4-2007

A Spectral Order for Infinite Dimensional Quantum Spaces: A Preliminary Report

Joe Mashburn
University of Dayton, joe.mashburn@udayton.edu

A Spectral Order for Infinite Dimensional Quantum Spaces
A Preliminary Report

Joe Mashburn[1]

Department of Mathematics
University of Dayton
Dayton, OH 45469-2316, USA

Abstract

In 2002 Coecke and Martin created a Bayesian order for the finite dimensional spaces of classical states in physics and used this to define a similar order, the spectral order on the finite dimensional quantum states. These orders gave the spaces a structure similar to that of a domain. This allows for measuring information content of states and for determining which partial states are approximations of which pure states. In a previous paper the author extended the Bayesian order to infinite dimensional spaces of classical states. The order on infinite dimensional spaces retains many of the characteristics important to physics, but loses the domain theoretic structure. It becomes impossible to measure information content in the same way that it is done for the finite dimensional spaces, and the sense of approximation is lost. In this paper, we will use the Bayesian order to define a spectral order on the infinite dimensional spaces of quantum states.

Keywords: Bayesian order, spectral order, classical states, quantum states, density operator, domain

1 Introduction

In 2002 Coecke and Martin ([1]) defined the *Bayesian order* on finite dimensional spaces of classical states in physics. This order reflected the important physical properties of such states and, because the spaces with this order have a structure similar to that of domains, it also provided a method for measuring the information content of the states and for determining which partial states approximated which total states. They then used the Bayesian order to define a *spectral order* on the finite dimensional spaces of quantum states. This order gives the quantum states a structure with the same sort of properties enjoyed by the classical states under the Bayesian order. It also provides the ability to measure and to approximate. Recently the author was able to extend the Bayesian order to infinite dimensional spaces of classical states. The order still reflects important physical properties in the infinite dimensional spaces, but the domain theoretic aspects of the structure are lost. In

[1] Email: joe.mashburn@udayton.edu

the following sections we will provide a candidate for the spectral order on infinite dimensional quantum states corresponding to that on finite dimensional quantum states. The remainder of this section will be devoted to giving some background on these orders and some associated concepts. In Section 2 the order will be defined and proven to be an order. In Section 3 we will show that the structure of this order is similar to that of the Bayesian order on finite dimensional spaces and that it has many properties similar to those of the spectral order on finite dimensional spaces.

The classical states can be viewed as strings of numbers which are probabilities that a particular value is obtained in an observation. The space Δ^n of classical states on n-dimensions is the set of functions $f : n \to [0,1]$ such that $\sum_{k\in n} f(k) = 1$. Here n is a positive natural number with $n = \{0,\ldots,n-1\}$. The Bayesian order moves in the direction of increasing certainty. That is, if $f \leq g$ then g should be a state that is consistent with f and is more certain in terms of which values will be observed. The idea of consistency is captured by requiring that there is a realignment of coordinates will change both states to decreasing functions. That is, in order to set $f \leq g$ we must know that there is a permutation σ of n such that $f \circ \sigma$ and $g \circ \sigma$ are both decreasing. Here we are using decreasing in the sense of nonincreasing. The idea of greater certainty (or more information) is obtained by requiring that the ratios of a coordinate with the succeeding coordinate is greater in the larger state. That is, given the permutation σ which rearranges f and g to decreasing sequences, we want $(f\circ\sigma)(k)(g\circ\sigma)(k+1) \leq (f\circ\sigma)(k+1)(g\circ\sigma)(k)$ for all $k \in n$. So we define $f \leq g$ if and only if there is a permutation σ on n such that $f\circ\sigma$ and $g\circ\sigma$ are both decreasing and $(f\circ\sigma)(k)(g\circ\sigma)(k+1) \leq (f\circ\sigma)(k+1)(g\circ\sigma)(k)$ for all $k \in n$.

The order for states on an infinite number of coordinates is defined in a similar fashion. Let Δ^ω be the set of all functions $f : \omega \to [0,1]$ such that $\sum_{n=0}^{\infty} f(n) = 1$. Here ω is the set of natural numbers. One difference between these functions and the finite functions used above is that when we rearrange the finite functions with a domain of size n the result is a function with a domain of size n. It is the same sort of sequence as the one with which we started. But if we rearrange a function whose domain is ω so that it is now decreasing, we could have a sequence that looks very different than the original one. The rearranged sequence could have a domain which, as an ordinal number, is larger than ω. For example, if $f(n) = 2^{-(n/2)-1}$ when n is even and $f(n) = 0$ when n is odd, then if we rearrange the terms of f into a decreasing sequence we must first list all the infinite number of positive terms before we can list any of the infinite number of zeros. For this reason, we do not consider permutations of ω for our rearrangements, but rather one-to-one functions from ω into ω which retain those coordinates on which the state is positive. We could drop the coordinates on which the state equals 0.

Definition 1.1 For every $f \in C$ let $S(f)$ be the set of one-to-one functions σ from ω into ω such that $f^{-1}[(0,1]] \subseteq \operatorname{ran}\sigma$ and $f\circ\sigma$ is decreasing.

If $n \in \omega$, $f(n) > 0$, and $\sigma \in S(f)$ then there is $m \in \omega$ such that $f(m) = n$. If $f(n) = 0$ then it may be that n is not in the range of σ. The order on Δ^ω is now defined as follows.

