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# A NOTE ON IRREDUCIBILITY AND WEAK COVERING PROPERTIES

J. D. Mashburn

## I. Introduction

A space  $X$  is irreducible if every open cover of  $X$  has a minimal open refinement. Interest in irreducibility began when Arens and Dugendji [1] used this property to show that metacompact countably compact spaces are compact. It was natural, then, to find out what other types of spaces would be irreducible and therefore compact in the presence of countable compactness or Lindelöf in the presence of  $\aleph_1$ -compactness. So the covering properties considered were weakened. A space  $X$  is  $\theta$ -refinable if for every open cover of  $X$  there is a sequence  $\{\mathcal{G}_n : n \in \omega\}$  of open refinements such that every element of  $X$  has finite order in at least one  $\mathcal{G}_n$ .  $X$  is weakly  $\theta$ -refinable if every open cover of  $X$  has an open refinement  $\mathcal{G} = \cup \mathcal{G}_n$  such that every element of  $X$  has finite order in at least one  $\mathcal{G}_n$ . If, besides this, the collection  $\{\mathcal{G}_n^* : n \in \omega\}$ , where  $\mathcal{G}_n^*$  represents the union of all elements of  $\mathcal{G}_n$ , is point finite, then  $X$  is weakly  $\bar{\theta}$ -refinable. Wicke and Worrel [16] stated and J. R. Boone [4] later proved that  $\theta$ -refinable spaces are irreducible. Then J. R. Boone [5] and J. C. Smith [14] showed that weakly  $\bar{\theta}$ -refinable spaces are irreducible. There are several examples of weakly  $\theta$ -refinable spaces that are not irreducible. See example 2.2 of Davis and Smith [9], van Douwen and Wicke [11], and deCaux [10] for three such spaces.

But the method of proof used for spaces with point-finite properties cannot be used for spaces with point-countable properties. To circumvent this difficulty, J. R. Boone [6] introduced the concept of irreducible of order  $\alpha$ . A space  $X$  is irreducible of order  $\alpha$  if, for every open cover  $\mathcal{U}$  of  $X$ , there is an open refinement  $\mathcal{V} = \bigcup_{a \in A} \mathcal{V}_a$  of  $\mathcal{U}$  and a family of discrete closed collections  $\{\mathcal{J}_a : a \in A\}$  where  $|A| < \alpha$  such that:

- 1) for each  $T \in \mathcal{J}_a$ ,  $\mathcal{V}_T = \{V \in \mathcal{V}_a : T \subset V\} \neq \emptyset$  and  $|\mathcal{V}_T| < \alpha$
- 2)  $\{V : V \in \mathcal{V}_T, T \in \mathcal{J}_a, a \in A\}$  covers  $X$

He also showed in the same paper that  $\delta\theta$ -refinable spaces are irreducible of order  $\aleph_1$ . A space is  $\delta\theta$ -refinable if for every open cover of  $X$  there is a sequence  $\{\mathcal{G}_n : n \in \omega\}$  of open refinements such that every element of  $X$  has countable order in at least one  $\mathcal{G}_n$ .  $X$  is weakly  $\overline{\delta\theta}$ -refinable if every open cover of  $X$  has an open refinement  $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$  such that each element of  $X$  has countable order in at least one  $\mathcal{G}_n$ , and  $\{\mathcal{G}_n^* : n \in \omega\}$  is point-finite. The covers specified above are called  $\delta\theta$ -covers and weak  $\overline{\delta\theta}$ -covers. In [7] J. R. Boone states that weakly  $\overline{\delta\theta}$ -refinable spaces are irreducible of order  $\aleph_1$ .

It is shown in this paper that  $T_1$   $\delta\theta$ -refinable spaces and  $T_1$  weakly  $\overline{\delta\theta}$ -refinable spaces are irreducible. Since examples of Lindelöf spaces that are neither  $T_1$  nor irreducible can be easily constructed, it is clear that the spaces must be  $T_1$ .

## II. $\theta$ -Refinable Spaces

In order to show that a given open cover  $\mathcal{U}$  of a space  $X$  is minimal, we must show that every element of the open cover contains a closed set which does not intersect any other element of the open cover (see J. R. Boone [5]). In fact, points will work just as well for us as closed sets. The collection of these closed sets, or points, is then discrete in  $X$ .

