspects of domain theory are lost.

Unlike the finite dimensional states, the infinite dimensional states don't come in easily recognizable levels or dimensions. We can, nonetheless, use projections to our advantage in our study of Δ^ω. Let $\mathcal{P}(f) = \{A \subseteq \omega : \sum_{n\in A} f(n) > 0\}$. Note that $\mathcal{P}(f) = \{A \subseteq \omega : \exists n \in A(f(n) > 0)\}$. If $A \in \mathcal{P}(f)$ then the projection $p_A(f)$ of f onto A is defined as follows.

Definition 4.3. $p_A(f) = \begin{cases} \frac{f(n)}{\sum_{m\in A} f(m)} & n \in A \\ 0 & n \notin A \end{cases}$

Theorem 4.5. For every $f, g \in \Delta^\omega$, $f \leq g$ if and only if $p_A(f) \leq p_A(g)$ for every $A \in \mathcal{P}(f) \cap \mathcal{P}(g)$.

In particular, the infinite Bayesian order reflects back to the finite Bayesian order, so that the infinite Bayesian order can be considered *the* natural extension of the finite Bayesian order to infinite dimensional states.

Definition 4.4. $\Delta^\omega_+ = \{f \in \Delta^\omega : \forall m \in \omega\,(f(m) > 0)\}$. For every $A \subseteq \omega$, $\Delta^\omega(A) = \{f \in \Delta^\omega : (\forall m \in \omega - A)(f(m) = 0)\}$ and $\Delta^\omega_+(A) = \{f \in \Delta^\omega : \forall m \in \omega\,(f(m) > 0 \iff m \in A)\}$.

Theorem 4.6. Let $n \in \omega$ and $A \subseteq \omega$.

1 If $|A| = n$ then $\Delta^\omega(A)$ is order isomorphic to Δ^n.
2 If $|A| = \omega$ then $\Delta^\omega(A)$ is order isomorphic to Δ^ω.
3 If $|A| = \omega$ then $\Delta^\omega_+(A)$ is order isomorphic to Δ^ω_+.

Definition 4.5. For every one-to-one function $\sigma : \omega \to \omega$ let Δ^ω_σ be the set of all $f \in \Delta^\omega$ such that $f \circ \sigma$ is decreasing. The set determined in this way by the identity function, that is the set of decreasing elements of Δ^ω, is denoted Λ^ω.

Theorem 4.7. For every one-to-one function $\sigma : \omega \to \omega$, Δ^ω_σ is order isomorphic to Λ^ω.

The function $s(f) = -\ln(f^+)$, where f^+ is the maximum value of f, provides a reasonable thermodynamic style measurement of entropy on Δ^ω. But Shannon entropy is not defined on all of Δ^ω due to the infinite number of coordinates. We say that f has finite Shannon entropy when the infinite series $S(f) = -\sum f(n) \ln f(n)$ converges, and that it has infinite Shannon entropy when the series diverges.

Theorem 4.8. Let $f, g \in \Delta^\omega$ with $f < g$. If f has finite Shannon entropy then g has finite Shannon entropy and $S(f) > S(g)$.

Every maximal element of Δ^ω, which automatically has Shannon entropy 0, is the limit of an increasing sequence of elements of Δ^ω which have infinite Shannon entropy.

A function ϕ defined on Δ^ω is said to be symmetric if and only if $\phi(f) = \phi(f \circ \sigma)$ for every one-to-one function σ from ω onto ω. In other words, we can rearrange the coordinates of f without changing the value of $\phi(f)$.

Theorem 4.9. If $\mu : \Delta^\omega \to [0, \infty)^*$ is symmetric and $\ker \mu = \max \Delta^\omega$ then μ is not a measurement of Δ^ω in the sense of Martin.

This means that neither of the types of entropy mentioned above are measurements in the sense of Martin.

5. Definition of the Spectral Order for Infinite Dimensional States

Let H be a countably infinite dimensional Hilbert space. We represent the states based on H as density operators (self-adjoint, positive linear operators of trace 1) on H. Let Ω^ω be the set of density operators on H. We want to follow the approach of Coecke and Martin in creating a sequence of eigenvalues of these operators which can then be treated as elements of Δ^ω. The problem is that different operators with the same eigenvalues could have very different eigenvectors. To differentiate between these different operators we will use orthonormal subsets of H arranged as sequences.

Definition 5.1. An *orthonormal sequence* is a one-to-one function $B : \omega \to H$ such that $\operatorname{ran} B$ is an orthonormal subset of H.

We sometimes abuse the notation by identifying the sequence with its range. Note that every orthonormal sequence can be extended to an orthonormal basis for H. If $r \in \Omega^\omega$ then we will use $E(r, \lambda)$ to denote the eigenspace of r corresponding to the eigenvalue λ. To fully understand the density operator we want to know its positive eigenvalues and their multiplicity.

Definition 5.2. Let $r \in \Omega^\omega$ and $B : \omega \to H$. The *coordinate function of* r *relative to* B is the function f^r_B given by $f^r_B(n) = \langle r(B(n)) | B(n) \rangle$.

If $B(n)$ is an eigenvector of r then $f^r_B(n)$ is the eigenvalue of r corresponding to $B(n)$.

Definition 5.3. The orthonormal sequence B is said to *label* the density operator r if and only if the following conditions are satisfied.

1 For every $n \in \omega$, $B(n)$ is an eigenvector of r.
2 For every $\lambda \in \operatorname{spec}^+(r)$ there is $M \subseteq \omega$ such that $B[M]$ is a basis for $E(r, \lambda)$.
3 $f^r_B \in \Lambda^\omega$.

Part 1 means that f_B^r is a list of eigenvalues of r. Part 2 means that the list contains every positive eigenvalue of r and that the number of times that a positive eigenvalue appears in the list equals the multiplicity of the eigenvalue. Part 3 means that the list is descending.

Theorem 5.1. Every element of Ω^ω is labeled by some orthonormal sequence.

Proof. This follows from the fact that every density function is compact. See (Fano 1971), page 291 and 376. $\square$

Since f_B^r is an element of Λ^ω we can now use the Bayesian order to compare various coordinate functions. For every density operator r let $L(r)$ denote the set of all orthonormal sequences that label r. Note that if B labels r and is not a basis for H then if A is a basis for H and $B \subseteq A$ every element of A which is not in B is an eigenvalue of r corresponding to 0. Also note that if $A, B \in L(r)$ for some $r \in \Omega^\omega$ then $f_A^r = f_B^r$.

Theorem 5.2. For every orthonormal sequence A of H and every $f \in \Delta^\omega$ there is $r \in \Omega^\omega$ such that A labels r and $f_A^r = f$.

Proof. Let r be the linear operator on H defined by setting $r(A(n)) = f(n)A(n)$ for all $n \in \omega$ and $f(\boldsymbol{\alpha}) = \mathbf{0}$ if $\boldsymbol{\alpha} \perp A$. Then r is obviously a positive linear operator on H and if we extend A to an orthonormal basis B for H then the trace of r along B is $\sum f(n) = 1$. It is easy to show that r is self-adjoint, therefore $r \in \Omega^\omega$. It is also easy to see that $f_A^r = f$. $\square$

Definition 5.4. The *spectral order* on Ω^ω is the relation $\sqsubseteq$ on Ω^ω defined by setting $r \sqsubseteq s$ if and only if there is $A \in L(r) \cap L(s)$ such that $f_A^r \leq f_A^s$, where $\leq$ is the Bayesian order on Δ^ω.

We say that an orthonormal sequence A witnesses $r \sqsubseteq s$ when A labels both r and s and $f_A^r \leq f_A^s$.

The following theorem comes directly from the fact, noted above, that every orthonormal sequence which labels a given density operator r produces the same coordinate function for r.

Theorem 5.3. Let $r, s \in \Omega^\omega$. If $r \sqsubseteq s$ and B is an orthonormal sequence that labels both r and s then $f_B^r \leq f_B^s$.

We follow the terminology of (Coecke and Martin 2002) and refer to the properties in the following theorem as degeneracy.

Theorem 5.4. For every $r, s \in \Omega^\omega$, if $r \sqsubseteq s$ then the following three properties hold.

1 $E(r, 0) \subseteq E(s, 0)$.
2 For every $\mu \in \operatorname{spec}^+ s$ there is $\lambda \in \operatorname{spec}^+ r$ such that $E(s, \mu) \subseteq E(r, \lambda)$.
3 If the subspace of H generated by the eigenvectors of s corresponding to positive eigenvalues is infinite dimensional then $E(s, 0) \subseteq E(r, 0)$.

Proof. Let A be an orthonormal sequence that witnesses $r \sqsubseteq s$. If $E(r, 0) = \emptyset$ then Part 1 follows immediately, so assume that $E(r, 0) \neq \emptyset$. Let $M = \{n \in \omega : A(n) \in$

$E(r,0)\}$. Of course, M could be empty. Regardless, we can expand $A[M]$ to an orthonormal basis B for $E(r,0)$. If $n \in M$ then $f_A^r(n) = 0$ and therefore $f_A^s(n) = 0$ since $f_A^r \le f_A^s$. If $\boldsymbol{\alpha} \in B - A[M]$ then $\boldsymbol{\alpha} \perp A(n)$ for all $n \in \omega$. Therefore $s(\boldsymbol{\alpha}) = \mathbf{0}$ and $\boldsymbol{\alpha} \in E(s,0)$. It follows that $E(r,0) \subseteq E(s,0)$.

Now let $\mu \in \operatorname{spec}^+ s$ and let $M \subseteq \omega$ such that $A[M]$ is a basis for $E(s,\mu)$. Then $f_A^s(n) = \mu$ for all $n \in M$. If M contains only one element then Part 2 follows immediately, so assume that M contains more than one element. Because $f_A^r \le f_A^s$ Theorem 4.1 implies that f_A^r is also constant on M. So there is $\lambda \in \operatorname{spec}^+ r$ such that $f_A^r(n) = \lambda$ for all $n \in M$. Thus $A[M] \subseteq E(r,\lambda)$ and $E(s,\mu) \subseteq E(r,\lambda)$.

Finally, assume that the subspace of H generated by the eigenvectors of s corresponding to positive eigenvalues is infinite dimensional. Then $f_A^s(n) > 0$ for every $n \in \omega$. If $\boldsymbol{\alpha} \in E(s,0)$ then $\boldsymbol{\alpha} \perp A(n)$ for every $n \in \omega$. If follows that $r(\boldsymbol{\alpha}) = \mathbf{0}$ so $\boldsymbol{\alpha} \in E(r,0)$. Therefore $E(s,0) \subseteq E(r,0)$. □

Note that the proof of part 2 above shows that if $f_A^s(n) > 0$ then $E(s, f_A^s(n)) \subseteq E(r, f_A^r(n))$.

Theorem 5.5. The relation $\sqsubseteq$ is an order.

Proof. The reflexivity and antisymmetry of $\sqsubseteq$ follow from the reflexivity and antisymmetry of the Bayesian order on Δ^ω. We just need to show that $\sqsubseteq$ is transitive. Let $r, s, t \in \Omega^\omega$ with $r \sqsubseteq s$ and $s \sqsubseteq t$, and let A and B be orthonormal sequences that witness $r \sqsubseteq s$ and $s \sqsubseteq t$ respectively. We construct an orthonormal sequence C that labels all of r, s, and t. Since $f_C^r = f_A^r \le f_A^s = f_B^s \le f_B^t = f_C^t$ we will then have $r \sqsubseteq t$. For every $\mu \in \operatorname{spec}^+ t$ let $N(t,\mu) \subseteq \omega$ such that $B[N(t,\mu)]$ is a basis for $E(t,\mu)$. For every $\lambda \in \operatorname{spec}^+ s$ let $N(s,\lambda) \subseteq \omega$ such that $B[N(s,\lambda)]$ is a basis for $E(s,\lambda)$. Since $f_A^s = f_B^s$ we also know that $A[N(s,\lambda)]$ is a basis for $E(s,\lambda)$. For every $\kappa \in \operatorname{spec}^+ r$ let $N(r,\kappa) \subseteq \omega$ such that $A[N(r,\kappa)]$ is a basis for $E(r,\kappa)$.

Let $n \in \omega$. If there is $\lambda \in \operatorname{spec}^+ s$ such that $n \in N(s,\lambda)$ then set $C(n) = B(n)$. If not, then set $C(n) = A(n)$. The function C is one-to-one on $\bigcup\{N(s,\lambda) : \lambda \in \operatorname{spec}^+ s\}$ and on $\omega - \bigcup\{N(s,\lambda) : \lambda \in \operatorname{spec}^+ s\}$. Also, $C[\bigcup\{N(s,\lambda) : \lambda \in \operatorname{spec}^+ s\}]$ and $C[\omega - \bigcup\{N(s,\lambda) : \lambda \in \operatorname{spec}^+ s\}]$ are both orthonormal subsets of H.

Let $N = \bigcup\{N(s,\lambda) : \lambda \in \operatorname{spec}^+ s\}$ and let $n \in \omega - N$. If $C(n) = A(n) \in E(r,0)$ then $A(n) \in E(s,0)$ so $C(n) \perp C[N]$. Assume that there is $\kappa \in \operatorname{spec}^+ r$ such that $C(n) \in E(r,\kappa)$. Let $m \in N$ and let $\lambda \in \operatorname{spec}^+ s$ such that $C(m) = B(m) \in E(s,\lambda)$. We show that $C(n) \perp C(m)$. By Theorem 5.4 there is $\gamma \in \operatorname{spec}^+ r$ such that $E(s,\lambda) \subseteq E(r,\gamma)$. If $\gamma \ne \kappa$ then $E(r,\gamma) \perp E(r,\kappa)$ so $C(n) \perp C(m)$. Assume that $\gamma = \kappa$. Now $A[N(s,\lambda)]$ is a basis for $E(s,\lambda)$, $A[N(r,\kappa)]$ is a basis for $E(r,\kappa)$, and $E(s,\lambda) \subseteq E(r,\kappa)$. Therefore $N(s,\lambda) \subseteq N(r,\kappa)$. Since $n \notin N(s,\lambda)$ and $m \in N(s,\lambda)$ we have $A(n) \in E(r,\kappa) - E(s,\lambda)$ and $B(m) \in E(s,\lambda)$. Thus $C(m) = B(m) \perp A(n) = C(n)$. It follows that C is one-to-one and that $\operatorname{ran} C$ is an orthonormal subset of H.

