6-2017

On Quasi-Uniform Box Products

Hope Sabao
North-West University (South Africa), hope@aims.edu.gh

Olivier Olela Otafudu
North-West University (South Africa), olivier.olelaotafudu@nwu.ac.za

On quasi-uniform box products

Hope Sabao* and Olivier Olela Otafudu

North-West University

32nd Summer Conference on Topology and its Applications

University of Dayton, Dayton, Ohio, USA.

June 27–30, 2017

Overview

- Introduction

Overview

- Introduction
- The box product

Overview

- Introduction
- The box product
- The Tychonov product topology

Overview

- Introduction
- The box product
- The Tychonov product topology
- The quasi-uniform box product

Overview

- Introduction
- The box product
- The Tychonov product topology
- The quasi-uniform box product
- Properties of filter pairs

Overview

- Introduction
- The box product
- The Tychonov product topology
- The quasi-uniform box product
- Properties of filter pairs
- Completeness in quasi-uniform box products

Introduction

Introduction

Suppose $(X_n)_{n\in\mathbb{N}}$ is a countable family of sets. Then the Cartesian product is defined by

$$\prod_{n\in\mathbb{N}} X_n = \left\{ x : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n : x(n) \in X_n \text{ for each } n \in \mathbb{N} \right\}.$$

Introduction

Suppose $(X_n)_{n\in\mathbb{N}}$ is a countable family of sets. Then the Cartesian product is defined by

$$\prod_{n\in\mathbb{N}} X_n = \left\{x : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n : x(n) \in X_n \text{ for each } n \in \mathbb{N}\right\}.$$

The topology on the product of a family of topological spaces was considered first by H. Tietze in 1923.

Introduction

Suppose $(X_n)_{n\in\mathbb{N}}$ is a countable family of sets. Then the Cartesian product is defined by

$$\prod_{n\in\mathbb{N}} X_n = \left\{ x : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n : x(n) \in X_n \text{ for each } n \in \mathbb{N} \right\}.$$

The topology on the product of a family of topological spaces was considered first by H. Tietze in 1923.

He defined the topology as follows:

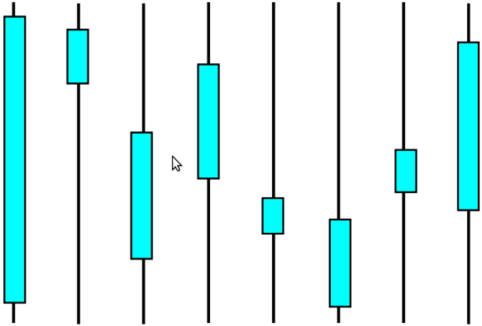
Definition

Suppose X_n is a topological space for each $n \in \mathbb{N}$ and G_n is open in X_n. Then a set of the form

$$G = \prod_{n \in \mathbb{N}} G_n$$

Is called an *open box*. The collection of all open boxes forms a basis for a topology on the product set called the box topology. The product space with this topology is called the box product

Box Product



Tychonov product topology

- Tietzes definition was not followed much because it was discovered that the box topology could not preserve interesting properties of the factor space such as Normality.

Tychonov product topology

- Tietzes definition was not followed much because it was discovered that the box topology could not preserve interesting properties of the factor space such as Normality.
- Therefore, in 1927, a new topology, called the Tychonov product topology, introduced by A. Tychonov became more popular and much work was done using this topology on the product space.

Tychonov product topology

- Tietzes definition was not followed much because it was discovered that the box topology could not preserve interesting properties of the factor space such as Normality.
- Therefore, in 1927, a new topology, called the Tychonov product topology, introduced by A. Tychonov became more popular and much work was done using this topology on the product space.

- Tietzes definition was not followed much because it was discovered that the box topology could not preserve interesting properties of the factor space such as Normality.
- Therefore, in 1927, a new topology, called the Tychonov product topology, introduced by A. Tychonov became more popular and much work was done using this topology on the product space.

