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The Specification Property and Infinite Entropy for Certain Classes of Linear Operators

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CHRISTOPHER NEWPORT UNIVERSITY

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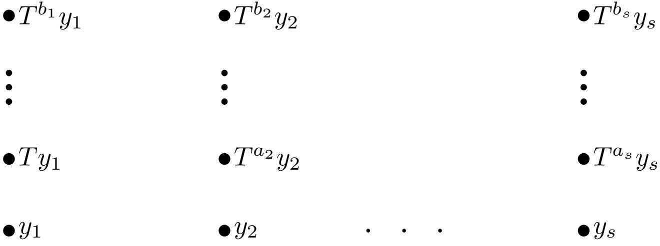
$$\{x, Tx, T^2x, T^3x, \ldots\}$$

  is dense in X.
- A point $x \in X$ is *periodic* if $T^n x = x$ for some $n \in \mathbb{N}$.
- T is *Devaney chaotic* if it is hypercyclic and has a dense set of periodic points.

Specification Property

$\bullet y_1 \qquad \bullet y_2 \qquad \cdots \qquad \bullet y_s$

Specification Property



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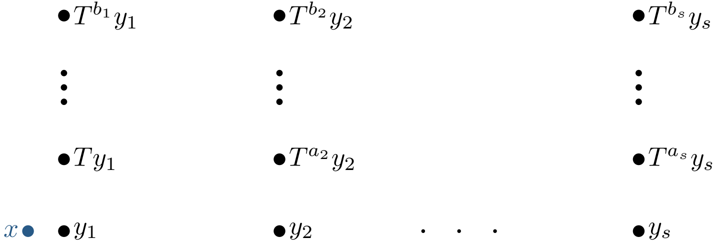

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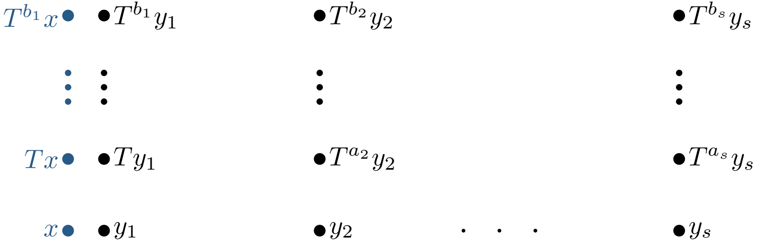

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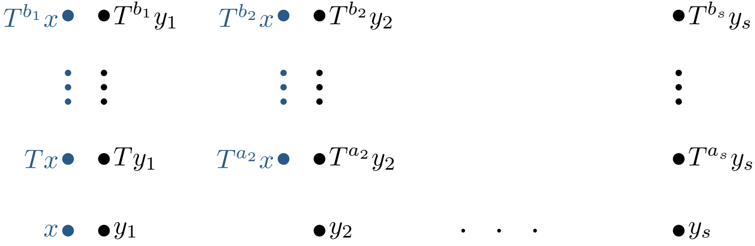

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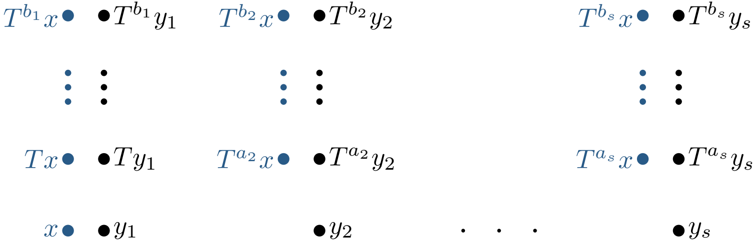

Specification Property

Definition

Let K be a compact, T-invariant, subset of X. We say that T has the *specification property* on K if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $s \in \mathbb{N}$, any points $y_1, \ldots, y_s \in K$, and any integers

$$0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$$

which satisfy $a_{i+1} - b_i \geq N$ for $i = 1, \ldots, s-1$, there exists a point $x \in K$ which is fixed by T^{N+b_s} and, for each $i = 1, \ldots, s$ and all integers k with $a_i \leq k \leq b_i$, we have

$$d(T^k x, T^k y_i) < \epsilon.$$

Specification Property

Definition

T has the *operator specification property* if there exists an increasing sequence $(K_n)_{n=1}^{\infty}$ of compact, T-invariant sets with $0 \in K_1$ and

$$\overline{\bigcup_{n=1}^{\infty} K_n} = X$$

such that T has the specification property on K_n for each $n \in \mathbb{N}$.

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such that T has the specification property on K_n for each $n \in \mathbb{N}$.

If T has the operator specification property, then

- T is Devaney chaotic.
- T has positive topological entropy.

