2008

Rearrangement on Conditionally Convergent Integrals in Analogy to Series

Edward J. Timko

Follow this and additional works at: http://ecommons.udayton.edu/mth_epumd

Part of the Mathematics Commons

eCommons Citation
http://ecommons.udayton.edu/mth_epumd/23

This Article is brought to you for free and open access by the Department of Mathematics at eCommons. It has been accepted for inclusion in Undergraduate Mathematics Day, Electronic Proceedings by an authorized administrator of eCommons. For more information, please contact frice1@udayton.edu, mschlangen1@udayton.edu.
Rearrangement on Conditionally Convergent Integrals in Analogy to Series

Edward J Timko
University of Dayton
Dayton OH 45469-2316
Email: timkoedz@notes.udayton.edu

Abstract

Rearrangements on conditionally convergent series suggests the existence of a similar process for integrals, here also referred to as rearrangement. In this document, a general theorem concerning rearrangement for conditionally convergent integrals is presented, as well as supporting theorems and a corollary to the general theorem. The corollary reads:

Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous function with an everywhere negative and monotone increasing derivative. If \( \int_{1}^{\infty} (-1)^{x} f(x) \, dx \) is conditionally convergent, then \( \forall z \in \mathbb{C} \), there exists an arrangement on \( \int_{1}^{\infty} (-1)^{x} f(x) \, dx \) such that \( z = \int_{1}^{\infty} (-1)^{x} f(x) \, dx \).
1 Preliminary Theorems and Lemmas

Note: Recall that $e^{iθ} = \cos(θ) + i\sin(θ)$ and $(-1)^n = e^{iπn}$.

Definition 1. For any function $f(x)$ integrable on $[1, \infty)$ such that
\[
\int_1^∞ f(x)dx = \int_0^1 \sum_{j=1}^∞ f(j + u) du
\]
a rearrangement on $\int_1^∞ f(x)dx$ is a rearrangement of the terms in the series
\[
\sum_{j=1}^∞ f(j + u)
\]
for each $u$ on $[0, 1]$.

Theorem 1. Let $\{a_j\}_{j=1}^∞$ and $\{b_j\}_{j=1}^∞$ be positive, real, decreasing sequences which converge to zero, where $a_j \leq b_j$ for all $j$. If
\[
a_j - a_{j+1} \leq b_j - b_{j+1} \quad j \in \mathbb{N}
\]
then
\[
\left| \sum_{j=k}^∞ (-1)^j a_j \right| \leq \left| \sum_{j=k}^∞ (-1)^j b_j \right|.
\]

Proof. From the hypothesis, both sequences are decreasing, and therefore $a_j - a_{j+1} > 0$ and $b_j - b_{j+1} > 0$. For $∀n \in \mathbb{N}$
\[
\sum_{j=0}^{2n-1} (-1)^j a_{j+1} = a_{l+1} + a_{l+2} - a_{l+3} + \ldots + a_{l+2n-2} - a_{l+2n-1}
\]
\[
\leq b_{l+1} + b_{l+2} - b_{l+3} + \ldots + b_{l+2n-2} - b_{l+2n-1}
\]
\[
= \sum_{j=0}^{2n-1} (-1)^j b_{j+1},
\]
with both sums being positive. Since $a_{2n} \leq b_{2n}$, it follows that
\[
0 < \sum_{j=0}^{2n} (-1)^j a_{j+l} \leq \sum_{j=0}^{2n} (-1)^j b_{j+l}
\]
and so $∀n \in \mathbb{N}$,
\[
0 < \sum_{j=0}^{n} (-1)^j a_{j+l} \leq \sum_{j=0}^{n} (-1)^j b_{j+l}
\]
which implies
\[
\left| \sum_{j=l}^{n+l} (-1)^j a_j \right| \leq \left| \sum_{j=l}^{n+l} (-1)^j b_j \right|.
\]
Since $a_j, b_j \to 0$ as $j \to \infty$, it follows that both series are convergent. Knowing this, let $l = k + 1$ and $n \to \infty$, to yield that

$$\left| \sum_{j=k+1}^{\infty} (-1)^j a_j \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j b_j \right|.$$ 

\[\square\]

**Theorem 2.** Let $f$ be a positive, decreasing function on $\mathbb{R}^+$, integrable such that $\int_1^{\infty} (-1)^x f(x)dx$ is conditionally convergent. Then $\sum_{j=1}^{\infty} (-1)^j f(j + u)$ is conditionally convergent $\forall u \in [0, 1]$.

