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On the Tightness and Long Directed Limits of Free Topological Algebras

Gábor Lukács
lukacs@topgroups.ca

Rafael Dahmen
Technische Universitat Darmstadt, dahmen@mathematik.tu-darmstadt.de

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- $\mathscr{A} \subseteq \mathscr{T}$, but the inclusion may be strict.
  - For $\lambda = \omega$ and $\mathbf{A} = \mathbf{Grp}(\mathbf{Top})$, one has $\mathscr{A} \neq \mathscr{T}$ in most cases (Yamasaki's Theorem, 1998).

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If the tightness of $(A, \mathscr{A}) = \operatorname*{colim}^{\mathbf{A}}_{\alpha<\lambda} A_\alpha$ is smaller than the cofinality of λ, then $\mathscr{A} = \mathscr{T}$.

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- Reason: If $E \subseteq A$ is such that $|E| \leq \tau$ and $\tau < \operatorname{cf}(\lambda)$, then $E \subseteq A_\alpha$ for some $\alpha < \lambda$.

Tightness of long limits

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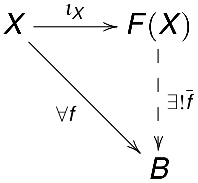
Corollary 2

For $\lambda = \omega_1$, if $(A, \mathscr{A})$ is countably tight, then $\mathscr{A} = \mathscr{T}$.

Free topological algebras

- For $X \in \mathbf{Top}$, the *free topological algebra on* X is $F(X) \in \mathbf{A}$ together with a cts $\iota_X \colon X \to F(X)$ such that for every $B \in \mathbf{A}$:



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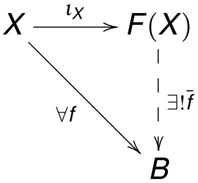

- If such an $F(X)$ exists for every $X \in$ Top, then one has a free algebra functor $F \colon \mathbf{Top} \to \mathbf{A}$ that is left adjoint to the forgetful functor $U \colon \mathbf{A} \to \mathbf{Top}$.

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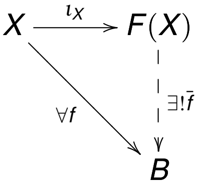

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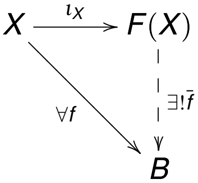

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  - If $X = \operatorname{colim}^{\mathbf{Top}}_{\alpha<\lambda} X_\alpha$, then $F(X) = \operatorname{colim}^{\mathbf{A}}_{\alpha<\lambda} F(X_\alpha)$.
  - If $t(F(X)) < \operatorname{cf}(\lambda)$, then $F(X) = \operatorname{colim}^{\mathbf{Top}}_{\alpha<\lambda} F(X_\alpha)$.

Examples

Take $X_\alpha = \alpha$ for $\alpha < \omega_1$.

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Main results (Dahmen & GL, 2017)

Theorem 7

For a Tychonoff space X, the natural cts homomorphism $A([-1,1] \times X) \longrightarrow V(X)$ induced by $(t,x) \mapsto te_x$ is a quotient in $\mathrm{Ab}(\mathrm{Top})$, and thus an open surjection.

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$V(\omega_1) = \mathrm{colim}^{\mathbf{TVS}}_{\alpha<\omega_1} V(\alpha) = \mathrm{colim}^{\mathbf{Top}}_{\alpha<\omega_1} V(\alpha).$

Counterexample to “Fact” on ArXiv

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If X is an uncountable Tychonoff space, then $V(X)$ has uncountable tightness and is not a k-space.

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- Hence, $V(\omega_1)$ is a k-space.