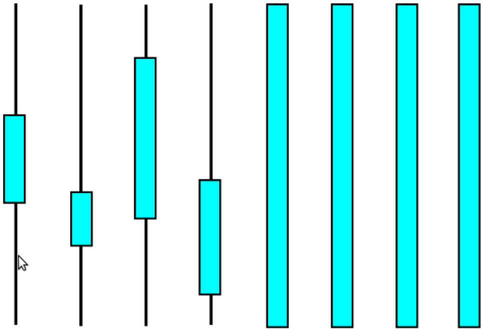
Definition

The Tychonov topology (or product topology) on $\prod_{n\in\mathbb{N}} X_n$ is obtained by taking as a base for open sets, sets of the form $\prod_{n\in\mathbb{N}} U_n$, where

(i) U_n is open in X_n for each $n \in \mathbb{N}$,

(ii) for all but finitely many coordinates, $U_n = X_n$.

Tychonov Product topology



The quasi-uniform box product

Definition

A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

(i) each member U of $\mathcal{U}$ contains the diagonal $\triangle = \{(x, x) : x \in X\}$ of X,

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = V \circ V = \{(x, z) \in X \times X : \text{ there is } y \in X \text{ such that } (x, y) \in V, (y, z) \in V\}$.

The members $U \in \mathcal{U}$ are called *entourages* of $\mathcal{U}$ and the elements of X are called *points*. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*.

Definition

A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

(i) each member U of $\mathcal{U}$ contains the diagonal $\triangle = \{(x,x) : x \in X\}$ of X,

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = V \circ V = \{(x,z) \in X \times X :$ there is $y \in X$ such that $(x,y) \in V, (y,z) \in V\}$.

The members $U \in \mathcal{U}$ are called *entourages* of $\mathcal{U}$ and the elements of X are called *points*. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*.

Remark

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, on $X \times X$, is also a quasi-uniformity on X. The quasi-uniformity $\mathcal{U}^{-1}$ is called the conjugate of $\mathcal{U}$. If $\mathcal{U} = \mathcal{U}^{-1}$, then $\mathcal{U}$ is called a uniformity on X. Furthermore, $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity.

Uniform topology

Definition

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of the point $x \in X$.

Definition

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of the point $x \in X$.

A subset A of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in A$, there is an entourage $U \in \mathcal{U}$ such that $U(x) \subset A$. Thus for each $x \in X$ and $U \in \mathcal{U}$, $U(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x.

Definition

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of the point $x \in X$.

A subset A of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in A$, there is an entourage $U \in \mathcal{U}$ such that $U(x) \subset A$. Thus for each $x \in X$ and $U \in \mathcal{U}$, $U(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x.

Note that $U(x)$ need not be $\tau(\mathcal{U})$-open in general. However, there is always a base $\mathcal{B}$ for $\mathcal{U}$ such that for each $B \in \mathcal{B}$ and $x \in X$, $B(x) \in \tau(\mathcal{U})$.

Definition

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of the point $x \in X$.

A subset A of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in A$, there is an entourage $U \in \mathcal{U}$ such that $U(x) \subset A$. Thus for each $x \in X$ and $U \in \mathcal{U}$, $U(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x.

Note that $U(x)$ need not be $\tau(\mathcal{U})$-open in general. However, there is always a base $\mathcal{B}$ for $\mathcal{U}$ such that for each $B \in \mathcal{B}$ and $x \in X$, $B(x) \in \tau(\mathcal{U})$.

Corollary

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\{U : U \in \mathcal{U}$ and U is $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$ open in $X \times X\}$ is a base for $\mathcal{U}$.

The quasi-uniform box product

Theorem

Let $(X, \mathcal{U})$ be a quasi-uniform space. For $U \in \mathcal{U}$, let

$$\overline{U} = \left\{ (x, y) \in \prod_{n \in \mathbb{N}} X \times \prod_{n \in \mathbb{N}} X : \forall n \in \mathbb{N},\ (x(n), y(n)) \in U \right\}$$

and define $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$. Then $\overline{\mathcal{U}}$ is a filter base generating a quasi-uniformity on $\prod_{n \in \mathbb{N}} X$.