Let $\mu \in \operatorname{spec}^+ t$. There is $\lambda \in \operatorname{spec}^+ s$ such that $E(t,\mu) \subseteq E(s,\lambda)$. Then $N(t,\mu) \subseteq N(s,\lambda)$ so $C[N(t,\mu)] = B[N(t,\mu)]$ is a basis for $E(t,\mu)$. Obviously, if $\lambda \in \operatorname{spec}^+ s$ then $C[N(s,\lambda)] = B[N(s,\lambda)]$ is a basis for $E(s,\lambda)$.

Let $\kappa \in \operatorname{spec}^+ r$. Let $M_0 = \{n \in N(r,\kappa) : \exists \lambda \in \operatorname{spec}^+ s\,(n \in N(s,\lambda))\} = N(r,\kappa) \cap N$

and let $M_1 = M(r,\kappa) - M_0$. Note that if $N(r,\kappa) \cap N(s,\lambda) \neq \emptyset$ then $E(s,\lambda) \subseteq E(r,\kappa)$. Now $A[M_0]$ and $C[M_0] = B[M_0]$ are both bases for the subspace of H generated by the union of all $E(s,\lambda)$ such that $\lambda \in \operatorname{spec}^+ s$ and $E(s,\lambda) \subseteq E(r,\kappa)$. Therefore $C[M_0 \cup M_1] = B[M_0] \cup A[M_1]$ is a basis for $E(r,\kappa)$.

So far we have shown that C satisfies parts 1 and 2 of the definition of a label for r, s, and t. It follows that $\sum f^r_C$, $\sum f^s_C$, and $\sum f^t_C$ all equal 1. To complete the proof we must show that f^r_C, f^s_C, and f^t_C are all decreasing. First, $f^s_C = f^s_B$ so f^s_C is obviously decreasing. Let $n \in \omega$. If $f^s_C(n) = 0$ then $n \in \omega - N$ so $C(n) = A(n)$. Therefore $f^r_C(n) = f^r_A(n)$. Also, $f^s_A(n) = f^s_B(n) = 0$, so $f^t_B(n) = 0$ and $A(n) \in E(s,0) \subseteq E(t,0)$. Thus $C(n) \in E(t,0)$ and $f^t_C(n) = 0 = f^t_B(n)$. If $f_C(n) > 0$ then $n \in N$ so $C(n) = B(n)$. Therefore $f^t_C(n) = f^t_B(n)$. Let $\lambda = f^s_C(n)$. Since $\lambda > 0$ there is $\kappa \in \operatorname{spec}^+ r$ such that $E(s,\lambda) \subseteq E(r,\kappa)$. Then $C(n) \in E(r,\kappa)$ so $f^r_c(n) = \kappa = f^r_A(n)$. Thus $f^r_C = f^r_A$ and $f^t_C = f^t_B$ so f^r_C and f^t_C are both decreasing. $\square$

Definition 5.5. For every nonzero $\boldsymbol{\alpha} \in H$ let $e_{\boldsymbol{\alpha}}$ be the density operator on H defined by setting $e_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ and $e_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \mathbf{0}$ for all $\boldsymbol{\beta} \in H$ that are orthogonal to $\boldsymbol{\alpha}$.

Theorem 5.6. For every $r \in \Omega^\omega$ and every nonzero $\boldsymbol{\alpha} \in H$, $r \sqsubseteq e_{\boldsymbol{\alpha}}$ if and only if $\boldsymbol{\alpha} \in E(r,\lambda)$ where $\lambda = \max \operatorname{spec}^+ r$.

The proof is easy. This means that the $e_{\boldsymbol{\alpha}}$'s are maximal in Ω^ω and every element in Ω^ω is less than or equal to at least one $e_{\boldsymbol{\alpha}}$. The maximal elements of Ω^ω can also be described as the density operators having an eigenvalue of 1.

6. Unitary Operators and the Spectral Order

The definition of the spectral order does not use a fixed basis for H because the eigenvectors of density operators do not come from a fixed set. It is possible to base the comparison of density operators on a single fixed basis. This process involves either rearranging the basis to fit certain operators (the passive approach), or rearranging the operators to fit the basis (the active approach). The rearranging is done by operators which are almost, but not quite, unitary.

Definition 6.1. A linear operator $U : H \to H$ is a *pseudo-unitary operator* if and only if $\langle U(\boldsymbol{\alpha}) | U(\boldsymbol{\beta}) \rangle = \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in H$.

A pseudo-unitary operator is one-to-one, and therefore is invertible, but it need not be onto when H is infinite dimensional, which is why it need not be unitary. If A is an orthonormal subset of H then so is $U[A]$.

Definition 6.2. Let A and B be orthonormal subsets of H with B a basis for H. Let $f : B \to A$ be a bijection. Then U_{BA} is the linear operator defined by setting $U_{BA}(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha})$ for every $\boldsymbol{\alpha} \in B$.

When A and B are orthonormal sequences of H the bijection we use is given by $f(B(n)) = A(n)$. The following lemma follows easily from the definition.

Lemma 6.1. Let A and B be orthonormal subsets of H with B a basis for H. If $f : B \to A$ is a bijection then U_{BA} is a pseudo-unitary operator.

Theorem 6.1 (Passive). Fix an orthonormal sequence B which is a basis for H. For every $r, s \in \Omega^\omega$, $r \sqsubseteq s$ if and only if there is a pseudo-unitary operator $U : H \to H$ such that $U \circ B$ witnesses $r \sqsubseteq s$.

Proof. Assume that $r \sqsubseteq s$ and let A witness this fact. Then U_{BA} is a pseudo-unitary operator and $U_{BA} \circ B = A$. The other direction is obvious. □

This means that $r \circ (U \circ B)$ and $s \circ (U \circ B)$ are both decreasing and satisfy the following inequality which defines the Bayesian order.

$$[r \circ (U \circ B)](n)[s \circ (U \circ B)](n+1) \leq [r \circ (U \circ B)](n+1)[s \circ (U \circ B)](n)$$

So, except for the orthogonal sequence B which is needed to give structure to H, the situation is the same as that for $f \circ \sigma$ and $g \circ \sigma$ in Δ^ω. The pseudo-unitary operator U performs the same function for r and s that σ does for f and g.

Lemma 6.2. Fix an orthonormal sequence B which is a basis of H. For every $r \in \Omega^\omega$ there is $r_B \in \Omega^\omega$ such that B labels r_B, $\operatorname{spec}^+ r_B = \operatorname{spec}^+ r$, and $\dim(E(r_B, \lambda)) = \dim(E(r, \lambda))$ for all $\lambda \in \operatorname{spec}^+ r$.

Proof. If $\operatorname{spec}^+ r$ is infinite let $\lambda(n)$ be a decreasing sequence whose range is $\operatorname{spec}^+ r$ and has the property that the number of times each $\lambda \in \operatorname{spec}^+ r$ appears in the sequence equals $\dim(E(r, \lambda))$. If $\operatorname{spec}^+ r$ is finite let $\lambda(n)$ be an infinite deceasing sequence of nonnegative real numbers whose range is $\operatorname{spec} r$ and which has the property that the number of times each $\mu \in \operatorname{spec} r$ appears in the sequence equals $\dim(E(r, \mu))$. The sequence $\lambda(n)$ is eventually zero. Let r_B be the linear operator determined by setting $r_B(B(n)) = \lambda(n)B(n)$ for all $n \in \omega$. Then $\operatorname{spec}^+ r_B = \operatorname{spec}^+ r$ and $\dim(E(r_B, \lambda)) = \dim(E(r, \lambda))$ for all $\lambda \in \operatorname{spec}^+ r$. It is also easy to see that B labels r_B. □

If $r \sqsubseteq s$ and A labels both r and s then $f_B^{r_B} = f_A^r \leq f_A^s = f_B^{s_B}$, so $r_B \sqsubseteq s_B$. But $r_B \sqsubseteq s_B$ is not enough to give $r \sqsubseteq s$ because r and s could be based on very different orthonormal sequences even when their eigenvalues form sequences that are comparable in Δ^ω. The following theorem overcomes this difficulty and shows the role played by pseudo-unitary operators.

Theorem 6.2 (Active). Fix an orthonormal sequence B which is a basis for H. For every $r, s \in \Omega^\omega$, $r \sqsubseteq s$ if and only if there is a pseudo-unitary operator U on H such that $U^{-1} \circ r \circ U$ and $U^{-1} \circ s \circ U$ are density operators and B witnesses $U^{-1} \circ r \circ U \sqsubseteq U^{-1} \circ s \circ U$.

Proof. Let A be an orthonormal sequence which witnesses $r \sqsubseteq s$ and set $U = U_{BA}$. Let $r_U = U^{-1} \circ r \circ U$ and $s_U = U^{-1} \circ s \circ U$. Then $B(n) \in \operatorname{spec} r_U$ for every $n \in \omega$. Furthermore, if $\lambda \in \operatorname{spec}^+ r$ there is $n \in \omega$ such that $A(n) \in E(r, \lambda)$. Then $r_U(B(n)) = \lambda B(n)$ so $B(n) \in E(r_U, \lambda)$. Therefore $\operatorname{spec}^+ r \subseteq \operatorname{spec}^+ r_U$. If $\lambda \in \operatorname{spec}^+ r_U$ then $U^{-1}(r(U(\boldsymbol{\alpha}))) = \lambda \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in H$. But then $r(U(\boldsymbol{\alpha})) = \lambda U(\boldsymbol{\alpha})$ so $\lambda \in \operatorname{spec}^+ r$. Thus $\operatorname{spec}^+ r_U = \operatorname{spec}^+ r$. Essentially the same argument shows that $\dim(E(r_U, \lambda)) = \dim(E(r, \lambda))$ for all $\lambda \in \operatorname{spec}^+ r$. Thus the trace of r_U is 1. It is also clear that r_U is a positive operator. It is a straightforward exercise to show that r_U is self-adjoint, so $r_U \in \Omega^\omega$. In the same way we can show that $s_U \in \Omega^\omega$.

It is clear from what we have done that $f_B^{r_U} = f_A^r$ and that $f_B^{s_U} = f_A^s$, so $f_B^{r_U}, f_B^{s_U} \in \Lambda^\omega$. Let $\lambda \in \operatorname{spec}^+ r$. There is $M \subseteq \omega$ such that $A[M]$ is a basis for $E(r, \lambda)$. But $B[M] \subseteq E(r_U, \lambda)$ and $\dim(E(r_U, \lambda)) = \dim(E(r, \lambda))$ so $B[M]$ is a basis for $E(r_U, \lambda)$. Thus B labels r_U. Similarly B labels s_U. Finally, $f_B^{r_U} = f_A^r \leq f_A^s = f_B^{s_U}$ so B witnesses $r_U \sqsubseteq s_U$.

To prove the other direction assume that there is a pseudo-unitary operator U such that $U^{-1} \circ r \circ U$ and $U^{-1} \circ s \circ U$ are both density operators and B witnesses $U^{-1} \circ r \circ U \sqsubseteq U^{-1} \circ s \circ U$. Again let $r_U = U^{-1} \circ r \circ U$ and $s_U = U^{-1} \circ s \circ U$, and set $A = U \circ B$. Then A is an orthonormal sequence. One can easily show that if $\lambda > 0$ then, for every $n \in \omega$, $A(n) \in E(r, \lambda)$ if and only if $B(n) \in E(r_U, \lambda)$. Therefore $\operatorname{spec}^+ r = \operatorname{spec}^+ r_U$ and $\dim E(r, \lambda) = \dim E(r_U, \lambda)$ for all $\lambda \in \operatorname{spec}^+ r$. It follows that A contains a basis for $E(r, \lambda)$ for all $\lambda \in \operatorname{spec}^+ r$. Also $f_A^r = f_B^{r_U} \in \Lambda^\omega$. Therefore A labels r. The same argument shows that A labels s and that $f_A^s = f_B^{s_U}$. Therefore A witnesses $r \sqsubseteq s$. □

Theorem 6.3. If $U : H \to H$ is a unitary operator then the function defined by $\phi_U(r) = U \circ r \circ U^{-1}$ is an order isomorphism from Ω^ω onto Ω^ω.

Proof. We first show that $\operatorname{ran} \phi_U \subseteq \Omega^\omega$. Let $r \in \Omega^\omega$ and set $t = \phi_U(r)$. It is straightforward to show that t is self-adjoint. If μ is an eigenvalue of t and $\boldsymbol{\beta} \in E(t, \mu)$ then $U(r(U^{-1}(\boldsymbol{\beta}))) = \mu\boldsymbol{\beta}$ or $r(U^{-1}(\boldsymbol{\beta})) = \mu U^{-1}(\boldsymbol{\beta})$. Therefore μ is an eigenvalue of r. It follows that t is a positive operator. If λ is an eigenvalue of r and $\boldsymbol{\alpha} \in E(r, \lambda)$ then $t(U(\boldsymbol{\alpha})) = U(r(U^{-1}(U(\boldsymbol{\alpha})))) = U(r(\boldsymbol{\alpha})) = \lambda U(\boldsymbol{\alpha})$. So r and t have the same eigenvalues. In order to show that the trace of t is 1 we show that $\dim E(t, \lambda) = \dim E(r, \lambda)$ for every $\lambda \in \operatorname{spec}^+ r$.

Let A be an orthonormal sequence that labels r and set $B = U \circ A$. Since U is unitary we know that B is an orthonormal sequence. We have already seen in the previous paragraph that the range of B consists of eigenvectors of t. Furthermore, $f_B^t = f_A^r$ so that $f_B^t \in \Lambda^\omega$.