The Frequent Hypercyclicity Criterion

Definition (Frequently Hypercyclic)

T is *frequently hypercyclic* if there exists a point $x \in X$ such that for every non-empty open set $U \subseteq X$,

$$\liminf_{n\to\infty} \frac{\operatorname{card}\left(\{k \in \mathbb{N} \colon T^k x \in U\} \cap [1, n]\right)}{n} > 0.$$

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Theorem (Frequent Hypercycicity Criterion, Bonilla and Grosse-Erdmann, 2007)

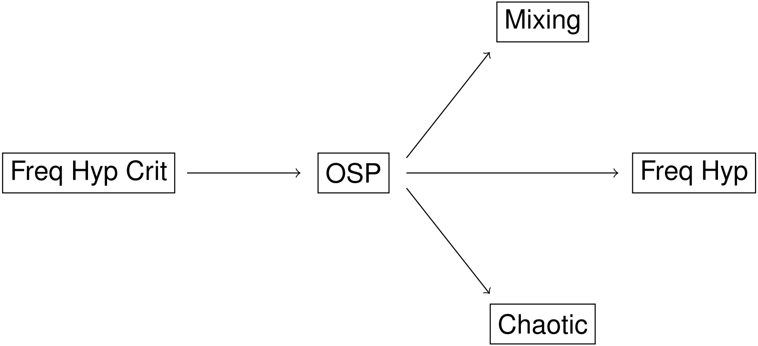
Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n \colon X_0 \to X$ such that, for each $x \in X_0$,

1. $\sum_{n=1}^{\infty} T^n x$ *converges unconditionally*
2. $\sum_{n=1}^{\infty} S_n x$ *converges unconditionally*
3. $T^n S_n x = x$*, and* $T^m s_n x = S_{n-m} x$ *for* $n > m$*.*

Then T is frequently hypercyclic.

The Frequent Hypercyclicity Criterion

(Adapted from a diagram by Bartoll, Martínez-Giménez, Peris)



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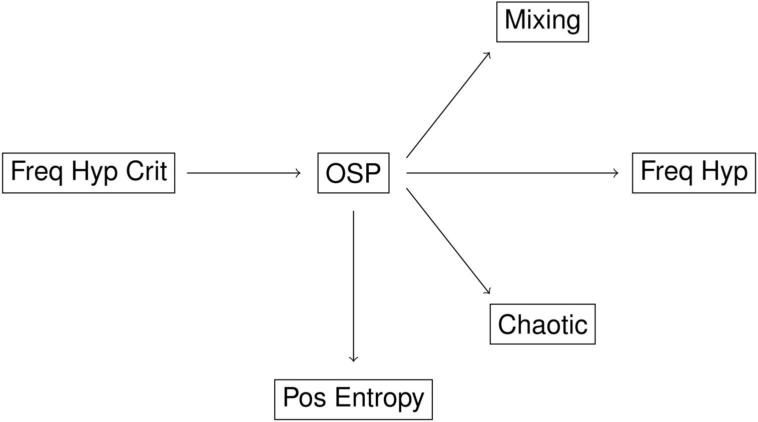

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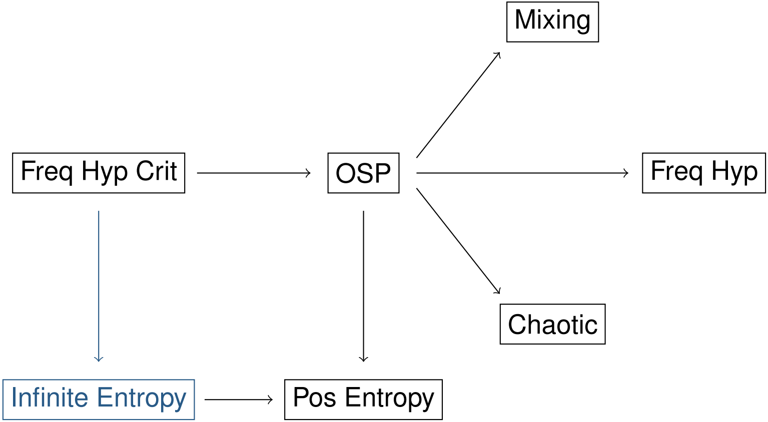

Topological Entropy

- Let $K \subseteq X$ be compact. Given $n \in \mathbb{N}$ and $\epsilon > 0$, a set $S \subseteq K$ is called (n, ϵ)-*separated* if for any $x, y \in S$ with $x \neq y$, we have $d\left(T^k x, T^k y\right) \geq \epsilon$ for some $0 \leq k \leq n$.

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  We denote the largest cardinality of an (n, ϵ)-separated subset of K by $s_{n,e}(T, K)$.