Theorem

Let $(X,\mathcal{U})$ be a quasi-uniform space. For $U \in \mathcal{U}$, let

$$\overline{U} = \left\{(x,y) \in \prod_{n\in\mathbb{N}} X \times \prod_{n\in\mathbb{N}} X : \forall n \in \mathbb{N},\ (x(n), y(n)) \in U\right\}$$

and define $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$. Then $\overline{\mathcal{U}}$ is a filter base generating a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$.

Definition

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the quasi-uniformity $\overline{\mathcal{U}}$ is called the constant quasi-uniformity, $\tau(\overline{\mathcal{U}})$ is called the constant quasi-uniform topology and the pair $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is called the quasi-uniform box product.

The quasi-uniform box product

Remark

If a quasi-uniform space $(X,\mathcal{U})$ is such $\mathcal{U} = \mathcal{U}^{-1}$, then $\overline{\mathcal{U}} = \overline{\mathcal{U}}^{-1} = \overline{\mathcal{U}}^{s}$. Therefore, the constant quasi-uniformity $\overline{\mathcal{U}}$ on $\prod_{n\in\mathbb{N}} X$ is exactly the constant uniformity in the sense of Bell.

The quasi-uniform box product

Remark

If a quasi-uniform space $(X, \mathcal{U})$ is such $\mathcal{U} = \mathcal{U}^{-1}$, then $\overline{\mathcal{U}} = \overline{\mathcal{U}}^{-1} = \overline{\mathcal{U}}^{s}$. Therefore, the constant quasi-uniformity $\overline{\mathcal{U}}$ on $\prod_{n\in\mathbb{N}} X$ is exactly the constant uniformity in the sense of Bell.

Remark

If $(X, \mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product, then the quasi-uniform space $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}^{-1}}\right)$ is again a quasi-uniform box product of $(X, \mathcal{U})$, where $\overline{\mathcal{U}^{-1}} = \{\overline{U^{-1}} : U \in \mathcal{U}\}$ is also a filter base generating a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$. Moreover, $\overline{\mathcal{U}^{-1}} \vee \overline{\mathcal{U}} = \overline{\mathcal{U}}^{s}$ is a filter base generating a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$ and the pair $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}^{s}\right)$ is a uniform box product of the uniform space $(X, \mathcal{U}^{s})$ which corresponds to the uniform box product in the sense of Bell.

The uniform box product

The uniform box product, introduced by Scott Williams, is a topology that sits between the Tychonov product topology and the box product. It extends the supremum metric to powers of compact spaces.

The uniform box product

The uniform box product, introduced by Scott Williams, is a topology that sits between the Tychonov product topology and the box product. It extends the supremum metric to powers of compact spaces.

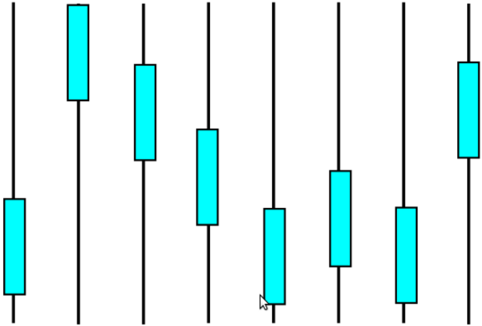
Definition

Let $(X, \mathcal{D})$ be a uniform space. For $D \in \mathcal{D}$, let

$$\overline{D} = \left\{ (x, y) \in \prod_{n \in \mathbb{N}} X \times \prod_{n \in \mathbb{N}} X : \forall n \in \mathbb{N}, (x(n), y(n)) \in D \right\}$$

and define $\overline{\mathcal{D}} = \{\overline{D} : D \in \mathcal{D}\}$. Then $\overline{\mathcal{D}}$ is a uniformity base called the constant uniformity base on the product $\prod_{n \in \mathbb{N}} X$. The topology $\tau(\overline{\mathcal{D}})$ is called the constant uniform topology and the pair $(\prod_{n \in \mathbb{N}} X, \overline{\mathcal{D}})$ is called the uniform box product.

