Uncountably Many Quasi-Isometry Classes of Groups of Type FP

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Joint work with

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and

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June 27, 2017
TOPOLOGY $\leadsto$ ALGEBRA

Space $X \leadsto \pi_1(X), H_n(X), \pi_n(X)$, etc.
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- $X$ is a CW-complex,
- $\pi_1(X) = G$,
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- $X$ is a CW-complex,
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We build $X = K(G, 1)$ as follows:
- $X$ has a single 0–cell,
- 1–cells of $X$ correspond to generators of $G$,
- 2–cells of $X$ correspond to relations of $G$,
- 3–cells of $X$ are added to kill $\pi_2(X)$,
- 4–cells of $X$ are added to kill $\pi_3(X)$,
- etc...
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If \(K(G, 1)\) has finitely many cells, group \(G\) is of type \(F\).

If \(X = K(G, 1)\), \(G\) acts cellularly on \(\tilde{X}\) and we have a long exact sequence
\[
\cdots \rightarrow C_i(\tilde{X}) \rightarrow \cdots \rightarrow C_1(\tilde{X}) \rightarrow C_0(\tilde{X}) \rightarrow \mathbb{Z} \rightarrow 0
\]
consisting of free \(\mathbb{Z}G\)-modules. This leads to a definition:
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A group \( G \) is **of type \( FP_n \)** if the trivial \( \mathbb{Z}G \)-module \( \mathbb{Z} \) has a projective resolution which is **finitely generated** in dimensions 0 to \( n \):
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If, in addition, all $P_i = 0$ for $i > N$, for some $N$, group $G$ is of type $\textbf{FP}$. Clearly,
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Clearly,

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FP_n \supset FP_{n+1} \quad \text{and} \quad F_n \supset F_{n+1}.
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Question 1: Are these inclusions strict?
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**Question 2:** How many groups are there of type $FP_2$?

**Answer 1:** Up to isomorphism: $2^\mathbb{N}$ (I.Leary’15)

**Answer 2:** Up to quasiisometry: $2^\mathbb{N}$ (R.Kropholler–I.Leary–S.’17)
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Bestvina–Brady machine:

**Input:** A flag simplicial complex $L$.

**Output:** A group $BB_L$ with nice properties:

- $L$ is $(n - 1)$–connected $\iff BB_L$ is of type $F_n$,
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$L$ is octahedron: $\pi_1(L) = 1, \pi_2(L) \neq 0$, $\implies$ Stallings’s example.
$L$ is $n$–dimensional octahedron (orthoplex) $\implies$ Bieri’s example.
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I.J. Leary’s groups $G_L(S)$

**Input:** A flag simplicial complex $L$, a finite collection $\Gamma$ of directed edge loops in $L$ that normally generates $\pi_1(L)$, a subset $S \subset \mathbb{Z}$.

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- **Generators:** directed edges of $L$, the opposite edge to $a$ being $a^{-1}$.
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**Theorem (I.J. Leary)**

If $L$ is a flag complex with $\pi_1(L) \neq 1$, then groups $G_L(S)$ form $2^{\aleph_0}$ isomorphism classes. If, in addition, $L$ is aspherical and acyclic, then groups $G_L(S)$ are all of type FP.

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Take the famous Higman’s group:

\[ H = \langle a, b, c, d \mid a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle. \]

Let \( K \) be its presentation complex. It is aspherical and acyclic. Take \( L \) to be the 2nd barycentric subdivision of \( K \). Then \( L \) is a flag simplicial complex with 97 vertices, 336 edges and 240 triangles. Thus,

\[ G_L(S) = \langle 336 \text{ gen’s} \mid 240 \times 2 \text{ triangle relators}, 1 \text{ long relator } \forall n \in S \rangle. \]
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**Theorem (R.Kropholler–Leary–S.)**

*Groups* \( G_L(S) \) *form* \( 2^{\aleph_0} \) *classes up to quasiisometry.*
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**Theorem (R.Kropholler–Leary–S.)**

Groups \( G_L(S) \) form \( 2^{\aleph_0} \) classes up to quasiisometry.

Recall that groups \( G_1, G_2 \) are **quasiisometric** (qi), if their Cayley graphs are qi as metric spaces, i.e. there exists \( f : Cay(G_1, d_1) \to Cay(G_2, d_2) \), and \( A \geq 1, B \geq 0, C \geq 0 \) such that for all \( x, y \in Cay(G_1) \):

\[
\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B,
\]

and for all \( z \in Cay(G_2) \) there exists \( x \in Cay(G_1) \) such that \( d_2(z, f(x)) \leq C \).
How to distinguish groups up to qi?

Bowditch’98: a concept of taut loops in Cayley graphs. These are the loops which are not consequences of shorter loops.
How to distinguish groups up to qi?

**Bowditch’98**: a concept of **taut loops** in Cayley graphs. These are the loops which are not consequences of shorter loops.

Let $TL(G)$ denote the spectrum of lengths of taut loops in the Cayley graph of a group $G$. Bowditch proves that if groups $G_1$ and $G_2$ are qi, then there exist constants $A, B, N > 0$ such that for every $l_1 \in TL(G_1)$, $l_1 > N$, there exist an $l_2 \in TL(G_2)$ such that $l_1 \in [Al_2, Bl_2]$ and vice versa.
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Goal: to engineer groups with taut loops spectra “wildly interspersed” in $\mathbb{N}$, this will make the linear relation above impossible.
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Bowditch does this for small cancellation groups: he proves that there exist continuously many qi classes of 2–generator small cancellation groups.
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In our case, groups $G_L(S)$ do not have the property of small cancellation, so instead we use CAT(0) geometry of branched covers of cubical complexes to get estimates for the taut loops spectra. This information, and the freedom to choose arbitrary subsets $S \subset \mathbb{Z}$ for groups $G_L(S)$ allow us to construct continuously many qi classes of these groups.


Thank you!