To motivate the proofs and point out some of the difficulties, let us consider a couple of naive approaches to the problem. We must try to construct a discrete set  $D$  and for each  $d \in D$  pick an element  $V(d)$  of the open cover  $\mathcal{U}$  containing  $d$ . Hopefully at the end of the construction  $\{V(d) : d \in D\}$  will cover the space. Then, if some elements of  $D$  are in more than one element of  $\{V(d) : d \in D\}$ , we can get rid of the overlap by subtracting elements of  $D$ .

There are two methods that quickly present themselves to accomplish such a construction. If a discrete set  $D'$  has been defined then, in order to pick the next point, we can either choose some point outside  $\bigcup_{d \in D'} \text{st}(d, \mathcal{U})$  or we can try to make sure that every element of  $\mathcal{U}$  containing a point of  $D'$  that is not already a subset of  $\bigcup_{d \in D'} V(d)$  has some point chosen from it. Both of these methods lead to difficulties. With the first method, we obtain a discrete  $D$ , but the set  $\{V(d) : d \in D\}$  may not cover  $X$ . In fact, every subcover of  $\mathcal{U}$  may have elements that do not contain any elements of  $D$ . On the other hand, after picking an infinite number of points by the second method, we might not have a discrete set.

What is needed is a blending of these two methods. For a given  $d \in D$  we must pick some points outside  $\text{st}(d, \mathcal{U})$  to insure discreteness, and at the same time we must continually return to the elements of  $\mathcal{U}$  containing  $d$  to guarantee a cover of  $X$  in the end. To do this we will line up  $\{U \in \mathcal{U}: d \in U\}$  in such a way that we will always have room to put new open sets between the elements of  $\{U \in \mathcal{U}: d \in U\}$ . We will use enmeshed sequences to enumerate each successive set. Let  $I_1$  be the odd positive integers. Of course, we have an infinite number of positive integers left over to use as indices of our next collection of open sets. Let  $I_2 = \{2(2k-1): k \in \omega\}$ . This gives us every other even positive integer, so we still have infinite number of possible indices left. In general, let  $I_n = \{2^{n-1}(2k-1): k \in \omega\}$ .

One more piece of terminology, borrowed from Aull [2], needs to be mentioned before we begin. We will say that a subset  $D$  of  $X$  distinguishes a collection  $\mathcal{U}$  of subsets of  $X$  if every element of  $\mathcal{U}$  contains exactly one element of  $D$  and every element of  $D$  is in exactly one element of  $\mathcal{U}$ .

*Lemma 1.* Let  $\mathcal{U}$  be an open cover of a  $T_1$  space  $X$  and let  $C = \{x \in X: \text{ord}(x, \mathcal{U}) \leq \aleph_0\}$ . Then for every  $y \in C$  there is a partial open refinement  $\mathcal{W}$  of  $\mathcal{U}$  and a subset  $D$  of  $C$  such that  $y \in D$ ,  $D$  distinguishes  $\mathcal{W}$ ,  $D$  is discrete in  $X$ , and  $\text{st}(d, \mathcal{U}) \cap C \subset \cup \mathcal{W}$  for every  $d \in D$ .

*Proof.* Set  $d_1 = y$  and  $A_1 = \{U \in \mathcal{U}: d_1 \in U\}$ . Let  $\{A_j: j \in I_1\}$  be an enumeration of  $A_1$ . Set  $V_1 = A_1$  and, for bookkeeping purposes, set  $s_1 = m(1) = 1$ .

Assume that  $A_1, \dots, A_n$  are subsets of  $\mathcal{U}$  and that  $\{A_i: i \in I_j \text{ and } i \geq s_j\}$  is an enumeration of  $A_j$  for  $j = 1, \dots, n$ . Further assume that  $V_1, \dots, V_n$  have been chosen from  $\bigcup_{j=1}^n A_j$  and that  $V_k = A_{m(k)}$  for  $k = 1, \dots, n$ . Finally, we assume that  $(\bigcup_{j=1}^n A_j^* \cap C) - \bigcup_{j=1}^n V_j \neq \emptyset$ . We must select  $V_{n+1}$  and construct  $A_{n+1}$ .

Let  $m(n+1) = \min\{i: 1 \leq j \leq n, i \in I_j \text{ and } A_i \cap C \not\subseteq \bigcup_{k=1}^n V_k\}$ . This can be done since the set in question is not empty by assumption. Let  $j(n+1) \leq n$  such that  $m(n+1) \in I_{j(n+1)}$ . Set  $V_{n+1} = A_{m(n+1)}$  and pick  $d_{n+1} \in (V_{n+1} \cap C) - \bigcup_{k=1}^n V_k$ .