Uniform box product



Example

If we equip the Fort space X with the Pervin quasi-uniformity $\mathcal{P}$ with the subbase $\mathcal{S} = \{S_A : A \subseteq V \text{ finite}\}$, where
$S_A = [A \times A] \cup [(X \setminus A) \times X]$.
Then $S_A^{-1} = [A \times A] \cup [X \times (X \setminus A)]$. Thus the basic neighbourhood of a point $x \in \prod_{n \in \mathbb{N}} X$ are given as follows:

Example

If we equip the Fort space X with the Pervin quasi-uniformity $\mathcal{P}$ with the subbase $\mathcal{S} = \{S_A : A \subseteq V \text{ finite}\}$, where $S_A = [A \times A] \cup [(X \setminus A) \times X]$.
Then $S_A^{-1} = [A \times A] \cup [X \times (X \setminus A)]$. Thus the basic neighbourhood of a point $x \in \prod_{n \in \mathbb{N}} X$ are given as follows:

$$\overline{S_A}(x) = \prod_{x(n) \in A} A \times \prod_{x(n) \notin A} X$$

Also,

Also,

$$\overline{S_A}^{-1}(x) = \prod_{x(n)\in A} A \times \prod_{x(n)\notin A} (X \setminus A).$$

Also,

$$\overline{S_A}^{-1}(x) = \prod_{x(n) \in A} A \times \prod_{x(n) \notin A} (X \setminus A).$$

Therefore,

Also,

$$\overline{S_A}^{-1}(x) = \prod_{x(n)\in A} A \times \prod_{x(n)\notin A} (X \setminus A).$$

Therefore,

$$\overline{S_A}(x) \cap \overline{S_A}^{-1}(x) = \prod_{x(n)\in A} A \times \prod_{x(n)\notin A} (X \setminus A).$$

$$\overline{W_F^{-1}}(x) = \prod_{x(n)\in F} \{x(n)\} \times \prod_{x(n)\notin F} X.$$

$$\overline{W_F^{-1}}(x) = \prod_{x(n)\in F} \{x(n)\} \times \prod_{x(n)\notin F} X.$$

Therefore,

$$\overline{W_F^{-1}}(x) = \prod_{x(n) \in F} \{x(n)\} \times \prod_{x(n) \notin F} X.$$

Therefore,

$$\overline{W_F}(x) \cap \overline{W_F^{-1}}(x) = \overline{D_A}(x) = \prod_{x(n) \in A} \{x(n)\} \times \prod_{x(n) \notin A} (X \setminus A).$$

In her PhD thesis, Jocelyn Bell showed that the uniform box product of countably many copies of a Fort-space is normal, countably paracompact and collectionwise Hausdorff.

In her PhD thesis, Jocelyn Bell showed that the uniform box product of countably many copies of a Fort-space is normal, countably paracompact and collectionwise Hausdorff.

Later on, in her paper, J.*R Bell, An infinite game with topological consequences, Topol. Appl. 175 (2014) 1–14*, Bell Showed that uniform box product of countably many copies of a Fort-space is collectionwise normal, countably paracompact and collectionwise Hausdorff.

In her PhD thesis, Jocelyn Bell showed that the uniform box product of countably many copies of a Fort-space is normal, countably paracompact and collectionwise Hausdorff.

Later on, in her paper, J.*R Bell, An infinite game with topological consequences, Topol. Appl. 175 (2014) 1–14*, Bell Showed that uniform box product of countably many copies of a Fort-space is collectionwise normal, countably paracompact and collectionwise Hausdorff.

In my PhD thesis, under the supervision of Olivier Olela Otafudu, we generalised the infinite game to the asymmetric setting and used this game to show that the quasi-uniform box product of countably many copies of a Fort-space is collectionwise normal, countably paracompact and collectionwise Hausdorff.