Definition 1.2 For every $f, g \in \Delta^\omega$ set $f \leq g$ if and only if there is $\sigma \in S(f)\cap S(g)$ such that $(f\circ\sigma)(n)(g\circ\sigma)(n+1) \leq (f\circ\sigma)(n+1)(g\circ\sigma)(n)$ for every $n \in \omega$.

It is shown in [3] that Δ^ω with this order satisfies the Mixing Law and Degeneracy, and that Δ^ω has an infinite number of maximal elements, but no minimal elements. This is the order which we will extend to quantum states. Degeneracy is an important property in the development of the spectral order, so we need to state exactly what it means in Δ^ω.

Theorem 1.3 *Let $f, g \in \Delta^\omega$. If $f \leq g$ and there is $n \in \omega$ such that $f(n) = 0$ then $g(n) = 0$. Also, if there are $m, n \in \omega$ with $g(m) = g(n) > 0$ then $f(m) = f(n)$.*

A special subset of Δ^ω is the set of all elements of Δ^ω which are already decreasing. We denote this set by Λ^ω and it is a typical subset of Δ^ω for the following reason. For every one-to-one function $\sigma : \omega \to \omega$ let Λ^ω_σ be the set of $f \in \Delta^\omega$ such that $f \circ \sigma$ is decreasing, that is, $f \circ \sigma \in \Lambda^\omega$. Then $\Delta^\omega = \bigcup_\sigma \Lambda^\omega_\sigma$ and Λ^ω_s is order isomorphic to Λ^ω for every σ.

The last topic in the introduction is that of approximation. This is where domain theory comes in. Coecke and Martin found that Δ^n is not quite a domain: it is not continuous under the way below relation. But if they modified the way below relation slightly, then they got something very much like a domain.

Definition 1.4 Let a and b be elements of an ordered set X. We say that a is *weakly way below* b if and only if for every directed subset D of X with $\sup D = b$ there is $c \in D$ such that $a \leq c$.

Under this relation Δ^n is *exact*, that is, every element of Δ^n is the directed supremum of the elements which are weakly way below it. Since Δ^n is directed complete and the weakly way below relation is interpolative in Δ^n, Δ^n looks very much like a domain. We could call it a weak domain. To the maximal elements of Δ^n, it looks just like a domain, since if b is maximal in X then a is weakly way below b if and only if a is way below b. And it is the maximal elements that we are interested in approximating. If $f, g \in \Delta^n$ and f is weakly way below g then f contains information that is essential to g. We cannot reach the state g without achieving a state h which contains at least as much information as f. This means that every path ending at g must contain an element which is greater than f. In this sense we can say that all paths to g start at f. Also, since g is the supremum of the elements which are weakly way below it, we can construct g from these elements.

However, it is shown in [3] that all this is lost when one moves to the infinite dimensional case. Here, no element is weakly way below any other one. There are just too many ways to approach a given state. It is still true that if f is a maximal element of Δ^ω then f is the supremum of the states that are strictly less than it, but none of these states can be considered essential any more. The situation should be the same with quantum states on infinite dimensions.

2 Quantum States

Throughout this paper H will represent an infinite dimensional Hilbert space. For the quantum states we will take the density operators on H.

Definition 2.1 A density operator is a self-adjoint positive linear operator whose trace is 1.

This means that if r is a density operator on H then all the eigenvalues of r are nonnegative real numbers and that if B is a basis for H then $\sum_{\beta\in B}\langle r(\beta)|\beta\rangle = 1$. If we choose B to be an orthonormal basis of eigenvectors of r then we see that the sum of all the eigenvalues of r, each repeated as many times as the dimension of its eigenspace, must be one. So the density operators look a bit like the elements of Δ^ω in this sense. But the elements of Δ^ω can be easily compared with one another because the set ω on which they are defined gives a fixed basis for comparison, while the density operators on H have no such fixed basis for comparison. An important part of the construction will be to give some structure to the bases of H. We will denote the set of all density operators on H by Ω. For every $r \in \Omega$ and every $\lambda \in \operatorname{spec} r$ we will let $E(r,\lambda)$ be the eigenspace of r corresponding to λ. We will use $\operatorname{spec}^+ r$ to represent the positive eigenvalues of r.

Definition 2.2 An *orthonormal sequence* is a one-to-one function $B : \omega \to H$ such that $\operatorname{ran} B$ is an orthonormal subset of H.

Note that the range of the orthonormal sequence need not be a basis, but can always be expanded to an orthonormal basis for H. We will abuse the notation by identifying B with its range and sometimes treating B as a subset of H. By enumerating elements of an orthonormal subset of H, these sequences provide the structure that we need to define the spectral order on Ω. The fact that $\operatorname{ran} B$ need not be a basis allows us the flexibility to eliminate the zeros from the spectrum of a density operator, just as we eliminated certain coordinates in the classical state which had the value of zero.

Definition 2.3 Let $r \in \Omega$ and let $B : \omega \to H$. The *coordinate function of r with respect to B* is the function $f^r_B : \omega \to \mathbb{C}$ given by $f^r_B(n) = \langle r(B(n))|B(n)\rangle$.