If $\lambda \in \operatorname{spec}^+ t$ then $\lambda \in \operatorname{spec}^+ r$ and there is $M \subseteq \omega$ such that $A[M]$ is a basis for $E(r, \lambda)$. Since $f_B^t = f_A^r$ we know that $B[M] \subseteq E(t, \lambda)$. If $\boldsymbol{\alpha} \in E(t, \lambda)$ and $\boldsymbol{\alpha} \notin \operatorname{span}(B[M])$ then $U^{-1}(\boldsymbol{\alpha}) \in E(r, \lambda)$ and $U^{-1}(\boldsymbol{\alpha}) \notin \operatorname{span}(A[M])$, which is impossible. Therefore $B[M]$ is a basis for $E(t, \lambda)$. We get two results from this. First, B labels t. Second, $\dim E(t, \lambda) = |B[M]| = |A[M]| = \dim E(r, \lambda)$. As a consequence of the second result we get that the trace of t is 1. Thus $t \in \Omega^\omega$. Since these properties hold for $\phi_{U^{-1}}$ we have $\operatorname{ran} \phi_U = \Omega^\omega$.

It is obvious that ϕ_U is one-to-one and that $\phi_U^{-1} = \phi_{U^{-1}}$. Let $r, s \in \Omega^\omega$ with $r \sqsubseteq s$. Let $t = \phi_U(r)$ and $v = \phi_U(s)$. Let A be an orthonormal sequence that witnesses $r \sqsubseteq s$ and set $B = U \circ A$. Then B labels both t and v. Furthermore, since $f_B^t = f_A^r$ and $f_B^v = f_A^s$ we have $f_B^t \leq f_B^v$ in Δ^ω. Thus $t \sqsubseteq v$. It follows that both ϕ_U and ϕ_U^{-1} are increasing, so ϕ_U is an order isomorphism. □

So unitary and pseudo-unitary operators simply rearrange the elements of Ω^ω while preserving the order relationship between the elements.

7. Decompositions of Ω^ω

Δ^ω can be decomposed into subsets consisting of sequences all of which can be made decreasing through the same rearrangement of their coordinates. These subsets are all order isomorphic to one another. In this section we will show that something similar can be done for Ω^ω.

Definition 7.1. For every orthonormal sequence B let $\Omega^\omega_B = \{r \in \Omega^\omega : B \in L(r)\}$.

Theorem 7.1. If A and B are orthonormal sequences then Ω^ω_A and Ω^ω_B are order isomorphic.

Proof. For every $r \in \Omega^\omega_A$ let $\psi_{AB}(r)$ be the linear operator on H determined by setting $\psi_{AB}(r)(\boldsymbol{\alpha}) = f^r_A(n)B(n)$ if $\boldsymbol{\alpha} = B(n)$ and $\psi_{AB}(\boldsymbol{\alpha}) = \mathbf{0}$ if $\boldsymbol{\alpha}$ is orthogonal to $B[\omega]$. Consider an arbitrary $r \in \Omega^\omega_A$ and to simplify the notation set $t = \psi_{AB}(r)$. By extending B to an orthonormal basis B' if necessary and recalling that $t(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha} \in B' - B$, one can show that t is self-adjoint.

It follows easily from the definition of $\psi_{AB}(r)$ that $\operatorname{spec}^+ t = \operatorname{spec}^+ r$ and we also see that $\dim E(t, \lambda) = \dim E(r, \lambda)$ for all $\lambda \in \operatorname{spec}^+ t$. Thus t is a positive operator of trace 1. It is also obvious that B labels t and that $f^t_B = f^r_A$. Therefore $\operatorname{ran} \psi_{AB} \subseteq \Omega^\omega_B$.

Now let $s \in \Omega^\omega_B$. It follows from our preceding arguments that $r = \psi_{BA}(s) \in \Omega^\omega_A$ and that $f^r_A = f^s_B$. Extend A to an orthonormal basis A' of H. If $\boldsymbol{\alpha} = A(n)$ for some n then $\psi_{AB}(r)(\boldsymbol{\alpha}) = f^r_A(n)B(n) = f^s_B(n)B(n) = s(\boldsymbol{\alpha})$. If $\boldsymbol{\alpha}$ is orthogonal to $B[\omega]$ then $\psi_{AB}(r)(\boldsymbol{\alpha}) = \mathbf{0} = s(\boldsymbol{\alpha})$. Therefore $\psi_{AB}(r) = s$ and $\operatorname{ran} \psi_{AB} = \Omega^\omega_B$. We have also shown that $\psi_{BA} = \psi^{-1}_{AB}$ so ψ_{AB} is one-to-one.

Let $r, s \in \Omega^\omega_A$ with $r \sqsubseteq s$. Let $t = \psi_{AB}(r)$ and $u = \psi_{AB}(s)$. Then $f^t_B = f^r_A \leq f^s_A = f^u_B$, so $\psi_{AB}(r) \sqsubseteq \psi_{AB}(s)$. We can apply this result to ψ_{BA}, so ψ_{AB} is an order isomorphism. □

So Ω^ω_A plays the same sort of role for Ω^ω that Δ^ω_σ does for Δ^ω.

Corollary 7.1. Fix an orthonormal sequence B of H. If U and V are pseudo-unitary operators on H then $\Omega^\omega_{U \circ B}$ is order isomorphic to $\Omega^\omega_{V \circ B}$.

There is also a decomposition of Ω^ω which follows the active approach to using pseudo-unitary operators to compare density operators.

Definition 7.2. Fix an orthonormal sequence B of H. If U is a pseudo-unitary operator on H then $\Omega^\omega_U = \{r \in \Omega^\omega : B \in L(U^{-1} \circ r \circ U)\}$.

Theorem 7.2. Fix an orthonormal sequence B for H. If U is a pseudo-unitary operator on H then $\Omega^\omega_U = \Omega^\omega_{U \circ B}$.

Proof. Let $r \in \Omega^\omega_U$. Then B labels $U^{-1} \circ r \circ U$. To simplify the notation let $s = U^{-1} \circ r \circ U$. Using an argument similar to that in the proof of Theorem 6.2 one can show that $(U \circ B)(n) \in E(r, f^s_B(n))$ for all $n \in \omega$. This means that $f^r_{U \circ B} = f^s_B$. Since $\sum_{n \in \omega} f^r_{U \circ B}(n) = \sum_{n \in \omega} f^s_B(n) = 1 = \sum_{\lambda \in \operatorname{spec}^+ r} \lambda \dim E(r, \lambda)$, it follows that $U \circ B$ labels r and that $r \in \Omega^\omega_{U \circ B}$.

Now let $r \in \Omega^\omega_{U \circ B}$. Then $U \circ B$ labels r. Again set $s = U^{-1} \circ r \circ U$. One can then show that $B(n) \in E(s, f^r_{U\circ B}(n))$. This means that $f^s_B = f^r_{U\circ B}$. Since $\sum_{n\in\omega} f^s_B(n) = \sum_{n\in\omega} f^r_{U\circ B}(n) = 1 = \sum_{\lambda\in\operatorname{spec}^+ s} \lambda \dim E(s,\lambda)$ it follows that B labels s and that $r \in \Omega^\omega_U$. $\square$

Corollary 7.2. Fix an orthonormal sequence B in H. If U and V are pseudo-unitary operators on H then Ω^ω_U is order isomorphic to Ω^ω_V.

8. A Comparison of Ω^ω and Δ^ω

The set Λ^ω is an important subset of Δ^ω because it is prototypical of subsets which decompose Δ^ω. If we can prove that Λ^ω satisfies a certain property then the property can generally be extended to all of Δ^ω. In this section we see that the subsets we used in the previous section to decompose Ω^ω are order isomorphic to Λ^ω.

Theorem 8.1. For every orthonormal sequence A of H, Ω^ω_A is order isomorphic to Λ^ω.

Proof. For every $r \in \Omega^\omega_A$ let $\phi(r) = f^r_A$. Then $\operatorname{ran}\phi = \Lambda^\omega$. If $r, s \in \Omega^\omega_A$ and $r \neq s$ then $f^r_A \neq f^s_A$. Thus ϕ is one-to-one. Also, $r \sqsubseteq s$ if and only if $f^r_A \leq f^s_A$. Therefore ϕ is an order isomorphism. $\square$

Corollary 8.1. Fix an orthonormal sequence B in H. For every pseudo-unitary operator U on H, Ω^ω_U is order isomorphic to Λ^ω.

We next show that Δ^ω itself is order isomorphic to a subset of Ω^ω.

Definition 8.1. For every orthonormal sequence A in H set Γ_A equal to the set of all density operators r on H which satisfy the following properties.

1 $A(n)$ is an eigenvector of r for every $n \in \omega$.
2 For every $\lambda \in \operatorname{spec}^+ r$ there is $M \subseteq \omega$ such that $A[M]$ is a basis for $E(r,\lambda)$.

The set Γ_A contains all density operators labeled by A, and hence is nonempty, but also contains some operators not labeled by A. If r is such an operator then f^r_A will contain the eigenvalues that we want in the sequence, but won't list them in descending order.

Theorem 8.2. For every orthonormal sequence A in H, Γ_A is order isomorphic to Δ^ω.

Proof. For every $r \in \Gamma_A$ let $\Phi(r) = f^r_A$. That $\Delta^\omega = \operatorname{ran}\Phi$ follows from Theorem 5.2. For every $n \in \omega$, $A(n) \in \operatorname{spec} r$ so $f^r_A(n) \in \mathbb{R}$ and $f^r_A(n) \geq 0$. Also, $\sum_{n\in\omega} f^r_A(n) \leq \sum_{\lambda\in\operatorname{spec} r} \lambda \dim E(r,\lambda) = 1$. But there is a one-to-one function $\rho : \omega \to \omega$ such that $A \circ \rho \in L(r)$. Therefore $\sum_{n\in\omega} f^r_A(n) \geq \sum_{n\in\omega} f^r_A(\rho(n)) = 1$. Thus $f^r_A \in \Delta^\omega$

Let $r, s \in \Gamma_A$ such that $\Phi(r) = \Phi(s)$, or $f^r_A = f^s_A$. Let ρ and σ be one-to-one functions from ω to ω such that $A \circ \rho$ labels r and $A \circ \sigma$ labels s. Now $\operatorname{ran}(A\circ\rho) \subseteq \operatorname{ran} A$ and $\operatorname{ran}(A\circ\sigma) \subseteq \operatorname{ran} A$ so $r(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha}$ orthogonal to $\operatorname{ran}(A\circ\rho)$ and $s(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha}$ orthogonal to $\operatorname{ran}(A\circ\sigma)$. Therefore

$$r(\boldsymbol{\alpha}) = \sum_{n\in\omega} f^r_A(n)\langle\boldsymbol{\alpha}|A(n)\rangle = \sum_{n\in\omega} f^s_A(n)\langle\boldsymbol{\alpha}|A(n)\rangle = s(\boldsymbol{\alpha})$$

for all $\boldsymbol{\alpha} \in H$ and Φ is one-to-one.

Let $r, s \in \Gamma_A$ such that $r \sqsubseteq s$. Let B be an orthonormal sequence that witnesses $r \sqsubseteq s$. Also, let ρ and σ be one-to-one functions from ω into ω such that $A \circ \rho$ labels r and $A \circ \sigma$ labels s. In order to show that $\Phi(r) \leq \Phi(s)$ we must show that $f_A^r \leq f_A^s$. As a first step towards this result we show that when f_A^s is constant and positive on a number of coordinates, then f_A^r is also constant and positive on those same coordinates.

For every $\lambda \in \operatorname{spec}^+ r$ there is $M_\lambda \subseteq \omega$ such that $B[M_\lambda]$ is a basis for $E(r, \lambda)$. If $n \in M_\lambda$ then $f_{A\circ\rho}^r(n) = f_B^r(n) = \lambda$ so $(A \circ \rho)(n) \in E(r, \lambda)$. Since $A \circ \rho$ is an orthonormal sequence, this means that $(A \circ \rho)[M_\lambda]$ is a basis for $E(r, \lambda)$.

We next show that if $n \in M_\lambda$ and $f_{A\circ\sigma}^s(n) > 0$ then $(A \circ \sigma)(n) \in (A \circ \rho)[M_\lambda]$. Let $n \in M_\lambda$ and let $f_{A\circ\sigma}^s(n) = \mu > 0$. Because $f_{A\circ\rho}^s(n) = f_B^s(n)$ we know that $B(n) \in E(s, \mu) \cap E(r, \lambda)$. Therefore, by part 2 of Theorem 5.4, $E(s, \mu) \subseteq E(r, \lambda)$ and $(A \circ \sigma)(n) \in E(r, \lambda)$. But if $(A \circ \sigma)(n) \notin (A \circ \rho)[M_\lambda]$ then $(A \circ \sigma)(n) \perp (A \circ \rho)[M_\lambda]$ which is impossible. Thus $(A \circ \sigma)(n) \in (A \circ \rho)[M_\lambda]$. Let $N_\lambda = \{n \in M_\lambda : f_{A\circ\sigma}^s(n) > 0\}$. Then $(A \circ \sigma)[N_\lambda] \subseteq (A \circ \rho)[M_\lambda]$ or $\sigma[N_\lambda] \subseteq \rho[M_\lambda]$.

Now we can begin rearranging the sequences. For every $\lambda \in \operatorname{spec}^+ r$ let $\tau_{\lambda\sigma}$ be the function σ restricted to the set N_λ and let $\tau_{\lambda\rho}$ be a one-to-one function from $M_\lambda - N_\lambda$ onto $\rho[M_\lambda] - \sigma[N_\lambda]$. Set $\tau_\lambda = \tau_{\lambda\sigma} \cup \tau_{\lambda\rho}$ and $M = \bigcup\{M_\lambda : \lambda \in \operatorname{spec}^+ r\}$. If $n \in \omega - M$ then $f_B^r(n) = 0$ and, since $f_B^r \leq f_B^s$, $f_B^s(n) = 0$. Let τ_0 be a one-to-one function from $\omega - M$ onto $\omega - \bigcup\{\tau_\lambda[M_\lambda] \lambda \in \operatorname{spec}^+ r\}$. Set $\tau = \left(\cup_{\lambda\in\operatorname{spec}^+ r} \tau_\lambda\right) \cup \tau_0$.