Let  $A_{n+1} = \{U \in \mathcal{U}: d_{n+1} \in U \text{ and } U \not\subseteq \bigcup_{j=1}^n A_j\}$ . We will use  $I_{n+1}$  to enumerate  $A_{n+1}$ , but to avoid backtracking let us start the enumeration at a point past  $m(n+1)$ . Let  $s_{n+1} = \min\{i \in I_{n+1}: i > m(n+1)\}$ . Enumerate  $A_{n+1}$  as  $\{A_i: i \in I_{n+1} \text{ and } i \geq s_{n+1}\}$ .

Let  $W_1 = V_1$  and, for every  $n > 1$ , let  $W_n = V_n - \{d_m: m < n\}$ . Let  $\mathcal{W} = \{W_n: n \in \omega\}$  and  $D = \{d_n: n \in \omega\}$ .  $\mathcal{W}$  is an open partial refinement of  $\mathcal{U}$ ,  $y \in D$ , and  $D$  distinguishes  $\mathcal{W}$ .

Let  $U \in \mathcal{U}$  and  $d_m \in U$ . Then  $U \in A_j$  for some  $j \leq m$ . Let  $U = A_i$ . Now  $\{m(k): k \in \omega\}$  is strictly increasing, so we can define  $n+1 = \min\{k \in \omega: m(k) > i\}$ . The notation  $n+1$  is used to better match the definition of  $m(n+1)$  above. Also,  $n+1 \geq 2$  since  $m(1) = 1$ . If  $n+1 \leq j$  then  $m(n+1) \leq m(j) < s_j \leq i$ , a contradiction. Thus  $j \leq n$  and  $i \in \bigcup_{k=1}^n I_k$ . By the definition of  $m(n+1)$ ,  $A_i \cap C \subseteq \bigcup_{k=1}^n V_k \subseteq U \mathcal{W}$ . This gives us two results.  $D$  is discrete because

every element of the open cover  $\mathcal{U}$  contains at most a finite number of elements of  $D$ , and  $\text{st}(d, \mathcal{U}) \subset \mathcal{U}$  for every  $d \in D$ .

*Lemma 2.* Let  $\mathcal{U}$  be an open cover of a  $T_1$  space  $X$  and let  $C \subset \{x \in X: \text{ord}(x, \mathcal{U}) \leq \aleph_0\}$ . Then there is an open partial refinement  $\mathcal{W}$  of  $\mathcal{U}$  covering  $C$  and a subset  $D$  of  $C$  such that  $D$  distinguishes  $\mathcal{W}$  and is discrete in  $X$ .

*Proof.* Set  $\mathcal{W}_0 = \{\emptyset\}$ . Let  $\beta$  be an ordinal number and assume that for every  $\alpha < \beta$ ,  $\mathcal{W}_\alpha$  is an open partial refinement of  $\mathcal{U}$ . Assume that  $C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^* \neq \emptyset$ . Pick  $y \in C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^*$ . By Lemma 1, there is an open partial refinement  $\mathcal{W}_\beta$  of  $\mathcal{U}$  and a subset  $D_\beta$  of  $C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^*$  such that  $D_\beta$  distinguishes  $\mathcal{W}_\beta$ ,  $y \in D_\beta$ ,  $D_\beta$  is discrete in  $X$ , and  $\text{st}(d, \mathcal{U}) \cap [C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^*] \subset \mathcal{W}_\beta^*$  for every  $d \in D$ .

For some ordinal number  $\sigma$  we must have  $C \subset \bigcup_{\alpha < \sigma} \mathcal{W}_\alpha^*$ . Let  $\mathcal{W} = \bigcup_{\alpha < \sigma} \mathcal{W}_\alpha$  and  $D = \bigcup_{\alpha < \sigma} D_\alpha$ . Clearly  $\mathcal{W}$  is an open partial refinement of  $\mathcal{U}$ .

We claim that every element of  $\mathcal{U}$  intersects at most one  $D_\alpha$ . Let  $U \in \mathcal{U}$  and let  $d \in U \cap D_\beta$ . Then  $U \cap C \subset \bigcup_{\alpha \leq \beta} \mathcal{W}_\alpha^*$ . So if  $\beta < \delta$  then  $U \cap D_\delta = \emptyset$ , since  $D_\delta \subset C - \bigcup_{\alpha < \delta} \mathcal{W}_\alpha^*$ . It then follows that  $U \cap D_\delta = \emptyset$  for all  $\delta < \beta$ , too.