If the range of B is an orthonormal set of eigenvectors of r then $f^r_B(n)$ gives us the eigenvalue corresponding to the eigenvector $B(n)$.

Definition 2.4 The orthonormal sequence B is said to *label* the density operator r if and only if the following conditions are satisfied.

(i) For every $n \in \omega$, $B(n)$ is an eigenvector of r.

(ii) For every $\lambda \in \operatorname{spec}^+ r$ there is $M \subseteq \omega$ such that $B[M]$ is a basis for $E(r,\lambda)$.

(iii) $f^r_B \in \Lambda^\omega$.

The first condition in this definition guarantees that each $f^r_B(n)$ is an eigenvalue of r. The second condition guarantees that each positive eigenvector appears the same number of times as the dimension of its eigenspace, which must be finite since r is a density operator. The last condition almost follows from the first two, since they will give $\sum_{n\in\omega} f^r_B(n) = 1$. But it also says that f^r_B is decreasing. This is a stronger condition than that used by Coecke and Martin for their labels. Note that if B is extended to an orthonormal basis $\hat{B}$ then $\hat{B} - B \subseteq E(r, 0)$. If B is an orthonormal basis of H and for every $\lambda \in \operatorname{spec}^+ r$ there is $A \subseteq B$ such that A is a basis for $E(r,\lambda)$ then B generates an orthonormal sequence that labels r. If A and B are both orthonormal sequences which label r then $f^r_A = f^r_B$.

If B is an orthonormal sequence that labels the density operator r then we can associate an observable e with B. An observable is a self-adjoint linear operator from H into H. Its eigenvectors form a basis for H. In this case, we begin with the basis and associate with it an observable. The observable acts on the density operator to produce the probability that a particular outcome is observed. This is the eigenvalue of r associated with that outcome. That is, the operation of the observable on r produces f^r_B. Any outcome associated with a coordinate n which is not in the image of B corresponds to the eigenvalue 0 of r. This means that it has no chance of being observed, so we can ignore its existence when it comes to the evaluation of r. This explains why B need not be onto.

Definition 2.5 For every $r \in \Omega$ let $L(r)$ be the set of orthonormal sequences which label r.

Theorem 2.6 *For every orthonormal sequence A and every $f \in \Lambda^\omega$ there is $r \in \Omega$ such that $A \in L(r)$ and $f^r_A = f$.*

Proof. Expand A to an orthonormal basis $\hat{A}$ of H. Let r be the linear operator on H determined by setting $r(\alpha) = f(n)\alpha$ if $\alpha = A(n)$ and $r(\alpha) = \mathbf{0}$ if $\alpha \in \hat{A} - A$. Let $\beta, \delta \in H$ and for every $\alpha \in \hat{A}$ let $b_\alpha, d_\alpha \in \mathbb{C}$ such that $\beta = \sum_{\alpha\in\hat{A}} b_\alpha \alpha$ and $\delta = \sum_{\alpha\in\hat{A}} d_\alpha \alpha$. If $\alpha = A(n)$ then we will use b_n and d_n in place of b_α and d_α. We must check that r is self-adjoint.

$$\begin{aligned}
\langle r(\beta)|\delta\rangle &= \left\langle r\left(\sum_{\alpha\in\hat{A}} b_\alpha\alpha\right) \middle| \sum_{\alpha\in\hat{A}} d_\alpha\alpha \right\rangle \\
&= \left\langle \sum_{n\in\omega} b_n f(n) A(n) \,|\, \sum_{\alpha\in\hat{A}} d_\alpha\alpha \right\rangle \\
&= \sum_{n\in\omega}\sum_{\alpha\in\hat{A}} b_n f(n) \overline{d}_\alpha \langle A(n)\,|\,\alpha\rangle \\
&= \sum_{n\in\omega} b_n f(n) \overline{d}_n \\
&= \sum_{\alpha\in\hat{A}}\sum_{n\in\omega} b_\alpha f(n) \overline{d}_n \langle \alpha\,|\,A(n)\rangle \\
&= \left\langle \sum_{\alpha\in\hat{A}} b_\alpha\alpha \middle| \sum_{n\in\omega} d_n f(n) A(n) \right\rangle \\
&= \left\langle \sum_{\alpha\in\hat{A}} b_\alpha\alpha \middle| r\left(\sum_{\alpha\in\hat{A}} d_\alpha\alpha\right) \right\rangle \\
&= \langle \beta\,|\,r(\delta)\rangle
\end{aligned}$$

It now follows from the definition of r that $r \in \Omega$, $A \in L(r)$, and $f^r_A = f$. □

Definition 2.7 Let $\sqsubseteq$ be the relation on Ω defined by setting $r \sqsubseteq s$ if and only if there is an orthonormal sequence A such that $A \in L(r) \cap L(s)$ and $f^r_A \le f^s_A$.

The order used to compare f^r_A and f^s_A is the Bayesian order on Δ^ω. This definition is saying that in order to compare two elements of Ω we must first find a common orthonormal sequence (or basis) for comparison. Once this standard has been set we use the sequences of eigenvalues of r and s in the same way that sequences are used in Δ^ω.