If $\lambda, \mu \in \operatorname{spec}^+ r$ with $\lambda \neq \mu$ then $M_\lambda \cap M_\mu = \emptyset$ and $\rho[M_\lambda] \cap \rho[M_\mu] = \emptyset$. Also, $M_\lambda \cap M = \emptyset$. Therefore τ is a one-to-one function from ω into ω. We show that τ witnesses $f_A^r \leq f_A^s$. We can do this by showing that $f_A^r \circ \tau = f_A^r \circ \rho$ and $f_A^s \circ \tau = f_A^s \circ \sigma$.

Let $n \in \omega$. If $f_{A\circ\sigma}^s(n) > 0$ then $f_B^s(n) > 0$ so $f_{A\circ\rho}^r(n) = f_B^r(n) > 0$. Therefore $n \in M_\lambda$ for some $\lambda \in \operatorname{spec}^+ r$ and $n \in N_\lambda$. So $\tau(n) = \sigma(n)$.

Assume that $f_{A\circ\sigma}^s(n) = 0$. We show that $f_A^s(\tau(n)) = 0$. First consider the case when $f_A^r(\rho(n)) > 0$. There is $\lambda \in \operatorname{spec}^+ r$ such that $n \in M_\lambda$. Since $f_A^s(\sigma(n)) = 0$ we know that $n \in M_\lambda - N_\lambda$. Therefore $\tau(n) = \tau_{\lambda\rho}(n) \in \rho[M_\lambda] - \sigma[N_\lambda]$.

If $f_A^s(\tau(n)) > 0$ then there is $k \in \omega$ such that $\sigma(k) = \tau(n)$. Since $f_A^s(\sigma(k)) > 0$ there is $\mu \in \operatorname{spec}^+ s$ such that $A(\sigma(k)) \in E(s, \mu)$. Let $\kappa \in \operatorname{spec}^+ r$ such that $E(s, \mu) \subseteq E(r, \kappa)$. But $A(\sigma(k)) = A(\tau(n)) \in (A \circ \rho)[M_\lambda] \subseteq E(r, \lambda)$ so $\kappa = \lambda$. Therefore $k \in N_\lambda$ and $\tau(n) = \sigma(k) \in \sigma[N_\lambda]$, a contradiction. It follows that $f_A^s(\tau(n)) = 0$.

Now assume that $f_A^r(\rho(n)) = 0$. Then $n \in \omega - M$ and $\tau(n) = \tau_0(n) \in \omega - \bigcup\{\tau_\lambda[M_\lambda] : \lambda \in \operatorname{spec}^+ r\}$. If $f_A^s(\tau(n)) > 0$ then there is $k \in \omega$ such that $\sigma(k) = \tau(n)$. Since $f_A^s(\sigma(k)) > 0$ there is $\mu \in \operatorname{spec}^+ s$ such that $A(\sigma(k)) \in E(s, \mu)$. Let $\lambda \in \operatorname{spec}^+ r$ such that $E(s, \mu) \subseteq E(r, \lambda)$. Then $A(\sigma(k)) \in E(r, \lambda)$ and $A(\sigma(k)) \in (A \circ \rho)[M_\lambda]$. Thus $\tau(n) = \sigma(n) \in \rho[M_\lambda] = \tau_\lambda[M_\lambda]$, a contradiction. It follows that $f_A^s(\tau(n)) = 0$.

We next show that $f_A^r \circ \tau = f_A^r \circ \rho$. Let $n \in \omega$. If $f_A^r(\rho(n)) > 0$ then there is $\lambda \in \operatorname{spec}^+ r$ such that $n \in M_\lambda$. So $\tau(n) = \tau_\lambda(n) \in \rho[M_\lambda]$ and $f_A^r(\tau(n)) = \lambda = f_A^r(\rho(n))$. If $f_A^r(\rho(n)) = 0$ then $n \in \omega - M$ so $\tau(n) = \tau_0(n) \in \omega - \bigcup\{\tau_\lambda[M_\lambda] : \lambda \in \operatorname{spec}^+ r\} = \omega - \bigcup\{\rho[M_\lambda] : \lambda \in \operatorname{spec}^+ r\}$. Therefore $f_A^r(\tau(n)) = 0$.

Now we have $f_A^r \circ \tau = f_{A\circ\rho}^r = f_B^r$ and $f_A^s \circ \tau = f_{A\circ\sigma}^s = f_B^s$. As a consequence, τ witnesses $f_A^r \leq f_A^s$ so $\Phi(r) \leq \Phi(s)$.

Finally, we must show that if $r, s \in \Gamma_A$ and $\Phi(r) \leq \Phi(s)$ then $r \sqsubseteq s$. If $\Phi(r) \leq \Phi(s)$

then $f_A^r \leq f_A^s$. Let $\sigma : \omega \to \omega$ be a one-to-one function which witnesses this relation. In particular, $f_A^s \circ \sigma, f_A^s \circ \sigma \in \Lambda^\omega$ and if $n \in \omega$ and either $f_A^r(n) > 0$ or $f_A^s(n) > 0$ then there is $m \in \omega$ such that $\sigma(m) = n$. We show that $A \circ \sigma$ witnesses $r \sqsubseteq s$. First we must show that $A \circ \sigma$ labels r. We already know that the range of A consists of eigenvectors of r, so the range of $A \circ \sigma$ is also a set of eigenvectors of r. Let $\lambda \in \operatorname{spec}^+ r$. There is a one-to-one function $\rho : \omega \to \omega$ such that $A \circ \rho \in L(r)$. So if $M_\lambda = \{n \in \omega : A(\rho(n)) \in E(r, \lambda)\}$ then $(A \circ \rho)[M_\lambda]$ is a basis for $E(r, \lambda)$. But $f_A^r(\rho(n)) = \lambda$ for all $n \in M_\lambda$, so $\rho[M_\lambda] \subseteq \sigma[\omega]$. Therefore, if we let $N_\lambda = \sigma^{-1}[\rho[M_\lambda]]$ then $(A \circ \sigma)[N_\lambda]$ is a basis for $E(r, \lambda)$. Thus $A \circ \sigma$ labels r. The same argument shows that $A \circ \sigma$ also labels s, so $A \circ \sigma$ witnesses $r \sqsubseteq s$. $\square$

So Ω^ω contains many copies of Δ^ω. But how do these copies sit within Ω^ω? They obviously overlap. But could they be open subsets or closed subsets of Ω^ω? The answer to both possibilities is no because Γ_A is neither increasing nor decreasing.

Let A be an orthonormal sequence in H and let $r \in \Gamma_A$ such that $f_A^r(0) = f_A^r(1) = 1/2$ and $f_A^r(n) = 0$ for all $n > 1$. Let B be the orthonormal sequence given by $B(0) = (1/\sqrt{2})A(0) + (1/\sqrt{2})A(1)$, $B(1) = (1/\sqrt{2})A(0) - (1/\sqrt{2})A(1)$, and $B(n) = A(n)$ for $n > 1$. Let s be the linear operator defined by setting $s(B(0)) = (3/4)B(0)$, $s(B(1)) = (1/4)B(1)$, and $s(\boldsymbol{\alpha}) = \mathbf{0}$ for every $\boldsymbol{\alpha} \in H$ that is orthogonal to $B(0)$ and $B(1)$. Then B is a labeling of both r and s which witnesses $r \sqsubseteq s$ but $s \notin \Gamma_A$ because $A(0)$ and $A(1)$ are not eigenvectors of s. The same sort of approach can be used to show that Γ_A is not decreasing. As we will see, Γ_A is closed under the suprema of directed subsets. But it can be reached by directed sets outside of Γ_A.

Theorem 8.3. For every orthonormal sequence B in H there is a function $\Psi_B : \Omega^\omega \to \Omega_B^\omega$ with the following properties.

1 Ψ_B is strictly increasing.
2 Ψ_B is Scott continuous.
3 Ψ_B is the identity function on Ω_B^ω.

Lemma 10.1 below shows that the Scott topology on Ω_B^ω is the same as the topology it inherits as a subspace of Ω^ω, so Properties 2 and 3 imply that Ψ_B is a retraction.

Proof. Fix an orthonormal sequence B. Let $r \in \Omega^\omega$. There is an orthonormal sequence A in H such that $r \in \Omega_A^\omega$. Set $\Psi_B(r) = \psi_{AB}(r)$, where ψ_{AB} is the function from Ω_A^ω onto Ω_B^ω defined in Theorem 7.1. So $\Psi_B(r)(\boldsymbol{\alpha}) = f_A^r(n)B(n)$ if $\boldsymbol{\alpha} = B(n)$ and $\psi_{AB}(r)(\boldsymbol{\alpha}) = \mathbf{0}$ if $\boldsymbol{\alpha}$ is orthogonal to $B[\omega]$. It was shown that ψ_{AB} is strictly increasing. We now show that Ψ_B is a function. Let $A, C \in L(r)$. Then $f_A^r(n) = f_C^r(n)$ for all $n \in \omega$. If $\boldsymbol{\alpha} = B(n)$ for some n then $\psi_{AB}(r)(\boldsymbol{\alpha}) = f_A^r(n)B(n) = f_C^r(n)B(n) = \psi_{CB}(r)(\boldsymbol{\alpha})$, and if $\boldsymbol{\alpha}$ is orthogonal to $B[\omega]$ then $\psi_{AB}(\boldsymbol{\alpha}) = \mathbf{0} = \psi_{CB}(r)(\boldsymbol{\alpha})$. So $\psi_{AB}(r) = \psi_{CB}(r)$ and Ψ_B is a function.

That Ψ_B is strictly increasing follows from the fact that each ψ_{AB} is strictly increasing. It is also obvious that Ψ_B is the identity on Ω_B^ω. In order to show that Ψ_B is Scott continuous we need only show that it preserves the suprema of directed sets. Let D be a directed subset of Ω^ω with supremum s. Then $\Psi_B(s)$ is an upper bound of $\Psi_B[D]$. Let t be an upper bound of $\Psi_B[D]$ in Ω_B^ω. We construct $u \in \Omega^\omega$ such that u is an upper bound of D and $\Psi_B(u) = t$. Then $s \sqsubseteq u$ and therefore $\Psi_B(s) \sqsubseteq \Psi_B(u) = t$. It follows that $\Psi_B(s) = \sup \Psi_B[D]$ and that Ψ_B is Scott continuous.

For every $r \in D$ let $A_r \in L(r)$. To simplify our notation set $\lambda_r(n) = f^r_{A_r}(n)$ for all $n \in \omega$. Then $\langle \lambda_r(n) : n \in \omega \rangle$ is a decreasing sequence of eigenvalues of r. Every positive eigenvalue appears in the sequence and the number of times it appears equals its multiplicity. If $\lambda_r(n) > 0$ for all $r \in D$ and all $n \in \omega$ then set $m = \omega$. Otherwise let m be the least natural number at which λ_r takes the value 0. Note that $m > 0$ in either case.

For every $n < m$ the set $\{E(r, \lambda_r(n)) : r \in D\}$ is a set of nontrivial finite dimensional subspaces of H which is directed under $\supseteq$. Therefore $H_n = \bigcap\{E(r, \lambda_r(n)) : r \in D\}$ is a nontrivial finite dimensional subspace of H. In fact, there is $r_n \in D$ such that $H_n = E(r_n, \lambda_{r_n}(n))$.

Let $i < j < m$. If $\lambda_{r_j}(i) = \lambda_{r_i}(j)$ then $H_j = E(r_j, \lambda_{r_j}(i))$, so $H_i \subseteq E(r_j, \lambda_{r_j}(i)) = H_j$. Now $E(r_i, \lambda_{r_i}(i)) = H_i \subseteq H_j \subseteq E(r_i, \lambda_{r_i}(j))$ so $\lambda_{r_j}(i) = \lambda_{r_i}(j)$. But then $H_j \subseteq E(r_i, \lambda_{r_i}(j)) = E(r_i, \lambda_{r_i}(i)) = H_i$. Therefore $H_i = H_j$. If $\lambda_{r_j}(i) > \lambda_{r_j}(j)$ then we have $E(r_j, \lambda_{r_j}(i)) \perp E(r_j, \lambda_{r_j}(j))$ so $H_i \perp H_j$.

It may be that many of the subspaces we have defined are duplicates of other ones. We now pick out only those that we really need. Set $n_0 = 0$. Let $j \in \omega$ and assume that $n_j < m$. If $H_i = H_{n_j}$ for $n_j \leq i < m$ then set $k = j + 1$ and $n_{j+1} = m$ and stop. If not, then let $n_{j+1} = \min\{i : n_i < i \text{ and } H_i \neq H_{n_j}\}$. If the sequence $\langle n_j \rangle$ is unbounded then set $k = \omega$. If $j < k$ and $n_j < i < n_{j+1}$ then $H_i = H_{n_j}$. We also know from the previous paragraph that if $i < j < k$ then $r_{n_i}(n_i) \neq r_{n_j}(n_j)$ and $H_{n_i} \perp H_{n_j}$.

We next show that the spacing of the n_j's is determined by the dimensions of the H_{n_j}'s. Fix an n_j. If $i < n_j$ then $H_i \neq H_{n_j}$ so $\lambda_{r_{n_j}}(i) \neq \lambda_{r_{n_j}}(n_j)$. If $n_j \leq i < n_{j+1}$ then $H_i = H_{n_j}$ so $\lambda_{r_{n_j}}(i) = \lambda_{r_{n_j}}(n_j)$. If $n_{j+1} \leq i$ then $H_i \neq H_{n_j}$. Therefore $A_{r_{n_j}}(i) \in H_{n_j}$ if and only if $n_j \leq i < n_{j+1}$. So $\dim H_{n_j} = n_{j+1} - n_j$.