**Proof.** Note that, for all $u \in [0, 1]$ and for all $x \in \mathbb{R}^+$,

$$[x] + u < x + 2.$$

This implies

$$f(x + 2) < f([x] + u).$$

Let $N \in \mathbb{N}$ such that $N > 2$. Since

$$\int_0^{N} f([x])dx = \sum_{j=1}^{N} f(j)$$

it follows that

$$\int_2^{N} f(x)dx < \sum_{j=1}^{N} f(j + u).$$

Taking the limit that $N \to \infty$ yields that

$$\int_2^{\infty} f(x)dx < \sum_{j=1}^{\infty} f(j + u).$$

Since $f$ is integrable on $\mathbb{R}^+$, it follows that $\int_1^{\infty} f(x)dx$ is finite. Therefore, if the integral $\int_1^{\infty} f(x)dx \to \infty$ then $\int_2^{\infty} f(x)dx \to \infty$, and therefore $\sum_{j=1}^{\infty} f(j + u) \to \infty$.

Suppose that $\int_1^{\infty} (-1)^x f(x)dx \to L$. It then follows $\forall \epsilon > 0, \exists M > 0$ such that

$$\left| \int_1^{y} (-1)^x f(x)dx - L \right| < \frac{\epsilon}{\pi \sqrt{2}} \quad y \geq M.$$

Thus, for $M \leq a \leq b$ it follows by the triangle inequality that

$$\left| \int_a^{b} (-1)^x f(x)dx \right| \leq \left| \int_1^{a} (-1)^x f(x)dx - L \right| + \left| \int_1^{b} (-1)^x f(x)dx - L \right| < \frac{\epsilon \sqrt{2}}{\pi}.$$
Since \( f(x) \) is decreasing, positive, and finite \( \forall x \in \mathbb{R}^+ \), it follows that \( f \) must converge to some value, say \( c \), as \( x \to \infty \). Since \( c \leq f(x) \), it follows for \( x \in [2n, 2n + 1/2] \) with \( n \in \mathbb{N} \) that

\[
-c \cos(\pi x)f(x) \leq c\cos(\pi x) \leq \cos(\pi x)f(x)
\]

from which follows that

\[
\left| \int_{2n}^{2n+1/2} c\cos(\pi x) \, dx \right| \leq \int_{2n}^{2n+1/2} \cos(\pi x)f(x) \, dx.
\]

Knowing that

\[
\int_{2n}^{2n+1/2} \cos(\pi x) \, dx = \frac{1}{\pi}
\]

it follows that

\[
\frac{c}{\pi} \leq \left| \int_{2n}^{2n+1/2} \cos(\pi x)f(x) \, dx \right|.
\]

A similar argument will show that the same is true for the sine function. From this, it follows that

\[
\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x) \, dx \right|.
\]

Let \( n \geq M \). Then it follows that

\[
\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x) \, dx \right| \leq \frac{\sqrt{2}}{\pi} \quad n \geq M
\]

and therefore \( c < \epsilon \). Therefore \( c = 0 \). Thus, \( f(x) \to 0 \) as \( x \to \infty \), and therefore \( f(u + j) \to 0 \), \( \forall u \in [0, 1] \) as \( j \to \infty \). From this, one can conclude that the series \( \sum_{j=1}^{\infty} (-1)^j f(j + u) \) is convergent. Thus, if \( \int_{1}^{\infty} (-1)^x f(x) \, dx \) is conditionally convergent, then \( \sum_{j=1}^{\infty} (-1)^j f(j + u) \) is conditionally convergent \( \forall u \in [0, 1] \).

\[\square\]

**Lemma 1.** Let \( f \) be a positive, decreasing, integrable function on every finite interval of \( \mathbb{R}^+ \). Then

\[
\int_{0}^{1} f(u + j) \cos(\pi u) \, du \geq 0
\]

for \( \forall j \in \mathbb{N} \).