Now the proof of Lemma 1 shows that for every  $U \in \mathcal{U}$  and every  $\alpha < \sigma$ ,  $U \cap D_\alpha$  is at most finite. Therefore every element of  $\mathcal{U}$  contains at most a finite number of elements of  $D$ . So  $D$  is discrete. Also, since  $\mathcal{W}_\alpha$  is a partial refinement of  $\mathcal{U}$ ,  $\mathcal{W}_\alpha^* \cap D_\beta \neq \emptyset$  if and only if  $\alpha = \beta$ . Therefore  $D$  distinguishes  $\mathcal{W}$ .

*Theorem 3. Every  $T_1$   $\delta\theta$ -refinable space is irreducible.*

*Proof.* Let  $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$  be a  $\delta\theta$ -cover of a  $T_1$  space  $X$ . Set  $W_0 = \{\emptyset\}$  and  $D_0 = \emptyset$ . Let  $n \in \omega$  and assume that  $W_0, \dots, W_{n-1}$  are open partial refinements of  $\mathcal{G}$ . Let  $C_n = \{x \in X: \text{ord}(x, \mathcal{G}_n) \leq \aleph_0\} - \bigcup_{m=0}^{n-1} W_m^*$ . By Lemma 2 there is an open partial refinement  $W_n$  of  $\mathcal{G}_n$  covering  $C_n$  and a subset  $D_n$  of  $C_n$  such that  $D_n$  distinguishes  $W_n$  and is discrete in  $X$ .

For every  $n \in \omega$  let  $W'_n = \{W - \bigcup_{m < n} D_m: W \in W_n\}$ . Let  $W = \bigcup_{n \in \omega} W'_n$  and let  $D = \bigcup_{n \in \omega} D_n$ . Then  $W$  is an open refinement of  $\mathcal{G}$ ,  $D$  distinguishes  $W$ , and  $D$  is discrete in  $X$ .

The next theorem is a corollary of Theorem 3.

*Theorem (Aull [2]). Every  $\aleph_1$ -compact  $T_1$   $\delta\theta$ -refinable space is Lindelöf.*

In [5] J. R. Boone proves that every  $\aleph_1$ -compact  $T_3$  irreducible space has the star-finite property. A space  $X$  has the star-finite property if every open cover of  $X$  has a star-finite open refinement.

*Corollary 4. Every  $\aleph_1$ -compact  $T_3$   $\delta\theta$ -refinable space has the star-finite property.*

A space  $X$  is  $[\alpha, \infty)$ -compact if every open cover of  $X$  has a subcover of cardinality  $< \alpha$ . In [6] J. R. Boone shows that if  $\alpha$  is a regular cardinal then every  $T_1$   $\alpha$ -compact space that is irreducible of order  $\alpha$  is  $[\alpha, \infty)$ -compact. It follows that if  $\alpha$  is a regular cardinal then every  $\alpha$ -compact  $T_1$   $\delta\theta$ -refinable space is  $[\alpha, \infty)$ -compact.



Corollary 5 generalizes this result to any infinite cardinal.

*Corollary 5. Every  $\alpha$ -compact  $T_1$   $\delta\theta$ -refinable space is  $(\alpha, \infty)$ -compact.*

### III. Weakly $\delta\theta$ -Refinable Spaces

*Lemma 6. Let  $X$  be a  $T_1$  space and let  $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$  be a weak  $\delta\theta$ -cover of  $X$ . Let  $\mathcal{H} = \{\mathcal{G}_n^* : n \in \omega\}$ . Let  $r \in \omega$ , let  $F \subset \omega$  such that  $|F| = r$ , and let  $U$  be an open subset of  $X$  such that  $\{x \in X : \text{ord}(x, \mathcal{H}) < r\} \subset U$ . Let  $C \subset (\bigcap_{n \in F} \mathcal{G}_n^*) - ((\bigcup_{n \in \omega - F} \mathcal{G}_n^*) \cup U)$ . Then for every  $y \in C$  there is an open partial refinement  $\mathcal{W}$  of  $\mathcal{G}$  and a subset  $D$  of  $C$  such that  $y \in D$ ,  $D$  distinguishes  $\mathcal{W}$ ,  $D$  is discrete in  $X$ , and  $\text{st}(d, \mathcal{G}_n) \cap C \subset \bigcup \mathcal{W}$  for any  $d \in D$  and any  $n \in F$  such that  $\text{ord}(d, \mathcal{G}_n) \leq \aleph_0$ .*

*Proof.* Set  $d_1 = y$  and  $A_1 = \{G : d_1 \in G, G \in \mathcal{G}_n, \text{ and } \text{ord}(d_1, \mathcal{G}_n) \leq \aleph_0\}$ . Let  $\{A_i : i \in I_1\}$  be an enumeration of  $A_1$ . Set  $V_1 = A_1$  and, for bookkeeping purposes, set  $s_1 = m(1) = 1$ .

