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On Roitman's Principle for Box Products

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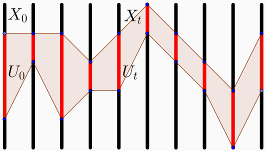
Box products

The **box product** of a family $\{X_i : i \in I\}$ is... $\square_{i \in I} X_i$



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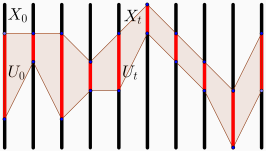

Problem: Is $\square(\omega+1)^\omega$ paracompact (normal)?

Here, $\omega+1 = \omega \cup \{\omega\} \simeq \{0\} \cup \{\frac{1}{n} : n \geq 1\}$.

Paracompactness $\equiv$ Every open cover has a locally finite open refinement.

Normality $\equiv$ Every two disjoint closed subsetes have a separation by disjoint open sets.

Relating $\square$ and ∇

On $\square_{i \in I} X_i$, define the relacion $x \sim y$ iff $\{i \in I : x(i) \neq y(i)\}$ is finite.

The ∇-**product** (Nabla product) of a family $\{X_i : i \in I\}$ is

$$\nabla_{i \in I} X_i = \square_{i \in I} X_i / \sim$$

Relating $\Box$ and ∇

On $\Box_{i\in I} X_i$, define the relacion $x \sim y$ iff $\{i \in I : x(i) \neq y(i)\}$ is finite.

The ∇-**product** (Nabla product) of a family $\{X_i : i \in I\}$ is

$$\nabla_{i\in I} X_i = \Box_{i\in I} X_i / \sim$$

Theorem (K. Kunen, 1978)
If X_n is compact, $n \in \omega$, then

$$\Box_{n\in\omega} X_n \text{ is paracompact iff } \nabla_{n\in\omega} X_n \text{ is paracompact.}$$

Problem: Is $\nabla(\omega+1)^\omega$ paracompact?

Negative results

Theorem (E. van Douwen, 1975)
$\omega^\omega \times \square(\omega+1)^\omega$ *is not normal.*

Thus, $\square(\omega+1)^\omega$ is not hereditary normal.

Theorem (B. Lawrence, 1996)
$\square(\omega+1)^{\omega_1}$ *is not normal.*

Positive results

$\Box(\omega+1)^\omega$ is paracompact in this models of ZFC:

(all of them via ∇, except M.E. Rudin)

- (M.E. Rudin, 1972) **CH**
- (Kunen, 1978) **MA**
- (Roitman, 1979) $\mathfrak{d}=\mathfrak{c}$: Dominating families of small size
- (van Douwen, 1980) $\mathfrak{b}=\mathfrak{d}$: Under a scale
- (Williams, 1984) $\mathfrak{d}=\omega_1$: With uniformities
- (Roitman, 2011) **MH**

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Definition

MH (The Model Hypothesis) $\equiv$ For some κ, $H(\omega_1) = \bigcup_{\alpha<\kappa} H_\alpha$, where each H_α is an elementary submodel of $(H(\omega_1), \in)$ and each $H_\alpha \cap \omega^\omega$ is not dominant.

An astonish principle of Roitman

Let $\omega^{\subset\omega} = \{x : A \to \omega : A \subseteq \omega$ is infinite-coinfinite$\}$.

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega \; \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega \;\wedge\; \neg\exists^\infty n \;\; x(n) \neq y(n)$
$\implies \; x \setminus y \not>^* F(y) \;\;\vee\;\; y \setminus x \not>^* F(x)$

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Roitman's principle $\Delta \equiv$
There is $F : \omega^{\subset\omega} \to \omega^\omega$ such that for $x, y \in \omega^{\subset\omega}$ with

1. $\{n \in dom(x) \cap dom(y) : x(n) \neq y(n)\}$ is finite,
2. $|x \setminus y| = |y \setminus x| = \omega$,

then, $(x \setminus y) \not>^* F(y)$ or $(y \setminus x) \not>^* F(x)$.

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Theorem (Roitman, 2011)
If Δ holds, then $\nabla(\omega+1)^\omega$ is paracompact.

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega \ \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega \ \wedge \ \neg\exists^\infty n \ \ x(n) \neq y(n)$
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$$\left.\begin{array}{l}\mathfrak{d} = \mathfrak{c} \\ \mathfrak{b} = \mathfrak{d} \\ \textbf{MH} \\ \Delta\end{array}\right\} \implies \nabla(\omega+1)^\omega \text{ is paracompact}$$

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Question: Suppose $\nabla(\omega+1)^\omega$ is paracompact. Does Δ hold?

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega\ \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega\ \wedge\ \neg\exists^\infty n\ \ x(n) \neq y(n)$
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An astonish principle of Roitman

Theorem (Roitman, 2011)

If $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} = \mathfrak{c}$ or MH holds, then Δ holds.

Poof:

- For $\mathfrak{b} = \mathfrak{d}$: Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ a scale in ω^ω. Let $X_\alpha = \{x \in \omega^{\subset\omega} : \alpha \text{ is least that } x \leq^* f_\alpha\}$. Then, $F(x) = f_\alpha$ iff $x \in X_\alpha$.

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- For $\mathfrak{d} = \mathfrak{c}$: Enumerate $\omega^{\subset\omega} = \{x_\alpha : \alpha < \mathfrak{d}\}$. By recursion, suppose Δ holds for $\{x_\alpha : \alpha < \beta\}$ with witness $F(x_\alpha) = f_\alpha$. Let $A = \{a_\alpha = dom(x_\alpha) \setminus dom(x_\beta) : \alpha < \beta\}$. Then $|A| < \mathfrak{d}$ and there is f_β not bounded by any $x_\alpha \restriction a_\alpha, \alpha < \beta$. Let $F(x_\beta) = f_\beta$.