Proposition

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\overline{\mathcal{F}}$ is a filter on $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then $\mathcal{F}$, defined by

$$\mathcal{F} = \left\{F : \prod_{n\in\mathbb{N}} F \in \overline{\mathcal{F}}\right\},$$

is a filter on $(X,\mathcal{U})$.

Proposition

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\overline{\mathcal{F}}$ is a filter on $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then $\mathcal{F}$, defined by

$$\mathcal{F}=\left\{F:\prod_{n\in\mathbb{N}} F\in\overline{\mathcal{F}}\right\},$$

is a filter on $(X,\mathcal{U})$.

Definition

Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $\mathcal{F}$ and $\mathcal{G}$ are filters on X. We say $(\mathcal{F}, \mathcal{G})$ is Cauchy filter pair provided that for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. A Cauchy filter pair on a quasi-uniform space $(X, \mathcal{U})$ is called *constant* provided that $\mathcal{F} = \mathcal{G}$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then the filter pair $(\mathcal{F},\mathcal{G})$, where $\mathcal{F}=\left\{F : \prod_{n\in\mathbb{N}} F \in \overline{\mathcal{F}}\right\}$ and $\mathcal{G}=\left\{G : \prod_{n\in\mathbb{N}} G \in \overline{\mathcal{G}}\right\}$, is a Cauchy filter pair on $(X,\mathcal{U})$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then the filter pair $(\mathcal{F},\mathcal{G})$, where $\mathcal{F}=\left\{F : \prod_{n\in\mathbb{N}} F \in \overline{\mathcal{F}}\right\}$ and $\mathcal{G}=\left\{G : \prod_{n\in\mathbb{N}} G \in \overline{\mathcal{G}}\right\}$, is a Cauchy filter pair on $(X,\mathcal{U})$.

Definition

A filter $\mathcal{G}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be a *D-Cauchy filter* if there is a filter $\mathcal{F}$ on X such that $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair. We call $\mathcal{F}$ a *cofilter* of $\mathcal{G}$.

Definition

We say a quasi-uniform space $(X,\mathcal{U})$ is quiet provided that for each $U \in \mathcal{U}$, there is an entourage $V \in \mathcal{U}$ such that if $\mathcal{F}$ and $\mathcal{G}$ are filters on X and x and y are points of X such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ and $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$, then $(x, y) \in U$. If V satisfies the above conditions, we say that V is quiet for U.

Definition

We say a quasi-uniform space $(X, \mathcal{U})$ is quiet provided that for each $U \in \mathcal{U}$, there is an entourage $V \in \mathcal{U}$ such that if $\mathcal{F}$ and $\mathcal{G}$ are filters on X and x and y are points of X such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ and $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair on $(X, \mathcal{U})$, then $(x, y) \in U$. If V satisfies the above conditions, we say that V is quiet for U.

Theorem

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n \in \mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ quiet, then $\left(\prod_{n \in \mathbb{N}} X, \overline{\mathcal{U}}\right)$ is quiet.

Completeness in quasi-uniform box products

Definition

A quasi-uniform space $(X,\mathcal{U})$ is called C-complete provided that each Cauchy filter pair $(\mathcal{F},\mathcal{G})$ converges.

Definition

A quasi-uniform space $(X,\mathcal{U})$ is called C-complete provided that each Cauchy filter pair $(\mathcal{F},\mathcal{G})$ converges.

Theorem

Let $(X,\mathcal{U})$ be a quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X,\mathcal{U})$ is C-complete, then $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is C-complete.

Definition

A quasi-uniform space $(X,\mathcal{U})$ is called *D- complete* if each *D*-Cauchy filter converges, that is, each second filter of the Cauchy filter pair $(\mathcal{F},\mathcal{G})$ converges with respect to $\tau(\mathcal{U})$.

Definition

A quasi-uniform space $(X, \mathcal{U})$ is called *D- complete* if each D-Cauchy filter converges, that is, each second filter of the Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ converges with respect to $\tau(\mathcal{U})$.

Theorem

Let $(X, \mathcal{U})$ be a quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ is D-complete, then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is D-complete.