Assume that  $A_1, \dots, A_n$  are subsets of  $\mathcal{G}$  and that  $\{A_i : i \in I_j \text{ and } i \geq s_j\}$  is an enumeration of  $A_j$  for  $j = 1, \dots, n$ . Further assume that  $V_1, \dots, V_n$  have been chosen from  $\bigcup_{j=1}^n A_j$  and that  $V_k = A_{m(k)}$  where  $m(k) \in I_j(k)$  for  $k = 1, \dots, n$ . Finally, assume that  $[(\bigcup_{j=1}^n A_j^*) \cap C] - \bigcup_{j=1}^n V_j \neq \emptyset$ . We must select  $V_{n+1}$  and construct  $A_{n+1}$ .

Let  $m(n+1) = \min\{i : 1 \leq j \leq n, i \in I_j, \text{ and } A_i \cap C \not\subset \bigcup_{k=1}^n V_k\}$ . This can be done since the set in question is nonempty by assumption. Then  $m(n+1)$  is in one of  $I_1, \dots, I_n$ .

Say  $m(n+1) \in I_{j(n+1)}$ . Set  $V_{n+1} = A_{m(n+1)}$  and pick  $d_{n+1} \in (V_{n+1} \cap C) - \bigcup_{k=1}^n V_k$ . Let  $A_{n+1} = \{G: d_{n+1} \in G, G \in \mathcal{G}_k, \text{ord}(d_{n+1}, \mathcal{G}_k) \leq \aleph_0, \text{ and } G \notin \bigcup_{j=1}^n A_j\}$ . Let  $s_{n+1} = \min\{i \in I_{n+1}: i > m(n+1)\}$ . Enumerate  $A_{n+1}$  as  $\{A_i: i \in I_{n+1} \text{ and } i \geq s_{n+1}\}$ .

Let  $W_1 = V_1$  and, for every  $n > 1$ , let  $W_n = V_n - \{d_m: m < n\}$ . Let  $\mathcal{W} = \{W_n: n \in \omega\}$  and  $D = \{d_n: n \in \omega\}$ . Clearly  $\mathcal{W}$  is an open partial refinement of  $\mathcal{G}$ ,  $y \in D$ , and  $D$  distinguishes  $\mathcal{W}$ .

Let  $d \in D$  and  $m \in F$  such that  $\text{ord}(d, \mathcal{G}_m) \leq \aleph_0$ . Let  $G \in \mathcal{G}_m$  such that  $d \in G$ . Then  $G \in A_j$  for some  $j \in \omega$ . Let  $i \in I_j$  such that  $G = A_i$ . Now  $\{m(k): k \in \omega\}$  is strictly increasing, so we can define  $n+1 = \min\{k: m(k) > i\}$ . If  $n+1 \leq j$  then  $m(n+1) \leq m(j) < s_j \leq i$ , a contradiction. Thus  $j \leq n$ . Then by the definition of  $m(n+1)$ ,  $A_i \cap C \subset \bigcup_{k=0}^n V_k \subset \mathcal{W}$ . Therefore  $\text{st}(d, \mathcal{G}_m) \cap C \subset \mathcal{W}$ .

Finally, we must show that  $D$  is discrete. It will be useful to break  $D$  up into a finite number of parts. For every  $m \in F$  let  $L_m = \{d \in D: \text{ord}(d, \mathcal{G}_m) \leq \aleph_0\}$ . Note that  $D = \bigcup_{m \in F} L_m$ . We will show that  $G \cap L_m$  is finite for every  $G \in \mathcal{G}_m$ . Let  $G \in \mathcal{G}_m$  and assume that  $d \in G \cap L_m$ . As before, let  $G = A_i$  and  $n = \min\{k: m(k) > i\}$ . Then  $G \cap C = A_i \cap C \subset \bigcup_{k=0}^n V_k$ . Therefore  $G \cap D$  is finite and, in particular,  $G \cap L_m$  is finite.

Now let  $x \in X$ . Assume that  $x \in \bigcap_{n \in F} \mathcal{G}_n^*$ . For every  $n \in F$  let  $G_n \in \mathcal{G}_n$  such that  $x \in G_n$ . Then  $\bigcap_{n \in F} G_n$  is a neighborhood of  $x$  that contains at most a finite number of elements of  $D$ . If  $x \notin \bigcap_{n \in F} \mathcal{G}_n^*$  then either  $x \in U$  or  $x \in \mathcal{G}_n^*$

for some  $n \in \omega - F$ . So either  $U$  or  $\mathcal{G}_n^*$  is a neighborhood of  $x$  that misses  $D$  entirely.