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega \ \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega \ \wedge \ \neg\exists^\infty n \ \ x(n) \neq y(n)$
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- For $\mathfrak{d} = \mathfrak{c}$: Enumerate $\omega^{\subset\omega} = \{x_\alpha : \alpha < \mathfrak{d}\}$. By recursion, suppose Δ holds for $\{x_\alpha : \alpha < \beta\}$ with witness $F(x_\alpha) = f_\alpha$. Let $A = \{a_\alpha = dom(x_\alpha) \setminus dom(x_\beta) : \alpha < \beta\}$. Then $|A| < \mathfrak{d}$ and there is f_β not bounded by any $x_\alpha \restriction a_\alpha, \alpha < \beta$. Let $F(x_\beta) = f_\beta$.
- For MH (Remember $H(\omega_1) = \bigcup_{\alpha<\kappa} H_\alpha \wedge H_\alpha \cap \omega^\omega$ not dominant): Let f_α witness that $H_\alpha \cap \omega^\omega$ is not dominant. If α is least that $x \in \omega^{\subset\omega} \cap H_\alpha$, let $F(x) = f_\alpha$.

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega \; \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega \;\wedge\; \neg\exists^\infty n \;\; x(n) \neq y(n)$
$\implies x \setminus y \not\geq^* F(y) \;\; \vee \;\; y \setminus x \not\geq^* F(x)$

What about models of $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$?

Roitman and Williams, 2015: *Paracompactness, normality, and related properties of topologies in infinite products*.

"...The simplest model where we don't know if it (Δ) holds is to first add ω_2 Cohen reals to a model of CH and to then add ω_3 random reals simultaneously..." (Here, $\mathbb{B}_{\omega_3} = \mathit{Baire}(2^{\omega_3})/\mathcal{N}$)

Notice that if $M \models CH$, then $M^{\mathbb{C}_{\omega_2} * \mathbb{B}_{\omega_3}} \models \begin{cases} \mathfrak{b} = \omega_1 \\ \mathfrak{d} = \omega_2 \\ \mathfrak{c} = \omega_3 \end{cases}$

Suppose $M \models CH$. Is it true that $M^{\mathbb{C}_{\omega_2} * \mathbb{B}_{\omega_3}} \models \Delta$?

$\Delta \equiv \exists F : \omega^{\subset \omega} \to \omega^{\omega}\ \forall x, y \in \omega^{\subset \omega}, |x \setminus y| = |y \setminus x| = \omega \ \wedge\ \neg \exists^{\infty} n\ \ x(n) \neq y(n)$
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Suppose $M \models CH$. Is it true that $M^{\mathbb{C}_{\omega_2} * \mathbb{B}_{\omega_3}} \models \Delta$? **Yes**

$\Delta \equiv \exists F : \omega^{\subset \omega} \to \omega^{\omega}\ \forall x, y \in \omega^{\subset \omega}, |x \setminus y| = |y \setminus x| = \omega\ \wedge\ \neg \exists^{\infty} n\ \ x(n) \neq y(n)$
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Ahother MH-like statement (due to J. Brendle):
If V is a model and for $\alpha < \mathfrak{d}$ ther is V_α models such that

- for $\alpha < \beta < \mathfrak{d}$, $V_\alpha \subseteq V_\beta \subseteq V$.
- $V \cap \omega^\omega = \bigcup_{\alpha < \mathfrak{d}} V_\alpha \cap \omega^\omega$.
- for $\alpha < \mathfrak{d}$, there is $f_\alpha \in V \cap \omega^\omega$ unbounded by $V_\alpha \cap \omega^\omega$.

Then Δ holds. ($F(x) = f_\alpha$ if α is least where $x \in V_\alpha$)

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Then Δ holds. ($F(x) = f_\alpha$ if α is least where $x \in V_\alpha$)

Proposition If $M \models CH$, then $M^{\mathbb{C}_{\omega_2} * \mathbb{B}_{\omega_3}} \models \Delta$.

Proof:

$$V = M^{\mathbb{C}_{\omega_2} * \mathbb{B}_{\omega_3}},$$

$$V_\alpha = M[\{c_\gamma : \gamma < \alpha\}][\{r_\gamma : \gamma < \omega_3\}], \alpha < \omega_2.$$

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Do not work with models

Let $Z = \omega^{\subset\omega} \cup \omega^\omega$ and suppose there is $Z_\alpha \subseteq Z, \alpha < \mathfrak{d}$ such that:

- Z_α is closed under set difference ($x \setminus y \in Z_\alpha$).
- $Z = \bigcup_{\alpha<\mathfrak{d}} Z_\alpha$ and $Z_\alpha \subseteq Z_\beta$ if $\alpha < \beta$.
- For $\alpha < \mathfrak{d}$, there is $f_\alpha \in \omega^\omega$ unbounded by Z_α.

Then Δ holds.

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega \; \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega \;\wedge\; \neg\exists^\infty n \;\; x(n) \neq y(n)$
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- For $\alpha < \mathfrak{d}$, there is $f_\alpha \in \omega^\omega$ unbounded by Z_α.

Then Δ holds.

Main question:

Is Δ true in ZFC?

$\Delta \equiv \exists F : \omega^{\subset\omega} \to \omega^\omega\ \forall x, y \in \omega^{\subset\omega}, |x \setminus y| = |y \setminus x| = \omega\ \wedge\ \neg\exists^\infty n\ \ x(n) \neq y(n)$
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Thanks for your attention!