For every $j < k$ and every i with $n_j \leq i < n_{j+1}$ set $F(i) = A_{r_{n_j}}(i)$. Then $\{F(i) : n_j \leq i < n_{j+1}\}$ is an orthonormal basis for H_{n_j}. Define a linear operator u on H by setting $u(F(i)) = f^t_B(i)F(i)$ for all $i < m$ and $u(\boldsymbol{\alpha}) = \mathbf{0}$ when $\boldsymbol{\alpha}$ is orthogonal to $\{F(i) : i < m\}$. Then u is a self-adjoint positive linear operator. If $m \leq i$ then there is $r \in D$ such that $f^r_{A_r}(i) = 0$. But $f^{\Psi_B(r)}_B(i) = f^r_{A_r}(i) = 0$ and $\Psi_B(r) \sqsubseteq t$ so $f^t_B(i) = 0$. Therefore, if $f^t_B(i) > 0$ then $i < m$. Expand $\{F(i) : i < m\}$ to an orthonormal basis G for H. Then $\sum_{\boldsymbol{\alpha} \in G} \langle r(\boldsymbol{\alpha}) | \boldsymbol{\alpha} \rangle = \sum_{i=0}^{m-1} f^t_B(i) = 1$. Therefore $u \in \Omega^\omega$. Let $r \in D$. If $m = \omega$ then $C = \langle F(i) : i < m \rangle$ is an orthonormal sequence that labels both u and r. Also, $f^u_C = f^t_B$ and $f^r_C = f^r_{A_r}$, so $f^r_C \leq f^u_C$ and $r \sqsubseteq u$. Assume that $m < \omega$. For all $i < m$ set $C(i) = F(i)$. If $\lambda_r(m-1) > \lambda_r(m)$ then $C(i) = A_r(i)$ for all $i \geq m$. Assume that $\lambda_r(m-1) = \lambda_r(m)$ and let $m_0 = \max\{n \in \omega : \lambda_r(n) = \lambda_r(m)\}$. We can choose an orthonormal set $\{C(i) : m \leq i \leq m_0\}$ so that $(\bigcup\{F(i) : \lambda_r(i) = \lambda_r(m)\}) \cup \{C(i) : m \leq i \leq m_0\}$ is a basis for $E(r, \lambda_r(m))$. Then set $C(i) = A_r(i)$ for all $i > m_0$. The resulting orthonormal sequence labels both r and u, and $f^r_C = f^{\Psi_B(r)}_B \leq f^t_B = f^u_C$. Therefore $r \sqsubseteq u$.

If C is an orthonormal sequence that labels u then $f^u_C(i) = f^t_B(i)$ for all $i < m$ and $f^u_C(i) = 0 = f^t_B(i)$ for all $i \leq m$. Thus $\Psi_B(u) = t$. $\square$

9. Domain Properties of Ω^ω

We are now ready to determine which of the domain-like properties Ω^ω satisfies. To do this we need the following lemma.

Lemma 9.1. Let δ be an ordinal number. If $\rho : \delta \to \Omega^\omega$ is increasing then there is an orthonormal sequence A which labels $\rho(\alpha)$ for every $\alpha \in \delta$.

Proof. For every $\alpha \in \beta$ let A_α be an orthonormal sequence which labels $\rho(\alpha)$. For every $\alpha \in \delta$ and every $k \in \omega$ let $\lambda_\alpha(k) = f^{\rho(\alpha)}_{A_\alpha}(k)$.

Let $\beta, \gamma \in \delta$ with $\beta < \gamma$ and let $B \in L(\rho(\beta)) \cap L(\rho(\gamma))$. Since $\rho(\beta) \sqsubseteq \rho(\gamma)$ we know from Theorem 5.4 that if $f^{\rho(\gamma)}_B(k) > 0$ then $f^{\rho(\beta)}_B(k) > 0$ and $E(\rho(\gamma), f^{\rho(\gamma)}_B(k)) \subseteq E(\rho(\beta), f^{\rho(\beta)}_B(k))$, or $E(\rho(\gamma), \lambda_\gamma(k)) \subseteq E(\rho(\beta), \lambda_\beta(k))$. Therefore if $\lambda_\beta(k) > 0$ for some $\beta \in \delta$ then the intersection of all $E(\rho(\beta), \lambda_\beta(k))$ such that $\beta \in \delta$ and $\lambda_\beta(k) > 0$ is a subspace of H with a positive finite dimension. In fact, there is $\alpha \in \delta$ such that this subspace equals $E(\rho(\alpha), \lambda_\alpha(k))$.

We define the orthonormal sequence A recursively. Since $\lambda_\alpha(0) > 0$ for all $\alpha \in \delta$ we know that $\bigcap\{E(\rho(\alpha), \lambda_\alpha(0)) : \alpha \in \delta\} \neq \emptyset$. Let $A(0)$ be a unit vector from $\bigcap\{E(\rho(\alpha), \lambda_\alpha(0)) : \alpha \in \delta\}$. Let $k \in \omega$ and assume that $A(j)$ is a unit vector of H for all $j \leq k$. Furthermore assume that if $j \leq k$ and $\alpha \in \delta$ such that $\lambda_\alpha(j) > 0$ then $A(j) \in E(\rho(\alpha), \lambda_\alpha(j))$. If $\lambda_\alpha(k+1) = 0$ for all $\alpha \in \delta$ then let $A(k+1)$ be a unit vector of H which is orthogonal to $A(j)$ for all $j < k$.

Assume that there is $\alpha \in \delta$ such that $\lambda_\alpha(k+1) > 0$. Choose $\beta \in \delta$ such that $E(\rho(\beta), \lambda_\beta(k+1)) = \bigcap\{E(\rho(\alpha), \lambda_\alpha(k+1))$ and $\lambda_\alpha(k+1) > 0\}$. Set $M = \{j \leq k : A(j) \in E(\rho(\beta), \lambda_{\beta(k+1)})\}$. If $j \leq k$ and $j \notin M$ then $A(j) \in E(\rho(\beta), \lambda)$ for some $\lambda \in \operatorname{spec}^+ \rho(\beta)$ other than $\lambda_\beta(k+1)$. So every element of $E(\rho(\beta), \lambda_\beta(k+1))$ is orthogonal to $A(j)$. But $|M| < \dim E(\rho(\beta), \lambda_\beta(k+1))$ so we can choose a unit vector $A(k+1)$ from $E(\rho(\beta), \lambda_\beta(k+1))$ which is orthogonal to $\{A(j) : j \leq k\}$.

We now show that the orthonormal sequence A that we have defined labels each $\rho(\alpha)$. Let $\alpha \in \delta$ and let $k \in \omega$. If $\lambda_\alpha(k) > 0$ then $A(k) \in E(\rho(\alpha), \lambda_\alpha(k))$ so $A(k)$ is an eigenvector of $\rho(\alpha)$ corresponding to $\lambda_\alpha(k)$. Assume that $\lambda_\alpha(k) = 0$. If $\lambda \in \operatorname{spec}^+ \rho(\alpha)$ then there is $j < k$ such that $\lambda = \lambda_\alpha(j)$. But $A(k)$ is orthogonal to $\{A(j) : j < k\}$ so $A(k)$ must be an eigenvector of $\rho(\alpha)$ corresponding to 0. It follows that $\{A(k) : \lambda_\alpha(k) = \lambda\}$ is a basis for $E(\rho(\alpha), \lambda)$ for all $\lambda \in \operatorname{spec}^+ \rho(\alpha)$ and that $f^{\rho(\alpha)}_A = \lambda_\alpha \in \Lambda^\omega$. $\square$

Theorem 9.1. If $\rho : \omega \to \Omega^\omega$ is increasing then $\operatorname{ran} \rho$ has a supremum.

Proof. By Lemma 9.1 there is an orthogonal sequence A which labels every $\rho(n)$. It follows from Theorem 19 of (Mashburn 2007b) that we can define an element f of Δ^ω by setting $f(k) = \lim_{n\to\infty} f^{\rho(n)}_A(k)$ for every $k \in \omega$. Furthermore, $f^{\rho(n)}_A \leq f$ for all $n \in \omega$. Define $r \in \Omega^\omega$ by setting $r(A(k)) = f(k)A(k)$ for every $k \in \omega$ and $r(\boldsymbol{\alpha}) = \mathbf{0}$ for every $\boldsymbol{\alpha} \in H$ which is orthogonal to $\operatorname{ran} A$. Then $r \in \Omega^\omega$ and A labels r. We show that $r = \sup \operatorname{ran} \rho$, but first we establish a property which is useful in this endeavor.

Let $k \in \omega$ such that $f(k) > 0$. Let $M \subseteq \omega$ such that $f[M]$ is a basis for $E(r, f(k))$. Now $f^{\rho(n)}_A(k) > 0$ and there is $M_n \subseteq \omega$ such that $f^{\rho(n)}_A[M_n]$ is a basis for $E(\rho(n), f^{\rho(n)}_A(k))$. If $j \in M$ then $f(j) = f(k)$ and, by Theorem 5.4, $f^{\rho(n)}_A(j) = f^{\rho(n)}_A(k)$. So $A(j) \in$

$E(\rho(n), f_A^{\rho(n)}(k))$ and $j \in M_n$. Therefore $M \subseteq M_n$ for all $n \in \omega$. Assume that $\bigcap_{n\in\omega} M_n \nsubseteq M$ and let $j \in \bigcap_{n\in\omega} M_n - M$. Then $f_A^{\rho(n)}(j) = f_A^{\rho(n)}(k)$ for all $n \in \omega$.

$$r(A(j)) = \left(\lim_{n\to\infty} f_A^{\rho(n)}(j)\right) A(j) = \left(\lim_{n\to\infty} f_A^{\rho(n)}(k)\right) A(j) = f(k)A(j)$$

Thus $A(j) \in E(r, f(k))$ and $j \in M$, a contradiction. Therefore $M = \bigcap_{n\in\omega} M_n$. This means that $A[M]$ is a basis for $\bigcap_{n\in\omega} E(\rho(n), f_A^{\rho(n)})$, so

$$E(r, f(k)) = \bigcap_{n\in\omega} E(\rho(n), f_A^{\rho(n)}(k))$$

Now we show that $r = \sup \operatorname{ran} \rho$. We already know that r is an upper bound for $\operatorname{ran} \rho$. Let s be an upper bound of $\operatorname{ran} \rho$ in Ω^ω. By Lemma 9.1 there is an orthonormal sequence B which labels s and every $\rho(n)$. We show that B also labels r.

Let $k \in \omega$ such that $f_A^r(k) > 0$. Now $B(k) \in E(\rho(n), f_B^{\rho(n)}(k))$ and this set equals $E(\rho(n), f_A^{\rho(n)}(k))$ for every $n \in \omega$, so $B(k) \in \bigcap_{n\in\omega} E(\rho(n), f_A^{\rho(n)}(k)) = E(r, f(k))$. Therefore $B(k)$ is an eigenvector of r corresponding to $f(k)$ and $f_B^r(k) = f(k)$. So if $\lambda \in \operatorname{spec}^+ r$ and $M \subseteq \omega$ such that $A[M]$ is a basis for $E(r, \lambda)$ then $B[M]$ is a basis for $E(r, \lambda)$ as well.

Let $k \in \omega$ such that $f(k) = 0$. Now $B(k)$ is orthogonal to $B(j)$ for all $j < k$. We know that if $f(j) > 0$ then $j < k$. Furthermore, if $\lambda \in \operatorname{spec}^+ r$ then a basis for $E(r, \lambda)$ can be found among $\{B(j) : j < k\}$. Thus $B(k)$ must be an eigenvector of r corresponding to 0, and $f_B^r(k) = f(k) = 0$.

We have established that $B(k)$ is an eigenvector of r for all $k \in \omega$ and that if $\lambda \in \operatorname{spec}^+ r$ then there is $M \subseteq \omega$ such that $B[M]$ is a basis for $E(r, \lambda)$. Since $f_B^r = f$, we know that B labels r. Also, $f_B^r(k) = f(k) = \lim_{n\to\infty} f_A^{\rho(n)}(k) = \lim_{n\to\infty} f_B^{\rho(n)}(k)$, so $f_B^r = \sup\{f_B^{\rho(n)} : n \in \omega\}$. But f_B^s is an upper bound of $\{f_B^{\rho(n)} : n \in \omega\}$ so $f_B^r \leq f_B^s$. Therefore $r \sqsubseteq s$ and $r = \sup \operatorname{ran} \rho$. □

It is shown in (Mashburn 2007b) that if $f < g$ in Δ^ω then $\max \operatorname{ran} f < \max \operatorname{ran} g$. It follows that the function $\xi : \Omega^\omega \to [0, \infty)^*$ given by $\xi(r) = 1 - \max \operatorname{spec} r$ is a strictly increasing function which preserves the suprema of increasing sequences in Ω^ω. The next theorem is a consequence of Theorem 2.2.1 of (Martin 2000).

Theorem 9.2. Ω^ω is directed complete and every nonempty directed subset D of Ω^ω contains a sequence whose supremum is the supremum of D.

In order to study exactness in Ω^ω we need to know some things about paths in Ω^ω.

Definition 9.1. Let $r, s \in \Omega^\omega$ with $r \sqsubseteq s$. The *path* from r to s is the function $\pi_{rs} : [0, 1] \to \Omega^\omega$ given by $\pi_{rs}(i) = (1 - i)r + is$.

Lemma 9.2. Let $r, s \in \Omega^\omega$. If $r \sqsubseteq s$ then for every $i \in [0, 1]$, $\pi_{rs}(i) \in \Omega^\omega$ and $r \sqsubseteq \pi_{rs}(i) \sqsubseteq s$.