*Lemma 7.* Let  $X$  be a  $T_1$  space and let  $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$  be a weak  $\overline{\delta\theta}$ -cover of  $X$ . Let  $\mathcal{H} = \{\mathcal{G}_n^* : n \in \omega\}$ . Let  $r \in \omega$ , let  $F \subset \omega$  such that  $|F| = r$ , and let  $U$  be an open subset of  $X$  such that  $\{x \in X : \text{ord}(x, \mathcal{H}) < r\} \subset U$ . Let  $C = (\bigcap_{n \in F} \mathcal{G}_n^*) - ((\bigcup_{n \in \omega - F} \mathcal{G}_n^*)U)$ . Then there is an open partial refinement  $\mathcal{W}$  of  $\mathcal{G}$  covering  $C$  and a subset  $D$  of  $C$  such that  $D$  distinguishes  $\mathcal{W}$  and is discrete in  $X$ .

*Proof.* Let  $\mathcal{W}_0 = \{\emptyset\}$  and  $D_0 = \emptyset$ . Assume that  $\mathcal{W}_\alpha$  is an open partial refinement of  $\mathcal{G}$  for all  $0 \leq \alpha < \beta$  and that  $C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^* \neq \emptyset$ . Let  $C_\beta = C - \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^*$  and pick  $y \in C_\beta$ . By Lemma 6 there is an open partial refinement  $\mathcal{W}_\beta$  of  $\mathcal{G}$  and a subset  $D_\beta$  of  $C_\beta$  such that  $y \in D_\beta$ ,  $D_\beta$  distinguishes  $\mathcal{W}_\beta$ ,  $D_\beta$  is discrete in  $X$ , and  $\text{st}(d, \mathcal{G}_n) \cap C_\beta \subset \bigcup \mathcal{W}_\beta$  for every  $d \in D_\beta$  and  $n \in F$  such that  $\text{ord}(d, \mathcal{G}_n) \leq \aleph_0$ .

For some ordinal number  $\sigma$  we must have  $C \subset \bigcup_{\alpha < \sigma} \mathcal{W}_\alpha^*$ . Before defining  $\mathcal{W}$ , we must show that  $D = \bigcup_{\alpha < \sigma} D_\alpha$  is discrete. For each  $n \in F$ , let  $L_n = \{d \in D : \text{ord}(d, \mathcal{G}_n) \leq \aleph_0\}$ . Notice that  $D = \bigcup_{n \in F} L_n$ . We claim that for every  $n \in F$  and  $G \in \mathcal{G}_n$ ,  $G \cap L_n$  is finite.

Let  $G \in \mathcal{G}_n$  and assume that  $G \cap L_n \neq \emptyset$ . Let  $d \in G \cap L_n \cap D_\beta$ . Now  $G \cap C \subset \text{st}(d, \mathcal{G}_n) \cap C \subset \bigcup_{\alpha < \beta} \mathcal{W}_\alpha^*$ . It follows that  $G \cap D_\alpha = \emptyset$  if  $\alpha > \beta$  and  $G \cap L_n \cap D_\alpha = \emptyset$  if  $\alpha < \beta$ . Therefore  $G \cap L_n = G \cap L_n \cap D_\beta$ , and the proof of Lemma 6 shows that  $G \cap L_n \cap D_\beta$  is finite.

Let  $x \in X$ . Assume that  $x \in \bigcap_{n \in F} \mathcal{G}_n^*$ . For each  $n \in F$  let  $G_n \in \mathcal{G}_n$  such that  $x \in G_n$ . Then  $\bigcap_{n \in F} G_n$  is a neighborhood

of  $x$  and  $|(\bigcap_{n \in F} G_n) \cap D| = |\bigcup_{n \in F} [(\bigcap_{n \in F} G_n) \cap L_n]| < \aleph_0$ .

If  $x \notin \bigcap_{n \in F} G_n^*$  then either  $x \in U$  or there is  $n \in \omega - F$  such that  $x \in G_n^*$ . So either  $U$  of  $G_n^*$  is a neighborhood of  $x$  that misses  $D$  entirely. Therefore  $D$  is discrete.