Lemma 2.8 *Let $r, s \in \Omega$. If $r \sqsubseteq s$ and $B \in L(r) \cap L(s)$ then $f^r_B \leq f^s_B$.*

Proof. Let A be an orthonormal sequence that witnesses $r \sqsubseteq s$. Since $\operatorname{ran} f^r_B = \operatorname{ran} f^r_A$ and both f^r_B and f^r_A are decreasing, we know that $f^r_B = f^r_A$. Similarly, $f^s_B = f^s_A$. Therefore $f^r_B \leq f^s_B$. $\square$

The next theorem shows that this relation on Ω satisfies degeneracy.

Theorem 2.9 (Degeneracy) *For every $r, s \in \Omega$ if $r \sqsubseteq s$ then $E(r, 0) \subseteq E(s, 0)$ and for every $\mu \in \operatorname{spec}^+ s$ there is $\lambda \in \operatorname{spec}^+ r$ such that $E(s, \mu) \subseteq E(r, \lambda)$.*

Proof. Assume that $r \sqsubseteq s$ and let A be an orthonormal sequence that witnesses this relation. Let $\mu \in \operatorname{spec}^+ s$ and let $M \subseteq \omega$ such that $A[M]$ is a basis for $E(s, \mu)$. Then $f^s_A(n) = \mu$ for all $n \in M$. It follows from the degeneracy of Δ^ω that there is $\lambda > 0$ such that $f^r_A(n) = \lambda$ for all $n \in M$. Then $\lambda \in \operatorname{spec}^+ r$ and $A[M] \subseteq E(r, \lambda)$. Therefore $E(s, \mu) \subseteq E(r, \lambda)$.

Assume that $E(r, 0) \neq \emptyset$. Let $M \subseteq \omega$ such that $A[M] \subseteq E(r, 0)$. (M could be empty.) We can expand $A[M]$ to a basis B for $E(r, 0)$. If $\alpha \in B$ and $n \in \omega - M$ then $\alpha \perp A(n)$. Thus $\alpha \perp E(s, \lambda)$ for every $\lambda \in \operatorname{spec}^+ s$. It follows that $\alpha \in E(s, 0)$ and that $E(r, 0) \subseteq E(s, 0)$. $\square$

We have shown that if A witnesses $r \sqsubseteq s$ and $n \in \omega$ then $E(s, f^s_A(n)) \subseteq E(r, f^r_A(n))$.

Theorem 2.10 *The relation $\sqsubseteq$ is an order on Ω.*

Proof. The reflexivity and antisymmetry of $\sqsubseteq$ follow from the reflexivity and antisymmetry of $\leq$ in Δ^ω. Let $r, s, t \in \Omega$ with $r \sqsubseteq s$ and $s \sqsubseteq t$. Let A and B be orthonormal sequences that witness $r \sqsubseteq s$ and $s \sqsubseteq t$, respectively. We will construct an orthonormal sequence C such that $C(n)$ is an eigenvector of both r and t, and $f^r_C = f^r_A$ and $f^t_C = f^t_B$. It follows that C labels both r and t and that $f^r_C = f^r_A \leq f^s_A = f^s_B \leq f^t_B = f^t_C$.

For every $\mu \in \operatorname{spec}^+ t$ let $N(t, \mu) \subseteq \omega$ such that $B[N(t, \mu)]$ is a basis for $E(t, \mu)$. For every $\lambda \in \operatorname{spec}^+ s$ let $N(s, \lambda) \subseteq \omega$ such that $B[N(s, \lambda)]$ is a basis for $E(s, \lambda)$. Since $f^s_A = f^s_B$ we also know that $A[N(s, \lambda)]$ is a basis for $E(s, \lambda)$. For every $\kappa \in \operatorname{spec}^+ r$ let $N(r, \kappa) \subseteq \omega$ such that $A[N(r, \kappa)]$ is a basis for $E(r, \kappa)$.

Let $n \in \omega$. If there is $\lambda \in \operatorname{spec}^+ s$ such that $n \in N(s, \lambda)$ then set $C(n) = B(n)$. If not, then set $C(n) = A(n)$. The function C is one-to-one on $\bigcup\{N(s, \lambda) : \lambda \in \operatorname{spec}^+ s\}$ and on $\omega - \bigcup\{N(s, \lambda) : \lambda \in \operatorname{spec}^+ s\}$. Also, $C[\bigcup\{N(s, \lambda) : \lambda \in \operatorname{spec}^+ s\}]$ and $C[\omega - \bigcup\{N(s, \lambda) : \lambda \in \operatorname{spec}^+ s\}]$ are both orthonormal subsets of H.

Let $N = \bigcup\{N(s, \lambda) : \lambda \in \operatorname{spec}^+ s\}$ and let $n \in \omega - N$. If $C(n) = A(n) \in E(r, 0)$ then $A(n) \in E(s, 0)$ so $C(n) \perp C[N]$. Assume that there is $\kappa \in \operatorname{spec}^+ r$ such that $C(n) \in E(r, \kappa)$. Let $m \in N$ and let $\lambda \in \operatorname{spec}^+ s$ such that $C(m) = B(m) \in E(s, \lambda)$. By Theorem 2.9 there is $\gamma \in \operatorname{spec}^+ r$ such that $E(s, \lambda) \subseteq E(r, \gamma)$. If $\gamma \neq \kappa$ then $E(r, \gamma) \perp E(r, \kappa)$ so $C(m) \perp C(n)$. Assume that $\gamma = \kappa$. Now $A[N(s, \lambda)]$ is a basis for $E(s, \lambda)$, $A[N(r, \kappa)]$ is a basis for $E(r, \kappa)$, and $E(s, \lambda) \subseteq E(r, \kappa)$. Therefore $N(s, \lambda) \subseteq N(r, \kappa)$. Since $n \notin N(s, \lambda)$ and $m \in N(s, \lambda)$ we have $A(n) \in E(r, \kappa) -$

$E(s,\lambda)$ and $B(m) \in E(s,\lambda)$. Thus $C(m) = B(m) \perp A(n) = C(n)$. It follows that C is one-to-one and that $\operatorname{ran} C$ is orthonormal.