Proof. Let $i \in [0, 1]$ and set $t = \pi_{rs}(i)$. It is straightforward to show that t is self-adjoint operator with trace 1. Let A be an orthonormal sequence that witnesses $r \sqsubseteq s$. Every vector that appears in A is an eigenvector of both r and s, and so is also an eigenvector of t. Furthermore, if A is not already a basis for H it can be extended to one,

B, by the addition of elements of H that are all orthogonal to the elements of A. If λ is an eigenvalue of t with corresponding eigenvector $\boldsymbol{\alpha}$ then $t(\boldsymbol{\alpha}) = \lambda\boldsymbol{\alpha}$ can be written as a linear combination of the operators r and s applied to the elements of B. But r and s are both $\mathbf{0}$ at all vectors orthogonal to the elements of A, so this linear combination reduces to one of r and s applied to the elements of A. If $\lambda \neq 0$ this means that $\boldsymbol{\alpha}$ is in the span of A. But the elements of A are already known to be eigenvectors of t so λ must equal the eigenvalue of some of these eigenvectors. Therefore t is a positive operator, and hence an element of Ω^ω. Also, every positive eigenvalue of t has a basis in A. Therefore A labels t. That $f^r_A \leq f^t_A \leq f^s_A$ follows from Theorem 4.2. □

These paths provide us with sequences that we can use to show that if $r, s \in \Omega^\omega$ then r is not weakly way below s. Since the weakly way below relation is therefore empty in Ω^ω it follows that Ω^ω cannot be exact.

Lemma 9.3. If $r, s \in \Omega^\omega$ with $r \sqsubseteq s$ then π_{rs} is Scott continuous.

Proof. Since $r \sqsubseteq s$ there is an orthonormal sequence A which labels both r and s. Then $r, s \in \Omega^\omega_A$ and $\phi_A : \Omega^\omega \to \Lambda^\omega$ is an order isomorphism and therefore Scott continuous. Now $\phi_A \circ \pi_{rs} = \pi_{\phi_A(r)\phi_A(s)}$, where $\pi_{\phi_A(r)\phi_A(s)}$ is the path in Λ^ω from $\phi_A(r)$ to $\phi_A(s)$. This function is Scott continuous, so π_{rs} must be Scott continuous. □

In particular, π_{rs} is increasing so $\operatorname{ran}\pi_{rs}$ is a chain in Ω^ω.

Theorem 9.3. If $r, s \in \Omega^\omega$ then r is not weakly way below s.

Proof. We may assume that $r \sqsubseteq s$, since otherwise it is automatic that r is not weakly way below s. Let A be an orthonormal sequence that labels both r and s. Then $\phi_A(r), \phi_A(s) \in \Lambda^\omega$ with $\phi_A(r) \leq \phi_A(s)$. It is shown in (Mashburn 2007b) that there is $f \in \Delta^\omega$ such that $f < \phi_A(s)$ but for no $i \in [0, 1]$. is $\phi_A(r) \leq \pi_{f\phi_A(s)}(i)$. Let $t \in \Omega^\omega_A$ such that $\phi_A(t) = f$. Then $t \sqsubseteq s$ so that π_{ts} is a chain whose supremum is s, but if $i \in [0, 1)$ then $r \not\sqsubseteq \pi_{ts}(i)$. Thus r is not weakly way below s. □

Theorem 9.4. For every orthonormal sequence A, Γ_A is closed under the suprema of directed subsets.

Proof. Let $\langle r_n \rangle$ be an increasing sequence in Γ_A and let $f_n = \Phi(r_n)$ for all $n \in \omega$. Then $\langle f_n \rangle$ is an increasing sequence in Δ^ω and has a supremum f given by $f(m) = \lim_{n\to\infty} f_n(m)$ for all $m \in \omega$. Furthermore, there is a one-to-one function $\sigma : \omega \to \omega$ which witnesses the fact that $\langle f_n \rangle$ is increasing and that $f_n \leq f$ for all $n \in \omega$. Let r be the density operator defined by setting $r(A(n)) = f(n)A(n)$ for all $n \in \omega$ and by setting $r(\boldsymbol{\alpha}) = \mathbf{0}$ for every $\boldsymbol{\alpha} \in H$ which is orthogonal to A. Each $A(n)$ is an eigenvector of r. Now $\operatorname{spec}^+ r = \{f(n) : f(n) > 0\}$ and if $\lambda \in \operatorname{spec}^+ r$ then $\{A(n) : f(n) = \lambda\}$ is a basis for $E(r, \lambda)$. But $M = f^{-1}(\lambda) \subseteq \operatorname{ran}\sigma$ so $A[M]$ is a basis for $E(r, \lambda)$. Also, $f^r_{A\circ\sigma} = f \circ \sigma \in \Lambda^\omega$.

Therefore $A \circ \sigma$ labels r. For the same reasons $A \circ \sigma$ labels r_n for every $n \in \omega$.

$$\begin{aligned} r(A(\sigma(m))) &= f(\sigma(m))A(\sigma(m)) \\ &= \lim_{n\to\infty} f_n(\sigma(m))A(\sigma(m)) \\ &= \lim_{n\to\infty} r_n(A(\sigma(m))) \end{aligned}$$

Thus $r = \sup r_n$. $\square$

This also shows that Φ^{-1} is an embedding of Δ^ω into Ω^ω.

Theorem 9.5. If A is an orthonormal sequence in H then there is $r \in \Gamma_A$ and a directed subset D of Ω^ω such that $D \cap \Gamma_A = \emptyset$ and $r = \sup D$.

Proof. Let r be the density operator defined by setting $r(A(0)) = A(0)$ and $r(\boldsymbol{\alpha}) = \mathbf{0}$ for every $\boldsymbol{\alpha} \in H$ which is orthogonal to $A(0)$. Let $\boldsymbol{\beta}$ be a unit vector in H which is orthogonal to $A(0)$ but is not a multiple of any $A(n)$ for $n > 0$. Let B be an orthonormal sequence such that $B(0) = A(0)$ and $B(1) = \boldsymbol{\beta}$. For every $n \in \omega$ let r_n be the linear operator defined by setting $r_n(B(0)) = (1 - 2^{-n-1})B(0)$ and $r_n(B(1)) = 2^{-n-1}B(1)$ and by setting $r_n(\boldsymbol{\alpha}) = \mathbf{0}$ for every $\boldsymbol{\alpha} \in H$ which is orthogonal to both $B(0)$ and $B(1)$. Then B witnesses the fact that $\langle r_n \rangle$ is an increasing sequence in Ω^ω and $r = \sup r_n$. Thus $D = \{r_n : n \in \omega\}$ is a directed subset of Ω^ω and $r = \sup D$. But $r \in \Gamma_A$ and for every $n \in \omega$, $r_n \notin \Gamma_A$ because there is no $M \subseteq \omega$ such that $A[M]$ is a basis for $E(r_n, 2^{-n-1})$. $\square$

10. Ω^ω, Entropy, and Measurements

One of the goals of defining an order on the quantum states is to have a structure which reflects the change in entropy or uncertainty from one state to another. A relation among sequences of real numbers which plays an important role in the study of entropy is that of majorization, defined below.

Definition 10.1. Let p and q be sequences of equal length of nonnegative real numbers such that the sum of the terms of p and q are each 1. Let $\hat{p}$ be a rearrangement of the terms of p into decreasing order and let $\hat{q}$ be a rearrangement of the terms of q into decreasing order. Let n denote the length of p and q. Here n could be ω. Then $p \prec q$ if and only if $\sum_{k=0}^{m} \hat{p}(k) \leq \sum_{k=0}^{m} \hat{q}(k)$ for $m = 0, \ldots, n$.

See (Uffink 1990), Section 1.3.3, (Marshall and Olkin 1979), Chapter 1, or (Hardy, Littlewood, and Pólya 1952), Sections 2.18–20, for some of the basics of majorization. See (Nielsen 1999) for an example of using majorization in the study of quantum physics. Note that majorization is a preorder and not an order. In his thesis (Uffink 1990), Uffink created a list of axioms which a reasonable measurement of the degree of certainty or predictability would satisfy. Among these axioms is the property of being Schur convex.

Definition 10.2. A function f defined on ${}^n\mathbb{R}$ (or ${}^\omega\mathbb{R}$) is said to be *Schur convex* if and only if $p \prec q$ implies $f(p) \leq f(q)$ for all p and q.

One can use the reciprocal of a measurement of the degree of certainty to obtain a measurement of uncertainty. See Section 1.5.2 of (Uffink 1990) where $M(f) = \exp \sum f \ln f$ is given as a measurement of the degree of certainty. Then Shannon entropy is $S(f) = \ln(1/M(f))$. If a measurement of the degree of certainty must be increasing, then the entropy function obtained from it must be decreasing. We will follow the convention of (Coecke and Martin 2002) and give the reverse order to the nonnegative real numbers, so that the entropy functions will be increasing, rather than decreasing.

Theorem 10.1. Let $f, g \in \Delta^n$ for some $n \leq \omega$. If $f \leq g$ then $f \prec g$.

Proof. First assume that $f, g \in \Delta^2$. There is a permutation σ of 2 such that $\hat{f} = f \circ \sigma$ and $\hat{g} = g \circ \sigma$ are decreasing and $\hat{f} \leq \hat{g}$. Then $\hat{f}(0) \leq \hat{g}(0)$ so $f \prec g$. Let $n \in \omega$ with $n \geq 2$ and assume that if $f, g \in \Delta^n$ with $f \leq g$ then $f \prec g$. Let $f, g \in \Delta^{n+1}$ with $f \leq g$. There is a permutation σ of $n+1$ such that $\hat{f} = f \circ \sigma$ and $\hat{g} = g \circ \sigma$ are decreasing and $\hat{f} \leq \hat{g}$. We know that $\sum_{k=0}^{n-1} \hat{f}(k) > 0$ and $\sum_{k=0}^{n-1} \hat{g}(k) > 0$ and that $\hat{f}(n) \geq \hat{g}(n)$. Therefore $f' = p_n(\hat{f})$ and $g' = p_n(\hat{g})$ are both defined and $f' \leq g'$. But f' and g' are both sequences of length n, that is, they are elements of Δ^n, so it follows by the inductive hypothesis that $f' \prec g'$. Let $m < n$. Then

$$\frac{\sum_{j=0}^{m} \hat{f}(j)}{\sum_{k=0}^{n-1} \hat{f}(k)} = \sum_{j=0}^{m} f'(j) \leq \sum_{j=0}^{m} g'(j) = \frac{\sum_{j=0}^{m} \hat{g}(j)}{\sum_{k=0}^{n-1} \hat{g}(k)}$$

But $\sum_{k=0}^{n-1} \hat{f}(k) = 1 - \hat{f}(n) \leq 1 - \hat{g}(n) = \sum_{k=0}^{n-1} \hat{g}(k)$ and therefore $\sum_{j=0}^{m} \hat{f}(j) \leq \sum_{j=0}^{m} \hat{g}(j)$. It follows that $f \prec g$.

We have now established the theorem for elements of Δ^n for all $n \in \omega$. We turn next to Δ^ω. Let $f, g \in \Delta^\omega$ such that $f \leq g$. There is a one-to-one function $\sigma : \omega \to \omega$ such that $\hat{f} = f \circ \sigma$ and $\hat{g} = g \circ \sigma$ are decreasing, $f^{-1}(0, \infty) \cap g^{-1}(0, \infty) \subseteq \operatorname{ran} \sigma$, and $\hat{f} \leq \hat{g}$. By Lemma 11 of (Mashburn 2007b) there is $n \in \omega$ such that $\hat{f}(m) < \hat{g}(m)$ for $m < n$ and $\hat{g}(m) \leq \hat{f}(m)$ for $m \geq n$. So if $m < n$ then $\sum_{j=0}^{m} \hat{f}(j) < \sum_{j=0}^{m} \hat{g}(j)$. Let $m \geq n$. Now $\sum_{j=0}^{m+1} \hat{f}(j)$ and $\sum_{j=0}^{m+1} \hat{g}(j)$ are both positive so $f' = p_A(\hat{f})$ and $g' = p_A(\hat{g})$ are both defined when $A = \{0, \ldots, m+1\}$. See Definition 4.3 and Theorem 4.5. Furthermore, $f' \leq g'$ and we can consider f' and g' as elements of Δ^{m+2}, so $f' \prec g'$. It then follows from an argument similar to that used above that $f \prec g$. $\square$

Therefore an entropy function arising from Uffink's axioms will be increasing as a function from Δ^n into $[0, \infty)^*$.

A problem of entropy measurements on infinite dimensional states is that they are not defined over all possibilities. This was noticed by Uffink in Section 1.5.5 of his thesis. Furthermore, it seems that under reasonable definitions of convergence, there are states with finite entropy which are limits of states with infinite entropy. It was shown in Lemma 45 of (Mashburn 2007b) that the maximal elements of Δ^ω, which have 0 Shannon entropy, are limits of elements with infinite entropy. The following theorem shows Ω^ω has the same property. The results follow immediately from similar properties of Shannon entropy under the Bayesian order. For the Shannon entropy equivalents see Section 6 of (Mashburn 2007b).

Theorem 10.2. Let $r, s \in \Omega^\omega$.

1 If $r \sqsubseteq s$ and r has finite von Neumann entropy then s has finite von Neumann entropy.
2 If r and s have finite von Neumann entropy and $r \sqsubset s$ then $V(r) > V(s)$.
3 If e is a maximal element of Ω^ω then $V(e) = 0$.
4 If X is the set of elements of Ω^ω which have finite von Neumann entropy then $V : X \to [0, \infty)^*$ is Scott continuous.
5 If $e \in \max \Omega^\omega$ then there is an increasing sequence $\langle r_n \rangle$ of elements of Ω^ω having infinite von Neumann entropy such that $\lim_{n\to\infty} r_n = e$.

We have seen that Φ_A^{-1} is an embedding of Δ^ω into Ω^ω for every orthonormal sequence A, so if $\varepsilon : \Omega^\omega \to [0, \infty)^*$ is an increasing Scott continuous function which is 0 exactly on the maximal elements (that is, ε looks like an entropy function) then the function $\varepsilon \circ \Phi_A^{-1}$ has the same properties. In particular, if S denotes Shannon entropy then $S = V \circ \Phi_A^{-1}$, although we must extend the range of V to $[0, \infty]^*$ and Scott continuity only holds where V is finite. So it seems that entropy functions on Ω^ω induce entropy functions on Δ^ω.