Uniform regularity

Definition

We say quasi-uniform space $(X, \mathcal{U})$ is uniformly regular if for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $cl_{\tau(\mathcal{U})} V(x) \subseteq U(x)$ whenever $x \in X$.

Uniform regularity

Definition

We say quasi-uniform space $(X, \mathcal{U})$ is uniformly regular if for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $cl_{\tau(\mathcal{U})} V(x) \subseteq U(x)$ whenever $x \in X$.

Lemma

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ is uniformly regular, then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is uniformly regular.

Definition

We say quasi-uniform space $(X, \mathcal{U})$ is uniformly regular if for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $cl_{\tau(\mathcal{U})} V(x) \subseteq U(x)$ whenever $x \in X$.

Lemma

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ is uniformly regular, then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is uniformly regular.

Corollary

Let $(X, \mathcal{U})$ be a D-complete uniformly regular quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}^{-1}\right)$ is D-complete.

Pair completeness

Definition

A quasi-uniform space is said to be *pair complete* provided that whenever $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair, there exists a point $p \in X$ such that the filter $\mathcal{G} \underset{\tau(\mathcal{U})}{\longrightarrow} p$ and $\mathcal{F} \underset{\tau(\mathcal{U}^{-1})}{\longrightarrow} p$

Pair completeness

Definition

A quasi-uniform space is said to be *pair complete* provided that whenever $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair, there exists a point $p \in X$ such that the filter $\mathcal{G} \underset{\tau(\mathcal{U})}{\longrightarrow} p$ and $\mathcal{F} \underset{\tau(\mathcal{U}^{-1})}{\longrightarrow} p$

Corollary

Let $(X, \mathcal{U})$ be a D-complete uniformly regular quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is pair complete.

Pair completeness

Definition

A quasi-uniform space is said to be *pair complete* provided that whenever $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair, there exists a point $p \in X$ such that the filter $\mathcal{G} \underset{\tau(\mathcal{U})}{\longrightarrow} p$ and $\mathcal{F} \underset{\tau(\mathcal{U}^{-1})}{\longrightarrow} p$

Corollary

Let $(X, \mathcal{U})$ be a D-complete uniformly regular quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is pair complete.

Corollary

Let $(X, \mathcal{U})$ be a D-complete quiet quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is C-complete.

Other forms of completeness in quasi-uniform box products

Definition

Let (X, τ) be a topological space and $(Y, \mathcal{V})$ be a quasi-uniform space. Furthermore, let $\mathcal{D}$ be a family of maps from X to Y. If $\mathcal{A}$ is a family of subsets of X, we denote by $\mathcal{V}_{\mathcal{A}}$ the quasi-uniformity on $\mathcal{D}$ which has, as subbase, the family of all relations of the form

$$(A, U) = \{(f, g) \in \mathcal{D} \times \mathcal{D} : (f(x), g(x)) \in U \text{ whenever } x \in A\}$$

whenever $A \in \mathcal{A}$ and $U \in \mathcal{V}$. The quasi-uniformity $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of uniform convergence* of $\mathcal{A}$.

Definition

Let (X,τ) be a topological space and $(Y,\mathcal{V})$ be a quasi-uniform space. Furthermore, let $\mathcal{D}$ be a family of maps from X to Y. If $\mathcal{A}$ is a family of subsets of X, we denote by $\mathcal{V}_{\mathcal{A}}$ the quasi-uniformity on $\mathcal{D}$ which has, as subbase, the family of all relations of the form

$$(A,U)=\{(f,g)\in\mathcal{D}\times\mathcal{D}:(f(x),g(x))\in U \text{ whenever } x\in A\}$$

whenever $A\in\mathcal{A}$ and $U\in\mathcal{V}$. The quasi-uniformity $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of uniform convergence* of $\mathcal{A}$.

If $\mathcal{A}=\{\mathbb{N}\}$ and $(Y,\mathcal{V})=(X,\mathcal{U})$, then the constant quasi-uniformity $\overline{\mathcal{U}}$ is exactly the quasi-uniformity of uniform convergence.