**Proof.** Note that

\[
\int_{0}^{1} f(u + j) \cos(\pi u) \, du = \int_{0}^{1/2} f(u + j) \cos(\pi u) \, du + \int_{1/2}^{1} f(u + j) \cos(\pi u) \, du \geq \int_{0}^{1/2} f(1/2 + j) \cos(\pi u) \, du + \int_{1/2}^{1} f(1/2 + j) \cos(\pi u) \, du = f(1/2 + j) \int_{0}^{1} \cos(\pi u) \, du = 0.
\]
Lemma 2. Let \( f(x) \) be positive, decreasing, and integrable on every finite subinterval of \( \mathbb{R}^+ \) such that

- \( \forall u \in [0, 1] \) and \( \forall j \in \mathbb{N} \), \( f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1) \);
- \( \forall u \in [0, 1] \) the series \( \sum_{j=0}^{\infty} (-1)^j f(j + u) \) is convergent.

Then \( (-1)^n \sum_{j=k}^{\infty} (-1)^j f(j + u) \) is integrable on \([0, 1]\) for each positive \( k \), and

\[
\begin{align*}
\sum_{j=1}^{\infty} \int_{0}^{1} (-1)^{u+j} f(u + j) \, du &= \int_{0}^{1} \sum_{j=1}^{\infty} (-1)^{u+j} f(u + j) \, du.
\end{align*}
\]

Proof. First to show integrability. Since \( \sum_{j=k}^{\infty} (-1)^j f(j + u) \) is convergent it follows that \( f(j + u) \to 0 \) as \( j \to \infty \). Thus, from the proof of Theorem 1 \( \forall n, k \in \mathbb{N} \) with \( n \geq k \),

\[
\left| \sum_{j=k}^{n} (-1)^j f(u + j) \right| \leq \left| \sum_{j=k}^{n} (-1)^j f(j) \right|.
\]

It follows from the hypothesis that \( f \) is integrable on \([j, j + 1]\), and therefore \( f(u + j) \) is integrable for \( u \in [0, 1] \). Fix \( k \). Define

\[
h_n(u) = \sum_{j=k}^{n} (-1)^j f(u + j).
\]

Since the finite sum of integrable functions is integrable, \( h_n(u) \) is integrable for each \( n \). Therefore \( \forall u \in [0, 1] \), there exists a finite \( h(u) \in \mathbb{R} \) such that \( \lim_{n \to \infty} h_n(u) = h(u) \). Also \( |h_n(u)| \leq |h_n(0)| \). Choose

\[
M = \sup\{ |h_k(0)|, |h_{k+1}(0)|, \ldots \}.
\]

Since \( h_n(0) \) is convergent and always finite, such a number exists. Therefore, \( |h_n(u)| \leq M \) for \( \forall u \in [0, 1] \) and \( \forall n \geq k \). Therefore, by Lebesgue Dominated Convergence Theorem, \( h(u) \) is integrable on \([0, 1]\).

Note that \( (-1)^n h(u) = \cos(\pi u) h(u) + i \sin(\pi u) h(u) \), and that \( \cos(\pi u) \) and \( \sin(\pi u) \) are integrable \([0, 1]\). Since the product of two integrable functions is integrable, it follows that \( \cos(\pi u) h(u) \) and \( \sin(\pi u) h(u) \) are integrable \([0, 1]\), and therefore \( (-1)^n h(u) \) is integrable on \([0, 1]\).

Now, in order to conserve space, define \( g(x) = (-1)^x f(x) \). Thus, for any finite \( k \)

\[
\sum_{j=1}^{\infty} \int_{0}^{1} g(u + j) \, du = \sum_{j=1}^{k} \int_{0}^{1} g(u + j) \, du + \sum_{j=k+1}^{\infty} \int_{0}^{1} g(u + j) \, du.
\]
and

\[ \int_0^1 \sum_{j=1}^\infty g(u + j)du = \int_0^1 \sum_{j=1}^k g(u + j)du + \int_0^1 \sum_{j=k+1}^\infty g(u + j)du \]

\[ = \sum_{j=1}^k \int_0^1 g(u + j)du + \int_0^1 \sum_{j=k+1}^\infty g(u + j)du. \]

Therefore

\[ \left| \sum_{j=k+1}^\infty \int_0^1 g(u + j)du - \int_0^1 \sum_{j=k+1}^\infty g(u + j)du \right| = \left| \sum_{j=1}^\infty \int_0^1 g(u + j)du - \int_0^1 \sum_{j=1}^\infty g(u + j)du \right| \]

for any finite positive \( k \).