We next consider Martin's measurements. Definitions 3.1, 3.2, and 3.3 are given for domains, but we can apply them to Ω^ω since it has the necessary ingredients to discuss measurements: maximal elements and the Scott topology.

Definition 10.3. A function μ defined on Ω^ω is symmetric if and only if $\mu(r) = \mu(s)$ for all $r, s \in \Omega^\omega$ satisfying the following two conditions.

1 $\operatorname{spec} r = \operatorname{spec} s$
2 There is an orthonormal sequence A in H such that $r, s \in \Gamma_A$.

It was shown in (Mashburn 2007b) that functions which are symmetric on Δ^ω and whose kernel is the set of maximal elements of Δ^ω cannot be a measurements of Δ^ω. This eliminates the functions which are intuitive candidates for measurements, such as the entropy functions. The situation for Ω^ω is analogous.

Theorem 10.3. If $\mu : \Omega^\omega \to [0, \infty)^*$ is symmetric and $\ker \mu = \max \Omega^\omega$ then μ is not a measurement of Ω^ω.

Proof. We may assume that μ is Scott continuous, since otherwise it is automatically not a measurement. Fix an orthonormal sequence A in H. For every $k \in \omega$ let r_k be the element of Ω^ω defined as follows.

1 $r_k(A(0)) = (1 - 2^{-k-1})A(0)$
2 $r_k(A(1)) = 2^{-k-1}A(1)$
3 $r_k(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha} \in H$ that are orthogonal to $A(0)$ and $A(1)$.

Then $\langle r_k : k \in \omega \rangle$ is an increasing sequence in Ω^ω and $\sup_{k\in\omega} r_k = e$, where $e(A(0)) = A(0)$ and $e(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha}$ orthogonal to $A(0)$. Clearly $e \in \max \Omega^\omega$.

Define a new sequence $\langle s_k \rangle$ as follows.

1 $s_k(A(0)) = (1 - 2^{-k-1})A(0)$
2 $s_k(A(k+1)) = 2^{-k-1}A(k+1)$
3 $s_k(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha} \in H$ that are orthogonal to $A(0)$ and $A(k+1)$.

Note that $\dim E(s_k, 2^{-k-1}) = 1$ for all $k \in \omega$ and that $r_k, s_k \in \Gamma_A$ for all $k \in \omega$. Let $K = \bigcup_{k\in\omega} \downarrow s_k$. Then K is clearly decreasing. We show that K is Scott closed. As a first step in this proof we show that for every $r \in \Omega^\omega$ the set $M = \{k \in \omega : r \sqsubseteq s_k\}$ is finite. For the sake of contradiction assume that there is $r \in \Omega^\omega$ for which M is infinite. We can write $M = \{k_j : j \in \omega\}$ using a one-to-one indexing. Let $j \in \omega$ and let B_j be an orthonormal sequence that labels both r and s_{k_j}. Then $B_j(1) \in E(s_{k_j}, 2^{-k_j-1})$ so $B_j(1)$ is a nonzero multiple of $A(k_j + 1)$. Let $\lambda = f^r_{B_j}(1)$. Then $A(k_i + 1) \in E(r, \lambda)$ for all $i \in \omega$. It follows that $E(r, \lambda)$ is infinite dimensional. The only way that this can happen is for λ to be 0. But this is impossible because then $f^r_{B_j} \leq f^{s_{k_j}}_{B_j}$, $f^r_{B_j}(1) = 0$, and $f^{s_{k_j}}_{B_j} = 2^{-k-1} > 0$.

To finish the proof that K is Scott closed we need only show that it is closed under the suprema of increasing sequences. Let $\langle t_n : n \in \omega\rangle$ be an increasing sequence in K. Then $\{k \in \omega : \exists n \in \omega(t_n \sqsubseteq s_k)\}$ is finite, so there is $m \in \omega$ such that $\{t_n : n \in \omega\} \subseteq\downarrow s_m$. Therefore $\sup_{n\in\omega} t_n \in\downarrow s_m \subseteq K$.

The set $U = \Omega^\omega - K$ is a Scott neighborhood of e. Let W be a Scott neighborhood of 0 in $[0,\infty)^*$. Since μ is Scott continuous there is $m \in \omega$ such that $r_m \in \mu^{-1}[W]$ for all $n \geq m$. But μ is symmetric so $s_n \in \mu^{-1}[W]$ for all $n \geq m$. Therefore $s_n \in \mu^{-1}[W] \cap \downarrow e$ for all $n \geq m$ and so $\mu^{-1}[W] \cap \downarrow e \nsubseteq U$. □

The reasonable entropy functions on Ω^ω are symmetric (that is one of Uffink's postulates) and equal to 0 on the maximal (pure) states, so the reasonable entropy functions cannot be measurements in the sense of Martin. But it would still be nice to know the relationship between measurements of Ω^ω and those of Δ^ω. We have seen that entropy functions on Ω^ω can induce entropy functions on Δ^ω. The situation is more complicated for measurements. We begin by considering Λ^ω, the decreasing classical states, rather than Δ^ω and need the following lemma.

Lemma 10.1. For every orthonormal sequence A in H the Scott topology on Ω^ω_A is the same as the subspace topology that Ω^ω_A inherits from the Scott topology of Ω^ω.

Proof. Let V be a Scott open subset of Ω^ω. Then $V \cap \Omega^\omega_A$ is increasing in Ω^ω_A. Let D be a directed subset of Ω^ω_A with $\sup_{\Omega^\omega_A} D \in V \cap \Omega^\omega_A$. Now $\sup_{\Omega^\omega} D = \sup_{\Omega^\omega_A} D$ so $D \cap V \neq \emptyset$. Thus $V \cap \Omega^\omega_A$ is Scott open in Ω^ω_A.

Let U be a Scott open subset of Ω^ω_A. We have seen in Theorem 8.3 that $\Phi_A^{-1}[U]$ is Scott open in Ω^ω and that $\Phi_A^{-1}[U] \cap \Omega^\omega_A = U$. Thus U is open in the subspace topology that Ω^ω_A inherits from the Scott topology on Ω^ω. □

Theorem 10.4. For every orthonormal sequence A in H, if $\mu : \Omega^\omega \to [0,\infty)^*$ is a measurement of Ω^ω then $\mu \restriction \Omega^\omega_A$ is a measurement of Ω^ω_A.

Since Ω^ω_A is order isomorphic to Λ^ω by Theorem 8.1 this means that measurements of Ω^ω induce measurements of Λ^ω.

Proof. Let $e \in \max \Omega^\omega_A$ and let U be a Scott neighborhood of e in Ω^ω_A. There is a Scott neighborhood V of e in Ω^ω such that $V \cap \Omega^\omega_A = U$. Since $e \in \max \Omega^\omega$ there is a neighborhood W of $\mu(e)$ such that $\mu^{-1}[W] \cap \downarrow_{\Omega^\omega} e \subseteq V$. Let $\nu = \mu \restriction \Omega^\omega_A$. Then

$\nu^{-1}[W] = \mu^{-1}[W] \cap \Omega^\omega_A$. But $(\downarrow_{\Omega^\omega} e) \cap \Omega^\omega_A = \downarrow_{\Omega^\omega_A} e$ so $\nu^{-1}[W] \cap \downarrow_{\Omega^\omega_A} e = (\mu^{-1}[W] \cap \downarrow_{\Omega^\omega} e) \cap \Omega^\omega_A \subseteq V \cap \Omega^\omega_A = U$. □

Theorem 10.4 relied on the fact that the subspace topology on Ω^ω_A is the same as the Scott topology on Ω^ω_A. We do not know either of these for Γ_A, which leads to the following questions.

Question 10.1. Is Γ_a a retract of Ω^ω?

Question 10.2. Does the Scott topology on Γ_A coincide with the subspace topology that Γ_A inherits from the Scott topology on Ω^ω?

Question 10.3. Does every measurement of Ω^ω induce a measurement of Δ^ω?

A positive answer to Question 10.1 implies a positive answer to Question 10.2 which in turn implies a positive answer to Question 10.3.

11. Projections

For every subspace G of H let Ω^ω_G be the set of density operators on G. Let P be the projection of H onto G. We know from Luder's Rule that if $r \in \Omega^\omega$ such that $\mathrm{tr}(P \circ r) \neq 0$ then $P'(r) = \dfrac{P \circ r \circ P}{\mathrm{tr}(P \circ r)}$ is a density operator on H. If $s = P'(r)$ then s maps G to G and is zero on vectors that are orthogonal to G. Thus the range of P' is clearly isomorphic to Ω^ω_G via the isomorphism which restricts the domain of an element of $\mathrm{ran}\, P'$ to G

Definition 11.1. Let $r \in \Omega^\omega$ and let G be a subspace of H. The projection $P : H \to G$ is *admitted* by r if and only if G is spanned by a set of eigenvectors of r, at least one of which corresponds to a nonzero eigenvalue.

If P is admitted by r then $\mathrm{tr}(P \circ r) \neq 0$ and r is in the domain of P'.

Theorem 11.1. For all density operators r and s on H, $r \sqsubseteq s$ if and only if $P'(r) \sqsubseteq P'(s)$ for every projection P admitted by both r and s.

Proof. The eigenvectors of r span H as do the eigenvectors of S. So if $P'(r) \sqsubseteq P'(s)$ for every projection P admitted by both r and s then $r = P'(r) \sqsubseteq P'(s) = s$ when P is the trivial projection of H onto itself.

Assume that $r \sqsubseteq s$ and let P be a projection that is admitted by both r and s. We know that such a projection exists because if $r \sqsubseteq s$ then there are a positive eigenvalue λ of r and a positive eigenvalue μ of s such that $E(s, \mu) \subseteq E(r, \lambda)$. The projection of H onto $E(s, \mu)$ is then admitted by both r and s. Set $G = \mathrm{ran}\, P$. We can find an orthonormal sequence A in H such that A witnesses $r \sqsubseteq s$ and a subset M of ω such that $A[M]$ is a basis for G. There is also a strictly increasing function $i : \omega \to \omega$ such that $M \subseteq \mathrm{ran}\, i = I$ and $B = A \circ i$ labels both $P'(r)$ and $P'(s)$. This means that if $A(n)$ corresponds to a positive eigenvector of $P'(r)$ or of $P'(s)$ and $n \notin M$ then $n \notin I$. Thus B is created from A by skipping those eigenvectors corresponding to positive eigenvalues which are not included in G. These are the ones whose probabilities are being set equal to 0. We take

$M \subseteq I$ rather than $M = I$ because G could be finite dimensional. To simplify notation set $u = P'(r)$ and $v = P'(s)$. We show that $f_B^u = p_M(f_A^r) \circ i$ and $f_B^v = p_M(f_A^s) \circ i$.

First consider $\operatorname{tr}(P \circ r)$. If $n \notin M$ then $A(n)$ is orthogonal to G.

$$\operatorname{tr}(P \circ r) = \sum_{n \in \omega} \langle A(n)|P(r(A(n)))\rangle = \sum_{n \in M} \langle A(n)|r(A(n))\rangle = \sum_{n \in M} f_A^r(n)$$

If $i(n) \notin M$ then $A(i(n))$ is orthogonal to G so $P(A(i(n))) = \mathbf{0}$ and $f_B^u(n)$ equals $\langle B(n)|u(B(n))\rangle = 0 = p_M(f_M^r)(i(n))$. If $i(n) \in M$ then $P(A(i(n))) = A(i(n))$ so $u(B(n))$ equals $r(A(i(n)))/\operatorname{tr}(P \circ r) = r(A(i(n)))/\sum_{m \in M} f_A^r(m)$.

$$\begin{aligned} f_B^u(n) &= \langle B(n)|u(B(n))\rangle \\ &= \frac{1}{\sum_{m \in M} f_A^r(m)} \langle A(i(n))|r(A(i(n)))\rangle \\ &= \frac{f_A^r(i(n))}{\sum_{m \in M} f_A^r(m)} \\ &= p_M(f_a^r)(i(n)) \end{aligned}$$

The same argument shows that $f_B^v = p_M(f_A^s) \circ i$.

Now $f_A^r \leq f_A^s$ so $p_M(f_A^r) \leq p_M(f_A^s)$. Since i is strictly increasing it follows that $p_M(f_A^r) \circ i \leq P_M(f_A^s) \circ i$. Therefore $f_B^u \leq f_B^v$ and B witnesses $P'(r) \sqsubseteq P'(s)$. □

12. Lattices of Birkhoff and von Neumann

In (Birkhoff and von Neumann 1936) Birkhoff and von Neumann show that propositions about physical characteristics of a classical or quantum system correspond to subspaces of a mathematical space. For classical physics these are subspaces of the phase space while for quantum physics these are subspaces of the underlying Hilbert space. The geometric and algebraic structure of these subspaces in turn give a logical structure to the propositional calculus of the physical propositions. The lattices derived from these subspaces capture one of the main differences between classical and quantum physics: the lattice of the classical space is distributive and the lattice of the quantum space is not. In (Coecke and Martin 2002) this relationship is shown to arise in a natural manner from the Bayesian and spectral orders of Δ^n and Ω^n respectively. There are subsets of Δ^n and Ω^n which, under the order they inherit, are order isomorphic to the lattices of Birkhoff and von Neumann. We show that a similar correspondence exists for the infinite dimensional classical and quantum states as long as the proposition under consideration results in a finite dimensional subspace of H.

A physical proposition essentially states that the value of an observable should fall within a given range of values. In a classical system, which uses ω as a fixed frame of reference, this restriction on the values of the observable selects a number of elements of ω on which the observable obtains those values. For a subset A of an ordered set X we use $\bigwedge A$ to denote the join or greatest lower bound of A in X.