Now let  $\mathcal{W}'_\beta = \{W - \bigcup_{\alpha < \beta} D_\alpha : W \in \mathcal{W}_\beta\}$  for every  $\beta < \sigma$ . Set  $\mathcal{W} = \bigcup_{\alpha < \sigma} \mathcal{W}'_\alpha$ . Then  $\mathcal{W}$  is an open partial refinement of  $\mathcal{G}$  covering  $C$  and  $D$  distinguishes  $\mathcal{W}$ .

*Lemma 8.* Let  $X$  be a  $T_1$  space and let  $\mathcal{G} = \bigcup_{n \in \omega} G_n$  be a weak  $\overline{\delta\theta}$ -cover of  $X$ . Let  $r \in \omega$ . Let  $\mathcal{H} = \{G_n^* : n \in \omega\}$  and let  $U$  be an open subset of  $X$  such that  $\{x \in X : \text{ord}(x, \mathcal{H}) < r\} \subset U$ . Let  $C = \{x \in X : \text{ord}(x, \mathcal{H}) = r\} - U$ . Then there is an open partial refinement  $\mathcal{W}$  of  $\mathcal{G}$  covering  $C$  and a subset  $D$  of  $C$  that distinguishes  $\mathcal{W}$  and is discrete in  $X$ .

*Proof.* Let  $\mathcal{J} = \{F \subset \omega : |F| = r\}$  and let  $\{F_n : n \in \omega\}$  be an enumeration of  $\mathcal{J}$ . Set  $\mathcal{W}_0 = \{\emptyset\}$  and  $D_0 = \emptyset$ .

Assume that  $\mathcal{W}_0, \dots, \mathcal{W}_{n-1}$  are open partial refinements of  $\mathcal{G}$ . Let  $C_n = (\bigcap_{m \in F_n} G_m^*) - [(\bigcup_{m \in \omega - F_n} G_m^*) \cup U \cup (\bigcup_{m=0}^{n-1} \mathcal{W}_m^*)]$ .

By Lemma 7 there is an open partial refinement  $\mathcal{W}_n$  of  $\mathcal{G}$  covering  $C_n$  and a subset  $D_n$  of  $C_n$  that distinguishes  $\mathcal{W}_n$  and is discrete in  $X$ .

For every  $n \in \omega$  let  $\mathcal{W}'_n = \{W - \bigcup_{m < n} D_m : W \in \mathcal{W}_n\}$ . Let  $\mathcal{W} = \bigcup_{n \in \omega} \mathcal{W}'_n$  and  $D = \bigcup_{n \in \omega} D_n$ . Then  $\mathcal{W}$  is an open partial refinement of  $\mathcal{G}$  and  $D$  distinguishes  $\mathcal{W}$ . To show that  $\mathcal{W}$  covers  $C$ , let  $x \in C$ , let  $F = \{n \in \omega : x \in G_n^*\}$ , and let  $m \in \omega$  such that  $F = F_m$ . If  $x \notin \bigcup_{k < m} \mathcal{W}'_k$ , then  $x \in C_m$ , so  $x \in \mathcal{W}'_m$ . Thus  $\mathcal{W}$  covers  $C$ .

It remains to show that  $D$  is discrete. Let  $x \in X$ . If  $x \in U$  then  $U$  is a neighborhood of  $x$  that misses  $D$

entirely. So we may assume that  $\text{ord}(x, \mathcal{H}) \geq r$  and  $x \notin U$ . If  $\text{ord}(x, \mathcal{H}) > r$  then  $F = \{n \in \omega : x \in \mathcal{G}_n^*\}$  has more than  $r$  elements, so  $\bigcap_{n \in F} \mathcal{G}_n^*$  is a neighborhood of  $x$  that misses  $D$  entirely. If  $\text{ord}(x, \mathcal{H}) = r$  then  $x \in C$ , so  $x \in U \cup W$ . Let  $W \in \mathcal{W}$  such that  $x \in W$ . Then  $W$  is a neighborhood of  $x$  that contains exactly one element of  $D$ . Thus  $D$  is discrete.

*Theorem 9.* Every  $T_1$  weakly  $\overline{\delta\theta}$ -refinable space is irreducible.

*Proof.* Let  $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$  be a weak  $\overline{\delta\theta}$ -cover of a  $T_1$  space  $X$ . Let  $\mathcal{H} = \{\mathcal{G}_n^* : n \in \omega\}$ . Set  $\mathcal{W}_0 = \{\emptyset\}$  and  $D_0 = \emptyset$ .