Let \( u \in [0, 1] \). Since \( f \) is decreasing and positive, it follows that \( 0 < f(u + j) - f(u + j + 1) \), and thus,

\[ 0 < f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1). \]

Therefore, by Theorem 1

\[ \left| \sum_{j=k+1}^\infty (-1)^j f(u + j) \right| \leq \left| \sum_{j=k+1}^\infty g(j) \right|. \]

Thus,

\[ - \sum_{j=k+1}^\infty g(j) \leq \sum_{j=k+1}^\infty (-1)^j f(j + u) \leq \sum_{j=k+1}^\infty g(j) \]

from which follows

\[ \left| \int_0^1 \cos(\pi u) \sum_{j=k+1}^\infty (-1)^j f(u + j)du \right| \leq \int_0^1 \cos(\pi u) \sum_{j=k+1}^\infty g(j)du \]

with the integrability of the left-hand side being given by Lemma 2. Evaluating the right-hand integral,

\[ \left| \int_0^1 \cos(\pi u) \sum_{j=k+1}^\infty (-1)^j f(u + j)du \right| \leq \frac{1}{\pi} \left| \sum_{j=k+1}^\infty g(j)du \right|. \]

The same follows identically for the sine function in place of the cosine function. Therefore, by the Triangle Inequality,

\[ \left| \int_0^1 \sum_{j=k+1}^\infty g(u + j)du \right| \leq \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^\infty g(j)du \right|. \]
Now, since \(|\cos \pi u| \leq 1\), it follows that
\[-f(j) \leq -f(u + j) \leq f(u + j) \cos(\pi u) \leq f(u + j) \leq f(j).\]
Integrating \(u\) over \([0, 1]\) yields
\[-f(j) \leq \int_0^1 f(u + j) \cos(\pi u) du \leq f(j).\]
Similarly, \(|\int_0^1 f(u + j) \sin(\pi u) du| \leq f(j)\). By a similar argument,
\[-[f(j) - f(j + 1)] \leq [f(u + j) - f(u + j + 1)] \cos(\pi u) \leq f(j) - f(j + 1)\]
and therefore
\[-[f(j) - f(j + 1)] \leq \int_0^1 [f(u + j) - f(u + j + 1)] \cos(\pi u) du \leq f(j) - f(j + 1)\]
with similar following for the sine function. Note, for \(u \in [0, 1], 0 \leq \sin(\pi u) \leq 1\),
and therefore \(0 \leq [f(u + j) - f(u + j + 1)] \sin(\pi u)\). Thus,
\[0 \leq \int_0^1 [f(u + j) - f(u + j + 1)] \sin(\pi u) du.\]
By Lemma 1, for \(\forall j \in \mathbb{N}\),
\[0 \leq \int_0^1 f(u + j) \cos(\pi u) du.\]
For every \(u \in [0, 1]\) and \(y \in \omega\)
\[f(u + j) - f(u + j + 1) \cos \pi u \leq f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1)\]
Therefore
\[\int_0^1 [f(u + j) - f(u + j + 1)] \cos \pi u du \leq f(j) - f(j + 1)\]
or
\[\int_0^1 f(u + j) \cos \pi u du - \int_0^1 f(u + j + 1) \cos \pi u du \leq f(j) - f(j + 1)\]
Letting \(a_j = \int_0^1 f(u + j) \cos \pi u du\) and \(b_j = f(j)\), it follows from Theorem 1
that
\[\left| \sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u + j) \cos \pi u du \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j f(j) \right|\]
for all \(k\). Similarly,
\[\left| \sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u + j) \sin \pi u du \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j f(j) \right|\]
By the Triangle Inequality
\[
\left| \sum_{j=k+1}^{\infty} \int_{0}^{1} g(u + j)du \right| \leq \sqrt{2} \left| \sum_{j=k+1}^{\infty} g(j) \right|
\]
and, again by the Triangle Inequality,
\[
\left| \sum_{j=k+1}^{\infty} \int_{0}^{1} g(u + j)du - \int_{0}^{1} \sum_{j=k+1}^{\infty} g(u + j)du \right| \leq \sqrt{2} \left( 1 + \frac{1}{\pi} \right) \left| \sum_{j=k+1}^{\infty} g(j) \right|.
\]
Since \( \sum_{j=1}^{\infty} g(j) \) is convergent, it follows that it is Cauchy, and therefore \( \forall \epsilon > 0, \exists N \in \mathbb{N} \) such that
\[
\left| \sum_{j=k+1}^{\infty} g(j) \right| < \frac{\epsilon}{\sqrt{2} \left( 1 + \frac{1}{\pi} \right)} \quad k \geq N.
\]
Thus
\[
\left| \sum_{j=k+1}^{\infty} \int_{0}^{1} g(u + j)du - \int_{0}^{1} \sum_{j=k+1}^{\infty} g(u + j)du \right| < \epsilon \quad k \geq N
\]
which implies
\[
\left| \sum_{j=1}^{\infty} \int_{0}^{1} g(u + j)du - \int_{0}^{1} \sum_{j=1}^{\infty} g(u + j)du \right| < \epsilon
\]
and therefore
\[
\sum_{j=1}^{\infty} \int_{0}^{1} g(u + j)du = \int_{0}^{1} \sum_{j=1}^{\infty} g(u + j)du.
\]

\[\square\]