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  We denote the largest cardinality of an (n, ϵ)-separated subset of K by $s_{n,e}(T, K)$.
- The *topological entropy of T restricted to the compact set K* is given by

$$h(T, K) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{n,\epsilon}(T, K).$$

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- The *topological entropy of* T is given by

$$h(T) = \sup\{h(T, K) \colon K \text{ is a compact subset of } X\}$$

Translation Operators on $L_v^p(\mathbb{R}_+)$

- Let $v : [0, \infty) \to [0, \infty)$ be a measurable function such that for every $b \geq 0$,

$$\int_0^b v(x)dx < \infty,$$

and for every $\alpha > 0$

$$\sup_{x>0} \frac{v(x)}{v(x+\alpha)} < \infty.$$

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- Then for each $1 < p < \infty$, we define

$$\begin{aligned} L^p_v(\mathbb{R}_+) &= \left\{ f : \mathbb{R}_+ \to \mathbb{R} \;\middle|\; \int_0^\infty |f(x)|^p v(x)dx < \infty \right\} \\ \|f\|_{L^p_v} &= \left(\int_0^\infty |f(x)|^p v(x)dx \right)^{1/p} \end{aligned}$$

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- For each $\alpha > 0$, we define the translation operator T_α on $L^p_v(\mathbb{R}_+)$ by

$$T_\alpha f(x) = f(x+\alpha).$$

Translation Operators on $L^p_v(\mathbb{R}_+)$

Theorem (Mangino and Murillo-Arcila, 2015)

The following are equivalent:

1. T_α *satisfies the Frequent Hypercyclicity Criterion.*
2. T_α *is frequently hypercyclic.*
3. T_α *has the operator specification property.*
4. T_α *is Devaney chaotic.*
5. $\int_0^\infty v(x)dx < \infty$.

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Theorem (Brian, K, Tennant)

If any of the equivalent conditions in the theorem above are satisfied, then $h(T_\alpha) = \infty$.

Translation Operators on $L^p_v(\mathbb{R}_+)$

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Theorem (Brian, K, Tennant)

If any of the equivalent conditions in the theorem above are satisfied, then $h(T_\alpha) = \infty$.

The converse does not hold.

The Backward Shift on l_v^p

- $l_v^p = \left\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \colon \sum_{n=1}^{\infty} |x_n|^p v_n < \infty\right\}$

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- $|x|_{l_v^p} = \left(\sum_{n=1}^{\infty} |x_n|^p v_n\right)^{1/p}$
- We define the backward shift B on l_v^p by

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Theorem (Bartoll, Martínez-Giménez, and Peris, 2015)

The following are equivalent:

1. $\sum_{n=1}^{\infty} v_n < \infty$
2. B *has the operator specification property.*
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Example of a non-summable weight sequence where $h(B) = \infty$.

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Suppose P_{j-1} and Q_{j-1} have been defined, and let $q = \max Q_j$. Then define

$$\begin{aligned} P_j &= \{q+1, q+2 \ldots, q+j^2\} \\ Q_j &= \{q+j^2+1, q+j^2+2 \ldots, q+10j^2\} \end{aligned}$$

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We define the weight sequence $(v_n)_{n=1}^{\infty}$ as follows:

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We define the weight sequence $(v_n)_{n=1}^{\infty}$ as follows:

If $n \in P_j$, then $v_n = 1/j^2$.

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We define the weight sequence $(v_n)_{n=1}^{\infty}$ as follows:

If $n \in P_j$, then $v_n = 1/j^2$.

If $n \in Q_j$, then $v_n = v_{n-1}/2$.

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$$v_1 = \frac{1}{1^2}$$

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v_2 &= \frac{1}{2 \cdot 1^2}
\end{aligned}
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$$\begin{aligned} v_1 &= \frac{1}{1^2} \\ v_2 &= \frac{1}{2 \cdot 1^2} \\ v_3 &= \frac{1}{2^2 \cdot 1^2} \\ &\vdots \\ v_{10} &= \frac{1}{2^9 \cdot 1^2} \end{aligned}$$

$$\begin{aligned} v_{11} &= \frac{1}{2^2} \\ &\vdots \end{aligned}$$

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The Backward Shift on l^p_v

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&\vdots \\
v_{14} &= \frac{1}{2^2} \\
v_{15} &= \frac{1}{2 \cdot 2^2} \\
&\vdots \\
v_{50} &= \frac{1}{2^{36} \cdot 1^2}
\end{aligned}
$$

The Backward Shift on l_v^p

If $(x_n)_{n=1}^{\infty}$ is bounded, and $x_n = 0$ for all $n \in \bigcup_{j=1}^{\infty} P_j$, then $(x_n)_{n=1}^{\infty} \in l_v^p$.

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Hence, for all $M \in \mathbb{N}$, we can embed the full shift on M symbols into l_v^p.

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Hence, for all $M \in \mathbb{N}$, we can embed the full shift on M symbols into l_v^p.

It follows that $h(B) \geq 0.9 \log M$ for all $M \in \mathbb{N}$.

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It follows that $h(B) \geq 0.9 \log M$ for all $M \in \mathbb{N}$.

Thus $h(B) = \infty$.

THANK YOU