We next show that each $C(n)$ is an eigenvector of both r and t. If $n \in N$ then $C(n) = B(n)$, which is an eigenvector of t, and there is also some $\kappa \in \operatorname{spec}^+ r$ such that $C(n) \in E(r,\kappa)$. Thus $C(n)$ is an eigenvector of r. If $n \in \omega - N$ then $C(n) = A(n)$, which is an eigenvector of r. We have also seen that $C(n)$ is orthogonal to $C[N]$, and therefore $t(n) = s(n) = 0$. Thus $C(n)$ is an eigenvector of t corresponding to the eigenvalue 0.

Finally, we will show that $f^r_C = f^r_A$ and $f^t_C = f^t_B$. If $n \in \omega - N$ then $C(n) = A(n)$ and $f^r_C(n) = f^r_A(n)$. Also, $B(n) \in E(s,0)$. Therefore $f^s_C(n) = f^s_A(n) = f^s_B(n) = 0$, so $A(n) \in E(s,0) \subseteq E(t,0)$. Thus $C(n) \in E(t,0)$ and $f^t_C(n) = 0 = f^t_B(n)$. If $n \in N$ then $C(n) = B(n)$ so $f^t_C(n) = f^t_B(n)$. Let $\lambda = f^s_C(n)$. Since $\lambda > 0$ there is $\kappa \in \operatorname{spec}^+ r$ such that $E(s,\lambda) \subseteq E(r,\kappa)$. Then $C(n) \in E(r,\kappa)$ and $f^r_C(n) = \kappa = f^r_A(n)$. Thus $f^r_C = f^r_A$ and $f^t_C = f^t_B$. □

3 The Structure of Ω

The space Δ^ω is the union of order isomorphic subsets, which allows us to consider the behavior of just one of those subsets when studying the behavior of Δ^ω. In this section we will see that Ω is also the union of order isomorphic subsets. The lack of a natural fixed basis for H makes it impossible for Ω to be order isomorphic to Δ^ω, but there are natural subsets of Ω which are order isomorphic to Δ^ω.

Definition 3.1 For every orthonormal sequence B let $\Omega_B = \{r \in \Omega : B \in L(r)\}$.

Ω is obviously the union of all Ω_B. The following theorem shows that all these pieces of Ω look alike.

Theorem 3.2 *If A and B are orthonormal sequences then Ω_A is order isomorphic to Ω_B.*

Proof. For every $r \in \Omega_A$ let $\psi_{AB}(r)$ be the linear operator on H defined by setting $\psi_{AB}(r)(\alpha) = f^r_A(n)B(n)$ if $\alpha = B(n)$ and $\psi_{AB}(r)(\alpha) = \mathbf{0}$ if $\alpha \perp B[\omega]$. Consider an arbitrary $r \in \Omega_A$ and to simplify the notation set $t = \psi_{AB}(r)$. We will first show that t is self-adjoint. To this end, extend B to an orthonormal basis $\hat{B}$ of H. Let $\alpha = \sum_{\gamma\in\hat{B}} a_\gamma \gamma$ and $\beta = \sum_{\gamma\in\hat{B}} b_\gamma \gamma$. If $\gamma = B(n)$ then we will use a_n and b_n in place of a_γ and b_γ.

$$
\begin{aligned}
\langle t(\alpha)|\beta\rangle &= \left\langle \sum_{n\in\omega} a_n f_A^r(n)B(n) \middle| \sum_{\gamma\in\hat{B}} b_\gamma \gamma \right\rangle \\
&= \sum_{n\in\omega}\sum_{\gamma\in\hat{B}} a_n f_A^r(n)\bar{b}_\gamma \langle B(n)|\gamma\rangle \\
&= \sum_{n\in\omega} a_n f_A^r(n)\bar{b}_n \\
&= \sum_{\gamma\in\hat{B}}\sum_{n\in\omega} a_\gamma \bar{b}_n f_A^r(n)\langle\gamma|B(n)\rangle \\
&= \left\langle \sum_{\gamma\in\hat{B}} a_\gamma\gamma \middle| \sum_{n\in\omega} b_n f_A^r(n)B(n) \right\rangle \\
&= \langle\alpha|t(\beta)\rangle
\end{aligned}
$$

It follows easily from the definition of ψ_{AB} that $\operatorname{spec}^+ t = \operatorname{spec}^+ r$ and that $\dim E(t,\lambda) = \dim E(r,\lambda)$ for all $\lambda \in \operatorname{spec}^+ t$. Thus $\psi_{AB}(r)$ is a positive operator of trace 1. It is also obvious that B labels t and that $f_B^t = f_A^r$. Therefore $\operatorname{ran}\psi_{AB} \subseteq \Omega_B$.