Definition 12.1. An element a of an ordered set X is said to be *irreducible* if and only if $\bigwedge[(\uparrow a) \cap \max X] = a$.

The set of irreducible elements of X is denoted $\mathrm{Ir}(X)$.

Theorem 12.1. For $f \in \Delta^\omega$ the following statements are equivalent.

1 f is irreducible.
2 There is a nonempty finite subset F of ω such that $f(n) = f(m)$ for all $m, n \in F$ and $f(n) = 0$ for all $n \in \omega - F$.
3 There is a finite subset X of $\max \Delta^\omega$ such that $f = \bigwedge X$.

Proof. First assume that f is irreducible. Let $F = \{n \in \omega : f \leq e_n\}$. F must be finite and nonempty by Theorem 16 of (Mashburn 2007b). Also, $f(n) = f^+$ for all $n \in F$. Now $(\uparrow f) \cap \max \Delta^\omega = \{e_n : n \in F\}$. Define g by $g(n) = 1/|F|$ for all $n \in F$ and $g(n) = 0$ if $n \notin F$. Then $g \in \Delta^\omega$ and $g \leq e_n$ for all $n \in F$. Therefore $g \leq f$ and so $f(n) = 0$ for all $n \notin F$.

Now assume that there is a nonempty finite subset F of ω such that $f(n) = f(m)$ for all $m, n \in F$ and $f(n) = 0$ for all $n \in \omega - F$. Then $f(n) = f^+$ for all $n \in F$ so $f \leq e_n$ for all $n \in F$. Let $g \in \Delta^\omega$ such that $g \leq e_n$ for all $n \in F$. Let σ be a one-to-one function from ω into ω such that $g \circ \sigma$ is decreasing and the only coordinates of g that are missing from $g \circ \sigma$ are some of those whose values are 0. Then $\sigma(n) \in F$ for all $n < |F|$. If $n+1 < |F|$ then $(g \circ \sigma)(n) = (g \circ \sigma)(n+1)$ and $(f \circ \sigma)(n) = (f \circ \sigma)(n+1)$ so $(g \circ \sigma)(n)(f \circ \sigma)(n+1) = (g \circ \sigma)(n+1)(f \circ \sigma)(n)$. If $n+1 > |F|$ then $(f \circ \sigma)(n+1) = 0$ so $(g \circ \sigma)(n)(f \circ \sigma)(n+1) = 0 \leq (g \circ \sigma)(n+1)(f \circ \sigma)(n)$. Therefore $g \leq f$. It follows that $f = \bigwedge\{e_n : n \in F\}$.

Finally, assume that there is a nonempty finite subset X of $\max \Delta^\omega$ such that $f = \bigwedge X$. Obviously $X \subseteq (\uparrow f) \cap \max \Delta^\omega$. Let $F = \{n \in \omega : e_n \in X\}$ and let g be the element of Δ^ω defined by $g(n) = 1/|F|$ if $n \in \omega$ and $g(n) = 0$ if $n \in \omega - F$. Then $g \leq e$ for all $e \in X$ so $g \leq f$. Therefore $f(n) = 0$ when $n \notin F$ and it follows that $X = (\uparrow f) \cap \max \Delta^\omega$. Thus f is irreducible. $\square$

It is possible to recover $\mathcal{P}(n)$, the power set or set of all subsets of n, from the irreducible elements of Δ^n. We are not able to recover $\mathcal{P}(\omega)$ from the irreducible elements of ω, but we can recover the lattice of nonempty finite subsets of ω. Let $\mathrm{Fin}(\omega)$ be the set of nonempty finite subsets of ω. We order $\mathrm{Fin}(\omega)$ by $X \leq Y$ if and only if $X \subseteq Y$. This preserves the idea of Birkhoff and von Neumann that the subset relation should reflect implication in physical propositions. But if $f, g \in \mathrm{Ir}\,\Delta^\omega$, $F = (\uparrow f) \cap \max \Delta^\omega$ and $G = (\uparrow g) \cap \max \Delta^\omega$, then $f \leq g$ if and only if $G \subseteq F$. We therefore need to reverse the order that $\mathrm{Ir}\,\Delta^\omega$ inherits from Δ^ω to make it match the order we wish to impose on $\mathrm{Fin}(\omega)$.

Theorem 12.2. $\mathrm{Ir}(\Delta^\omega)^*$ is order isomorphic to $\mathrm{Fin}(\omega)$.

Proof. Define $i : \mathrm{Ir}(\Delta^\omega)^* \to \mathrm{Fin}(\omega)$ by $i(f) = \{n \in \omega : f \leq e_n\}$ for all $f \in \mathrm{Ir}\,\Delta^\omega$. It is easy to see that i is an order isomorphism. $\square$

The ordered sets $\mathrm{Ir}(\Delta^\omega)^*$ and $\mathrm{Fin}(\omega)$ are not quite lattices because by omitting $\emptyset$ we have omitted what would have been the join of many pairs of nonempty finite subsets of ω. We can rectify this by including $\emptyset$ in $\mathrm{Fin}(\omega)$ and introducing a new element $\perp$ into $\mathrm{Ir}(\Delta^\omega)^*$ which is declared to be less than all elements of $\mathrm{Ir}(\Delta^\omega)^*$. There is no element of Δ^ω corresponding to this new element, for any such element of Δ^ω which is larger than all elements of $\operatorname{Ir}\Delta^\omega$ (in the order of Δ^ω, not the reverse order we use in $\operatorname{Ir}(\Delta^\omega)^*$) could not be smaller than any maximal element of Δ^ω, an impossible situation.

We cannot obtain the full lattice $\mathcal{P}(\omega)$ in this way. The infinite subsets of ω would have to correspond to elements of Δ^ω which lie below infinitely many maximal elements of Δ^ω. If f were such an element then f would be constant on an infinite subset of ω, which is impossible.

The next theorem shows that there is a strong and natural connection between the irreducible elements of Ω^ω and those of Δ^ω.

Theorem 12.3. A state r in Ω^ω is irreducible if and only if $f_A^r \in \operatorname{Ir}\Delta^\omega$ for every orthonormal sequence A that labels r.

Proof. Assume that $r \in \operatorname{Ir}\Omega^\omega$ and let A be an orthonormal sequence that labels r. Let $F = \{n \in \omega : f_A^r(n) = \max\operatorname{spec} r\}$. Then $f_A^r(m) = f_A^r(n)$ for all $m, n \in F$. Let s be the element of Δ^ω defined by setting $s(A(n)) = (1/|F|)A(n)$ for all $n \in F$ and $s(\boldsymbol{\alpha}) = \mathbf{0}$ for all $\boldsymbol{\alpha}$ orthogonal to $\{A(n) : n \in F\}$. Let $e \in (\uparrow r) \cap \max\Omega^\omega$. There is $n \in \omega$ such that $A(n) \in E(e, 1)$. The comments at the end of Section 5 show that $s \sqsubseteq e$. Thus $s \sqsubseteq r$ since r is irreducible.

Let B be an orthonormal sequence that witnesses $s \sqsubseteq r$. Since f_B^s is decreasing we know that $f_B^s(n) = \max\operatorname{spec} s = 1/|F|$ for $n < |F|$ and $f_B^s(n) = 0$ for $n \geq |F|$ and that f_B^r has the same property. But $f_B^s \leq f_B^r$ so $f_B^r(n) = 0$ for $n \geq |F|$. By Theorem 5.3 $f_A^r(n) = f_B^r(n) = 0$ for $n \geq |F|$. Thus $f_A^r \in \operatorname{Ir}\Delta^\omega$.

Let $r \in \Omega^\omega$ and let A be an orthonormal sequence that labels r with $f_A^r \in \operatorname{Ir}\Delta^\omega$. There is a nonempty finite subset F of ω such that $f_A^r(m) = f_A^r(n)$ for all $m, n \in F$ and $f_A^r(n) = 0$ for $n \in \omega - F$. Thus $\operatorname{spec} r = \{0, \lambda\}$ for some $\lambda > 0$. Let $s \in \Omega^\omega$ such that $s < e$ for all $e \in (\uparrow r) \cap \max\Omega^\omega$. If $n \in F$ then $A(n)$ is an eigenvector of s corresponding to its maximum eigenvalue. We can therefore find an orthonormal sequence B that labels s such that $B(n) = A(n)$ for all $n \in F$. Therefore B labels r as well and $f_B^s \leq e_n$ for all $n \in F$. But $f_B^r = f_A^r = \bigwedge[(\uparrow f_A^r) \cap \max\Delta^\omega] = \bigwedge\{e_n : n \in F\}$ so $f_B^s \leq f_B^r$. So $s \sqsubseteq r$ and r is irreducible in Ω^ω. □

Let $\mathbb{F}$ be the nontrivial finite dimensional subspaces of H.

Theorem 12.4. $\operatorname{Ir}(\Omega^\omega)^*$ is order isomorphic to $\mathbb{F}$.

Proof. For every $r \in \operatorname{Ir}(\Omega^\omega)^*$ let $\lambda_r = \max\operatorname{spec} r$. It is easy to show that the function $i : \operatorname{Ir}(\Omega^\omega)^* \to \mathbb{F}$ given by $i(r) = E(r, \lambda_r)$ is an order isomorphism. □

We can make these ordered sets lattices by adding the trivial subspace $\{\mathbf{0}\}$ to $\mathbb{F}$ and a least element to $\operatorname{Ir}(\Omega^\omega)^*$. Our ordered set again fails to capture all possible physical propositions. Those which allow an infinite number of possible values, for example those which claim that the value of some observable is less than a given value, cannot be represented by one of our irreducibles because that proposition is associated with an infinite dimensional subspace. No density operator can lie below an infinite number of pure states in the spectral order on Ω^ω for then the operator would have to take on a value of a nonzero constant on an infinite orthogonal subset of H.

13. Conclusion

The spectral order defined here for infinite dimensional quantum states retains many of the desirable characteristics of the spectral order defined by Coecke and Martin in (Coecke and Martin 2002) for finite dimensional quantum states, but it fails to provide a true domain-like structure. Also, the reasonable entropy functions fail to be measurements in the sense of Martin. This leaves us with the following question.

Question 13.1. Is there an order structure for the infinite dimensional quantum states which will provide a domain or weak domain setting, will provide a meaningful model of the quantum states, and in which reasonable entropy functions will be measurements in the sense of Martin?

The spectral order does seem to be the natural extension of Coecke and Martin's order, so something very different may be needed.

References

Abramsky, S. and Jung, A.(1994) Domain Theory. In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume III, Oxford University Press.

Birkhoff, G. and von Neumann, J. (1936) The logic of quantum mechanics. *Annals of Mathematics* **37**, 823–843.

Brukner, C. and Zeilinger, A. (1999) Operationally Invariant Information in Quantum Measurements. *Physical Review Letters* **83** (17), 3354.

Brukner, C. and Zeilinger, A. (2001) Conceptual inadequacy of the Shannon information in quantum measurements. *Physical Review A* **63** 022113.

Bub, J. (2005)Quantum theory is about quantum information. *Foundations of Physics* **35** (4), 541–560.

Clifton, R., Bub, J., and Halvorson, H. (2003) Characterizing Quantum Theory in terms of Information-Theoretic Constraints. *Foundations of Physics* **33**, 1561–1591, arXive:quant-ph/0211089.

Coecke, B. and Martin, K. (2002) A partial order on classical and quantum states. *Oxford University Computing Laboratory, Research Report PRG-RR-02-07, August 2002,* `http://web.comlab.ox.ac.uk/oucl/publications/tr/rr-02-07.html`.

Coecke, B. and Martin, K. (2004) Partiality in Physics. Invited paper in *Quantum theory: reconsideration of the foundations 2*, Vaxjo, Sweden, Vaxjo University Press.

van Enk, S.J. (2007) A Toy Model for Quantum Mechanics. *Foundations of Physics* **37**, 1447–1460.

Fano, G. (1971) *Mathematical Methods of Quantum Mechanics*, New York, McGraw Hill.

Fuchs, C. *Quantum Mechanics as Quantum Information (and only a little more)*, arXive:quant-ph/0205039, 2002.

Hagar, A. and Hemmo, M. (2006) Explaining the Unobserved - Why Quantum Mechanics Ain't Only About Information. *Foundations of Physics* **36** (9), 1295–1324.

Hardy, G.H., Littlewood, J.E., and Pólya, G. (1952) *Inequalities*, Second Edition, Cambridge University Press, London and New York.

Marshall, A.W. and Olkin, I. (1979) Inequalities: Theory of Majorization and Its Applications. *Mathematics in Science and Engineering* Volume 143, Academic Press, New York.

Martin, K. (2000) *A Foundation for Computing*, Ph.D. Thesis, Tulane University, (http://www.math.tulane.edu/ martin/).

Martin, K. (2004) A continuous domain of classical states. *Research Report RR-04-06*, Oxford University. Computing Laboratory, Programming Research Group (http://www.math.tulane.edu/ martin/).

Mashburn, J. (2008) A comparison of three topologies on ordered sets. *Topology Proceedings* **31**, Number 1, 197–217.

Mashburn, J. (2008) An Order Model for Infinite Classical States. *Foundations of Physics* **38**, Number 1, 47–75.

Nielsen, M.A. (1999) Conditions for a class of entanglement transformations. *Physical Review Letters* **83** (2), 436–439.

Scott, D. (1970) *Outline of a mathematical theory of computation*, Technical Monograph PRG-2, November 1970.

Spekkens, R. (2007) In the defense of the epistemic view of quantum states: a toy theory. *Physical Reviews A* **75**, 032110.

Timpson, C.G. (2003) On a supposed conceptual inadequacy of the Shannon information in quantum mechanics. *Stud. Hist. Phil. Mod. Phys.* **35** (4), 441–468.

Uffink, J.B.M. (1990) *Measures of Uncertainty and the Uncertainty Principle*, Ph.D. Thesis, University of Utrecht, (http://www.phys.uu.nl/igg/jos/publications/proefschrift.pdf).