Künzi and Romaguera studied the completeness of the quasi-uniformity of uniform convergence with the aim of obtaining an appropriate quasi-uniform generalisation of the classical result that if X is a topological space and $(Y, \mathcal{U})$ is a complete uniform space, then the uniformity of uniform convergence is complete. Since the constant quasi-uniformity is a particular case of the quasi-uniformity of uniform convergence, we adapt the results of Künzi and Romaguera to the framework of quasi-uniform box products.

Definition

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is called:

(i) $(X, \mathcal{U})$ is left (right) K-complete provided that each left (right) K-Cauchy filter is $\tau(\mathcal{U})$ convergent

(ii) $(X, \mathcal{U})$ is half complete provided that each $\mathcal{U}^s$-Cauchy filter is $\tau(\mathcal{U})$-convergent.

(iii) $(X, \mathcal{U})$ is bicomplete provided that the uniform space $(X, \mathcal{U}^s)$ is complete

(iv) $(X, \mathcal{U})$ is strongly D-complete provided that if $(\mathcal{F}, \mathcal{G}) \to 0$, then the filter $\mathcal{F}$ has a $\tau(\mathcal{U})$-cluster point

(v) $(X, \mathcal{U})$ is S-complete provided that each stable Cauchy pair of filters $(\mathcal{F}, \mathcal{G})$ converges to a point $x \in X$, that is, $\mathcal{G}$ is $\tau(\mathcal{U})$-convergent to x and $\mathcal{F}$ is $\tau(\mathcal{U}^{-1})$-convergent to x

(vi) $(X, \mathcal{U})$ is U-complete provided that each stable Cauchy pair of ultrafilters is convergent to a point $x \in X$.

Theorem

Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then

(i) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is half complete if and only if $(X, \mathcal{U})$ is half complete.

(ii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is bicomplete if and only if $(X, \mathcal{U})$ is bicomplete.

(iii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is right K-complete if and only if $(X, \mathcal{U})$ is right K-complete.

(iv) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is S-complete if and only if $(X, \mathcal{U})$ is S-complete.

(v) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is U-complete if and only if $(X, \mathcal{U})$ is U-complete.

Theorem

Suppose $(X, \mathcal{U})$ is a strongly D-complete quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is D-complete.

References

- J.R. Bell, The uniform box product, Proc. Amer. Math. Soc. 142 (2014), 2161–2171.
- J.R. Bell, An infinite game with topological consequences, Topology Appl. 175 (2014), 1–14.
- J.R. Bell, The uniform box product problem, PhD thesis, SUNY at Buffalo, 2010.
- D. Doitchinov, On completeness of quasi-uniform spaces, C.R. Acad. Bulgare Sci. 41 (4) (1988), 5-8.
- P. Fletcher, W. Hunsaker, A note on totally bounded quasi-uniformities, Serdica Math. J. 24 (1998), 95-98.
- P. Fletcher and W. Hunsaker, Completeness using pairs of filters, Topology Appl. 44 (1992), 149-155.
- G. Gruenhage, Infinite games and generalizations of first-countable spaces, General Topology and Appl. 6 (1976),

References Cont'

- H.-P. A. Künzi, An Introduction to quasi-uniform spaces. Contemp. Math. 486 (2009) 239-304.
- J. Roitman, Paracompactness and normality in box products: old and new. Set theory and its Applications, Contemp. Math. 533 (2011) 157-181.
- M. E. Rudin, A normal space X for which $X \times I$ is not normal, Fund. Math., 73 (1971) 179-186.
- M. E. Rudin, The box topology, Proceedings of The University of Houston Point Set Topology Conference, Univ Houston, Houston TX, (1971) 191-199.
- R. Telgarsky, Spaces defined by topological games, Fund. Math. 88 (1975) 193-223.
- S.W. Williams, Box Products, Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan, Eds., Elsevier, (1984) 169-200

Thank You for Your Attention