Now let $s \in \Omega_B$. It follows from our preceding arguments that $r = \psi_{BA}(s) \in \Omega_A$ and that $f_A^r = f_B^s$. Extend A to an orthonormal basis $\hat{A}$ of H. If $\alpha = A(n)$ for some n then $\psi_{AB}(r)(\alpha) = f_A^r(n)B(n) = f_B^s(n)B(n) = s(\alpha)$. If $\alpha \perp B[\omega]$ then $\psi_{AB}(r)(\alpha) = \mathbf{0} = s(\alpha)$. Therefore $\psi_{AB}(r) = s$ and $\operatorname{ran}\psi_{AB} = \Omega_B$. We have also shown that $\psi_{BA} = \psi_{AB}^{-1}$ so ψ_{AB} is one-to-one.

Let $r, s \in \Omega_A$ with $r \sqsubseteq s$. Let $t = \psi_{AB}(r)$ and $u = \psi_{AB}(s)$. Then $f_B^t = f_A^r \leq f_A^s = f_B^u$, so $\psi_{AB}(r) \sqsubseteq \psi_{AB}(s)$. We can apply this result to ψ_{BA}, so ψ_{AB} is an order isomorphism. □

The next theorem connects these pieces of Ω to Δ^ω.

Theorem 3.3 *For every orthonormal sequence A of H, Ω_A is order isomorphic to Λ^ω.*

Proof. For every $r \in \Omega_A$ let $\phi(r) = f_A^r$. Then $\operatorname{ran}\phi = \Lambda^\omega$ by Theorem 2.6. If $r, s \in \Omega_A$ and $r \neq s$ then $f_A^r \neq f_A^s$. Thus ϕ is one-to-one. Also, $r \sqsubseteq s$ if and only if $f_A^r \leq f_A^s$. Therefore ϕ is an order-isomorphism. □

If we consider functions with all the properties of f_A^r except that they need not be decreasing then we will obtain a subset of Ω that is order isomorphic to Δ^ω.

Definition 3.4 For every orthonormal sequence A let Γ_A be the set of all $r \in \Omega$ which satisfy the following conditions.

(i) $A(n)$ is an eigenvector of r for every $n \in \omega$.

(ii) There is a one-to-one function $\rho : \omega \to \omega$ such that $A \circ \rho$ labels r.

Theorem 3.5 *For every orthonormal sequence A, Γ_A is order isomorphic to Δ^ω.*

Proof. For every $r \in \Gamma_A$ let $\phi(r) = f^r_A$. For every $n \in \omega$, $A(n) \in \operatorname{spec} r$ so $f^r_A(n) \in \mathbb{R}$ and $f^r_A(n) \geq 0$. Also, $\sum_{n\in\omega} f^r_A(n) \leq \sum_{\lambda\in\operatorname{spec} r} \lambda = 1$. But there is a one-to-one function $\rho : \omega \to \omega$ such that $A \circ \rho \in L(r)$. Therefore $\sum_{n\in\omega} f^r_A(n) \geq \sum_{n\in\omega} f^r_A(\rho(n)) = 1$. Thus $f^r_A \in \Delta^\omega$. By Theorem 2.6 for every $f \in \Delta^\omega$ there is $r \in \Omega$ such that $A \in L(r)$ and $f^r_A = f$. So $\operatorname{ran}\phi = \Delta^\omega$.

Let $r, s \in \Gamma_A$ such that $\phi(r) = \phi(s)$, or $f^r_A = f^s_A$. Let ρ and σ be one-to-one functions from ω into ω such that $A \circ \rho \in L(r)$ and $A \circ \sigma \in L(s)$. Now $\operatorname{ran}(A \circ \rho) \subseteq \operatorname{ran} A$ and $\operatorname{ran}(A \circ \sigma) \subseteq \operatorname{ran} A$, so $r(\alpha) = \mathbf{0}$ for all $\alpha \in H$ such that $\alpha \perp \operatorname{ran}(A \circ \rho)$ and $s(\alpha) = \mathbf{0}$ for all $\alpha \in H$ such that $\alpha \perp \operatorname{ran}(A \circ \sigma)$. Therefore

$$r(\alpha) = \sum_{n\in\omega} f^r_A(n)\langle\alpha|A(n)\rangle = \sum_{n\in\omega} f^s_A(n)\langle\alpha|A(n)\rangle = s(\alpha)$$

for all $\alpha \in H$ and ϕ is one-to-one.

Now assume that $r \sqsubseteq s$ and let B be an orthonormal sequence that witnesses $r \sqsubseteq s$. In order to show that $\phi(r) \leq \phi(s)$ we must show that $f^r_A \leq f^s_A$. Since A itself no longer necessarily labels r or s we will have to rearrange the terms of f^r_A and f^s_A in such a way that decreasing sequences are produced. We will show that when f^s_A is constant and positive on a finite number of coordinates, then f^r_A is also constant and positive on those same coordinates. This will allow us to first arrange the coordinates on which f^s_A is positive, then arrange the remaining coordinates in order to achieve decreasing sequences.