Assume that  $\mathcal{W}_0, \dots, \mathcal{W}_{n-1}$  are open partial refinements of  $\mathcal{G}$  and that  $\{x \in X : \text{ord}(x, \mathcal{H}) < n\} \subset \bigcup_{m=0}^{n-1} \mathcal{W}_m^*$ . Set  $C_n = \{x \in X : \text{ord}(x, \mathcal{H}) = n\} - \bigcup_{m=0}^{n-1} \mathcal{W}_m^*$ . By Lemma 8, there is an open partial refinement  $\mathcal{W}_n$  of  $\mathcal{G}$  covering  $C_n$  and a subset  $D_n$  of  $C_n$  such that  $D_n$  distinguishes  $\mathcal{W}_n$  and is discrete in  $X$ .

For every  $n \in \omega$  let  $\mathcal{W}'_n = \{W - \bigcup_{m < n} D_m : W \in \mathcal{W}_n\}$ .

Set  $\mathcal{W} = \bigcup_{n \in \omega} \mathcal{W}'_n$  and  $D = \bigcup_{n \in \omega} D_n$ . Then  $\mathcal{W}$  is an open refinement of  $\mathcal{G}$  and  $D$  distinguishes  $\mathcal{W}$ . Therefore  $D$  is discrete and  $\mathcal{W}$  is minimal.

Theorem 9 generalizes the result by J. C. Smith [14] that  $\aleph_1$ -compact, countably compact, weakly  $\overline{\delta\theta}$ -refinable spaces are metacompact and hence irreducible.

*Corollary 10.* Every  $\aleph_1$ -compact  $T_3$  weakly  $\overline{\delta\theta}$ -refinable space has the star-finite property.

*Corollary 11.* Every  $\alpha$ -compact  $T_1$  weakly  $\overline{\delta\theta}$ -refinable space is  $[\alpha, \infty)$ -compact.

In particular, every  $\aleph_1$ -compact  $T_1$  weakly  $\overline{\delta\theta}$ -refinable space is Lindelöf (J. C. Smith [14]).

Theorem 9 also provides a partial answer to a question raised by J. C. Smith in [15] regarding the shrinkability of weakly  $\overline{\delta\theta}$ -refinable spaces. He defines property  $S^*$  to mean that every minimal open cover is shrinkable to a linearly closure-preserving closed collection. Then any  $T_1$  weakly  $\overline{\delta\theta}$ -refinable (or  $\delta\theta$ -refinable) space that satisfies property  $S^*$  is shrinkable.

## References

- [1] R. Arens and J. Dugundji, *Remarks on the concept of compactness*, Portugaliae Math. 9 (1950), 141-143.
- [2] C. E. Aull, *A generalization of a theorem of Aquaro*, Bull. Austral. Math. Soc. 9 (1973), 105-108.
- [3] H. Bennett and D. Lutzer, *A note on weak  $\theta$ -refinability*, Gen. Topology Appl. 2 (1972), 49-54.
- [4] J. R. Boone, *On irreducible spaces*, Bull. Austral. Math. Soc. 12 (1975), 143-148.
- [5] \_\_\_\_\_, *On irreducible spaces II*, Pac. J. Math. 62 (1976), 351-357.
- [6] \_\_\_\_\_, *On the cardinality relationships between discrete collections and open covers*, Pac. J. Math. 62 (1976), 359-364.
- [7] \_\_\_\_\_, *A note on  $\delta\theta$ -refinable spaces*, Set Theoretic Topology, Academic Press, New York, 1977.
- [8] S. W. Davis, *Spaces with linearly ordered local bases*, Top. Proc. 3 (1978), 37-51.
- [9] S. Davis and J. Smith, *The paracompactness of pre-paracompact spaces*, Top. Proc. 4 (1979), 345-360.
- [10] P. DeCaux, *A collectionwise normal weakly  $\theta$ -refinable Dowker space which is neither irreducible nor real-compact*, Top. Proc. 1 (1976), 67-77.
- [11] E. Van Douwen and H. Wicke, *A real, weird topology on the reals*, Houston J. Math. 3 (1977), 141-152.

- [12] R. Hodel and J. Vaughan, *A note on  $[\alpha, \beta]$ -compactness*, Gen. Topology Appl. 4 (1974), 179-189.
- [13] J. C. Smith, *Properties of weak  $\bar{\theta}$ -refinable spaces*, Proc. Am. Math. Soc. 53 (1975), 511-517.
- [14] \_\_\_\_\_, *A remark on irreducible spaces*, Proc. Am. Math. Soc. 57 (1976), 133-139.
- [15] \_\_\_\_\_, *Applications of shrinkable covers*, Proc. Am. Math. Soc. 73 (1979), 379-387.
- [16] J. Worrell and H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), 820-830.

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