2 Main Theorem

**Theorem 3** (Main Theorem). Let \( f \) be a positive, decreasing function integrable on any finite subinterval of \( \mathbb{R}^+ \). If
- \( \forall u \in [0, 1] \) and \( \forall j \in \mathbb{N} \) it holds that \( f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1) \)
- \( \int_{1}^{\infty} (-1)^{x} f(x)dx \) is conditionally convergent

then \( \forall z \in \mathbb{C} \) there exists a rearrangement on \( \int_{1}^{\infty} (-1)^{x} f(x)dx \) such that \( z = \int_{1}^{\infty} (-1)^{x} f(x)dx \).
Proof. Note
\[
\int_{1}^{\infty} (-1)^{x} f(x) dx = \sum_{j=1}^{\infty} \int_{j}^{j+1} (-1)^{x} f(x) dx.
\]
Making the substitution that \( u = j + x \), it follows that
\[
\int_{1}^{\infty} (-1)^{x} f(x) dx = \sum_{j=1}^{\infty} \int_{0}^{1} (-1)^{u+j} f(u+j) du.
\]
By Theorem 2, \( \forall u \in [0,1] \) it holds that \( \sum_{j=1}^{\infty} (-1)^{j} f(j+u) \) is conditionally convergent. Therefore, by Lemma 2
\[
\int_{1}^{\infty} (-1)^{x} f(x) dx = \sum_{j=1}^{\infty} (-1)^{u+j} f(u+j) du.
\]
Since \( \int_{1}^{\infty} f(u+j)(-1)^{j} \) is conditionally convergent \( \forall u \in [0,1] \) it follows that \( \forall h(u) \in \mathbb{R} \), there exists a rearrangement of the terms in the series such that \( h(u) = \sum_{j=1}^{\infty} f(u+j)(-1)^{j} \). This constitutes a rearrangement on \( \int_{1}^{\infty} (-1)^{x} f(x) dx \).

Choose \( z \in \mathbb{C} \) such that \( z = |z|e^{i\theta} \).

Now choose a rearrangement of the terms in \( \sum_{j=1}^{\infty} f(u+j)(-1)^{j} \) for \( \forall u \in [0,1] \) such that
\[
\sum_{j=1}^{\infty} f(u+j)(-1)^{j} = 2|z| \cos(\pi u - \theta).
\]
Then
\[
\int_{1}^{\infty} (-1)^{x} f(x) dx = z.
\]

Corollary 1. Let \( f \) be a positive function, integrable on any finite subinterval of \( \mathbb{R}^{+} \), with an everywhere negative and increasing derivative. If \( \int_{1}^{\infty} (-1)^{x} f(x) dx \) is conditionally convergent, then \( \forall z \in \mathbb{C} \), there exists a rearrangement on \( \int_{1}^{\infty} f(x) dx \) such that \( z = \int_{1}^{\infty} (-1)^{x} f(x) dx \).

Proof. It suffices to show that \( f \) satisfies the requirements of the Main Theorem. The requirement of conditional convergence is obviously met. It is also clear that \( f \) is decreasing.

Since \( f \) is continuous, it follows by the Mean Value Theorem that there exists \( c_{j} \in (j, j+u) \) such that
\[
f(u+j) - f(j) = f'(c_{j})u.
\]
Since \( (j, j+u) \cap (j+1, j+u+1) = \emptyset \), it follows that \( c_{j} < c_{j+1} \), and \( f'(c_{j}) < f'(c_{j+1}) \). Thus,
\[
f(u+j) - f(j) < f(u+j+1) - f(j+1)
\]
and
\[ f(u + j) - f(u + j + 1) < f(j) - f(j + 1). \]
Thus, \( f \) satisfies the requirements of the Main Theorem. 
\[
\square
\]