For every $\lambda \in \operatorname{spec}^+ r$ there is $M_\lambda \subseteq \omega$ such that $B[M_\lambda]$ is a basis for $E(r, \lambda)$. If $n \in M_\lambda$ then $f^r_{A\circ\rho}(n) = f^r_B(n) = \lambda$ so $(A \circ \rho)(n) \in E(r, \lambda)$. Since $A \circ \rho$ is an orthonormal sequence, this means that $(A \circ \rho)[M_\lambda]$ is a basis for $E(r, \lambda)$.

We will next show that if $n \in M_\lambda$ and $f^s_{A\circ\sigma}(n) > 0$ then $(A\circ\sigma)(n) \in (A\circ\rho)[M_\lambda]$. Let $n \in M_\lambda$ and let $f^s_{A\circ\sigma}(n) = \mu > 0$. Because $f^s_{A\circ\sigma}(n) = f^s_B(n)$ we know that $B(n) \in E(s, \mu) \cap E(r, \lambda)$. Therefore $E(s, \mu) \subseteq E(r, \lambda)$ and $(A \circ \sigma)(n) \in E(r, \lambda)$. But if $(A \circ \sigma)(n) \notin (A \circ \rho)[M_\lambda]$ then $(A \circ \sigma)(n) \perp (A \circ \rho)[M_\lambda]$ which is impossible. Thus $(A \circ \sigma)(n) \in (A \circ \rho)[M_\lambda]$. Let $N_\lambda = \{n \in M_\lambda : f_{A\circ\sigma}(n) > 0\}$. Then $(A \circ \sigma)[M_\lambda] \subseteq (A \circ \rho)[M_\lambda]$ or $\sigma[N_\lambda] \subseteq \rho[M_\lambda]$.

Now we can begin arranging the sequences. For every $\lambda \in \operatorname{spec}^+ r$ let $\tau_{\lambda\sigma}$ be the restriction of σ to N_λ and let $\tau_{\lambda\rho}$ be a one-to-one function from $M_\lambda - N_\lambda$ onto $\rho[M_\lambda] - \sigma[N_\lambda]$. Set $\tau_\lambda = \tau_{\lambda\sigma} \cup \tau_{\lambda\rho}$, and $M = \bigcup\{M_\lambda : \lambda \in \operatorname{spec}^+ r\}$. If $n \in \omega - M$ then $f^r_B(n) = 0$ and, since $f^r_B \leq f^s_B$, $f^s_B(n) = 0$. Let τ_0 be a one-to-one function from $\omega - M$ onto $\omega - \bigcup\{\tau_\lambda[M_\lambda] : \lambda \in \operatorname{spec}^+ r\}$. Set $\tau = \left(\bigcup_{\lambda\in\operatorname{spec}^+ r} \tau_\lambda\right) \cup \tau_0$.

If $\lambda, \mu \in \operatorname{spec}^+ r$ with $\lambda \neq \mu$ then $M_\lambda \cap M_\mu = \emptyset$ and $\rho[M_\lambda] \cap \rho[M_\mu] = \emptyset$. Also, $M_\lambda \cap M = \emptyset$. Therefore τ is one-to-one function from ω to ω. We will show that τ witnesses $f^r_A \leq f^s_A$. We can do this by showing that $f^r_A \circ \tau = f^r_A \circ \rho$ and $f^s_A \circ \tau = f^s_A \circ \sigma$. Let $n \in \omega$. If $f_{A\circ\sigma}(n) > 0$ then $f^s_B(n) > 0$ so $f^r_{A\circ\rho}(n) = f^r_B(n) > 0$. Therefore $n \in M_\lambda$ for some $\lambda \in \operatorname{spec}^+ r$ and $n \in N_\lambda$. So $\tau(n) = \sigma(n)$ and $(f^s_A \circ \tau)(n) = (f^s_A \circ \sigma)(n)$.

Assume that that $f^s_{A\circ\sigma}(n) = 0$. We will show that $f^s_A(\tau(n)) = 0$. First consider the case when $f^r_A(\rho(n)) > 0$. There is $\lambda \in \operatorname{spec}^+ r$ such that $n \in M_\lambda$. Since $f^s_A(\sigma(n)) = 0$ we know that $n \in M_\lambda - N_\lambda$. Therefore $\tau(n) = \tau_{\lambda\rho} \in \rho[M_\lambda] - \sigma[N_\lambda]$. If $f^s_A(\tau(n)) > 0$ then there is $k \in \omega$ such that $\sigma(k) = \tau(n)$. Since $f^s_A(\tau(n)) > 0$ there is $\mu \in \operatorname{spec}^+ s$ such that $A(\sigma(k)) \in E(s, \mu)$. Let $\kappa \in \operatorname{spec}^+ r$ such that

$E(s,\mu) \subseteq E(r,\kappa)$. But $A(\sigma(k)) = A(\tau(n)) \in (A \circ \rho)[M_\lambda] \subseteq E(r,\lambda)$ so $\kappa = \lambda$. Therefore $k \in N_\lambda$ and $\tau(n) = \sigma(k) \in \sigma[N_\lambda]$, a contradiction. It follows that $f^s_A(\tau(n)) = 0$.

Now assume that $f^r_A(\rho(n)) = 0$. Then $n \in \omega - M$ and $\tau(n) = \tau_0(n) \in \omega - \bigcup\{\tau_\lambda[M_\lambda] : \lambda \in \operatorname{spec}^+ r\}$. If $f^s_A(\tau(n)) > 0$ then there is $k \in \omega$ such that $\sigma(k) = \tau(n)$. Since $f^w_A(\sigma(k)) > 0$ there is $\mu \in \operatorname{spec}^+ s$ such that $A(\sigma(k)) \in E(s,\mu)$. Let $\lambda \in \operatorname{spec}^+ r$ such that $E(s,\mu) \subseteq E(r,\lambda)$. Then $A(\sigma(k)) \in E(r,\lambda)$ and $A(\sigma(k)) \in (A \circ \rho)[M_\lambda]$. Thus $\tau(n) = \sigma(k) \in \rho[M_\lambda] = \tau_\lambda[M_\lambda]$ a contradiction. It follows that $f^s_A(\tau(n)) = 0$.

We will next show that $f^r_A \circ \tau = f^r_A \circ \rho$. Let $n \in \omega$. If $f^r_A(\rho(n)) > 0$ then there is $\lambda \in \operatorname{spec}^+ r$ such that $n \in M_\lambda$. So $\tau(n) = \tau_\lambda(n) \in \rho[M_\lambda]$ and $f^r_A(\tau(n))) = \lambda = f^r_A(\rho(n))$. If $f^r_A(\rho(n)) = 0$ then $n \in \omega - M$ so $\tau(n) = \tau_0(n) \in \omega - \bigcup\{\tau_\lambda[M_\lambda] : \lambda \in \operatorname{spec}^+ r\} = \omega - \bigcup\{\rho[M_\lambda] : \lambda \in \operatorname{spec}^+ r\}$. Therefore $f^r_A(\tau(n)) = 0$.

Now we have $f^r_A \circ \tau = f^r_{A\circ\rho} = f^r_B$ and $f^s_A \circ \tau = f^s_{A\circ\sigma} = f^s_B$. As a consequence, τ witnesses $f^r_A \leq f^s_A$ and $\phi(r) \leq \phi(s)$.

Finally, we must show that if $r, s \in \Gamma_A$ and $\phi(r) \leq \phi(s)$ then $r \sqsubseteq s$. If $\phi(r) \leq \phi(s)$ then $f^r_A \leq f^s_A$. Let $\sigma : \omega \to \omega$ be a one-to-one function which witnesses this relation. In particular, $f^r_A \circ \sigma, f^s_A \circ \sigma \in \Lambda^\omega$ and if $n \in \omega$ such that either $f^r_A(n) > 0$ or $f^s_A(n) > 0$ then there is $m \in \omega$ such that $\sigma(m) = n$. We will show that $A \circ \sigma$ witnesses $r \sqsubseteq s$. First show that $A \circ \sigma$ labels r. We already know that the range of A consists of eigenvectors of r, so the range of $A \circ \sigma$ will also be a set of eigenvectors of r. Let $\lambda \in \operatorname{spec}^+ r$. There is a one-to-one function $\rho : \omega \to \omega$ such that $A \circ \rho \in L(r)$. So if $M_\lambda = \{n \in \omega : A(\rho(n)) \in E(r,\lambda)\}$ then $(A \circ \rho)[M_\lambda]$ is a basis for $E(r,\lambda)$. But $f^r_A(\rho(n)) = \lambda$ for all $n \in M_\lambda$, so $\rho[M_\lambda] \subseteq \sigma[\omega]$. Therefore, if we let $N_\lambda = \sigma^{-1}[\rho[M_\lambda]]$ then $(A \circ \sigma)[N_\lambda]$ is a basis for $E(r,\lambda)$. Thus $A \circ \sigma$ labels r. The same argument shows that $A \circ \sigma$ labels s, so σ witnesses $r \sqsubseteq s$. $\square$

4 Summary

We have seen that the infinite dimensional space Ω of quantum states has a spectral order corresponding to the spectral order of finite dimensional spaces Ω^n of quantum states. The spectral order on Ω is created from the Bayesian order on Δ^ω in same way that the spectral order on Ω^n is created from the Bayesian order on Δ^n. As a result, Ω contains copies of Δ^ω and can be written as the union of subsets each of which is isomorphic to Λ^ω. The spectral order provides Ω with the property of degeneracy, and it is anticipated that the other properties given to Δ^ω by the Bayesian order will also hold in Ω. It is also anticipated that the weakly way below relation will utterly fail in Ω as it does in Δ^ω. But details of these conjectures have yet to be checked. Here are some questions which need answers. What physical significance is there, if any, to the fact that the weak domain structure of Δ^n and Ω^n completely breaks down in Δ^ω and Ω? Is it possible to define an order model for infinite dimensional quantum states which has the basic physical properties needed for such a model but also provides a sense of partiality or approximation found in domain-like structures?

References

[1] B. Coecke and K. Martin, *A partial order on classical and quantum states*, Oxford University Computing Laboratory, Research Report PRG-RR-02-07, August 2002, URL: http://web.comlab.ox.ac.uk/oucl/publications/tr/rr-02-07.html.

[2] B. Coecke and K. Martin, *Partiality in Physics*, Invited paper in "Quantum theory: reconsideration of the foundations 2," Vaxjo, Sweden, Vaxjo University Press.

[3] J. Mashburn, *An Order Model for Infinite Classical States*, U. of Dayton, Dept. of Math., preprint