6-2017

A Trace Formula for Foliated Flows (working paper)

Jesús A. Álvarez López
Universidade de Santiago de Compostela, jesus.alvarez@usc.es

Yuri A. Kordyukov

Eric Leichtnam

Follow this and additional works at: http://ecommons.udayton.edu/topology_conf
Part of the Geometry and Topology Commons, and the Special Functions Commons

eCommons Citation
http://ecommons.udayton.edu/topology_conf/20

This Topology + Dynamics and Continuum Theory is brought to you for free and open access by the Department of Mathematics at eCommons. It has been accepted for inclusion in Summer Conference on Topology and Its Applications by an authorized administrator of eCommons. For more information, please contact frice1@udayton.edu, msclangen1@udayton.edu.
A trace formula for foliated flows
work in progress
joint with Yuri Kordyukov & Eric Leichtnam

J.A. Álvarez López

Universidade de Santiago de Compostela

32nd Summer Conference on Topology and its Applications
Dayton, 2017
The trace formula
Case of non-singular foliated flows
General case
• $M$ a closed manifold, $\dim M = n$.
• $\mathcal{F}$ a foliation on $M$, $\text{codim} \mathcal{F} = 1$.
• $\phi = (\phi^t)$ a foliated flow on $M$, leaves $\to$ leaves.
• $M^0$ the union of leaves with fixed points.
• $M^1 = M \setminus M^0$. 
Foliated flows

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a foliation on $M$, $\operatorname{codim} \mathcal{F} = 1$.
- $\phi = (\phi^t)$ a foliated flow on $M$, leaves $\rightarrow$ leaves.
- $M^0$ the union of leaves with fixed points.
- $M^1 = M \setminus M^0$. 
Foliated flows

- $M$ a closed manifold, $\dim M = n$.  
- $\mathcal{F}$ a foliation on $M$, $\text{codim} \mathcal{F} = 1$.  
- $\phi = (\phi^t)$ a foliated flow on $M$, leaves $\rightarrow$ leaves.  
- $M^0$ the union of leaves with fixed points.  
- $M^1 = M \setminus M^0$.  

J.A. Álvarez López  |  A trace formula for foliated flows
Foliated flows

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a foliation on $M$, $\text{codim} \mathcal{F} = 1$.
- $\phi = (\phi^t)$ a foliated flow on $M$, leaves $\to$ leaves.
- $M^0$ the union of leaves with fixed points.
- $M^1 = M \setminus M^0$. 
Foliated flows

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a foliation on $M$, $\text{codim} \mathcal{F} = 1$.
- $\phi = (\phi^t)$ a \textit{foliated flow} on $M$, leaves $\rightarrow$ leaves.
- $M^0$ the union of leaves with fixed points.
- $M^1 = M \setminus M^0$. 
Hypotheses

1. The closed orbits are simple: $c$, any period $\ell$, $x \in c$,

$$\det(\operatorname{id} - \phi^\ell_*: T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0,$$

$\implies \epsilon_\ell(c) = \operatorname{sign} \det.$

2. The fixed points are simple: $p$,

$$\det(\operatorname{id} - \phi^t_*: T_pM \to T_pM) \neq 0 \quad \forall t \neq 0,$$

$\implies \epsilon_p = \operatorname{sign} \det.$

$\implies \phi^t_* = e^{\kappa_p t}$ on $N_p\mathcal{F} := T_pM / T_p\mathcal{F}, \quad \kappa_p \neq 0.$

$\implies M^0$ is a finite union of compact leaves.

3. $\phi^t \cap \mathcal{F}$ on $M^1 := M \setminus M^0.$
The trace formula
Case of non-singular foliated flows
General case

Hypotheses

1. The closed orbits are simple: $c$, any period $\ell$, $x \in c$,

\[
\det(\text{id} - \phi_{\ast}^{\ell} : T_xF \to T_xF) \neq 0,
\]

$\Rightarrow \epsilon_{\ell}(c) = \text{sign det}.$

2. The fixed points are simple: $p$,

\[
\det(\text{id} - \phi_{\ast}^{t} : T_pM \to T_pM) \neq 0 \quad \forall t \neq 0,
\]

$\Rightarrow \epsilon_p = \text{sign det}.$

$\Rightarrow \phi_{\ast}^{t} = e^{\kappa_p t}$ on $N_pF := T_pM / T_pF$, $\kappa_p \neq 0$.

$\Rightarrow M^0$ is a finite union of compact leaves.

3. $\phi^{t} \pitchfork F$ on $M^1 := M \setminus M^0$. 

J.A. Álvarez López

A trace formula for foliated flows
Hypotheses

1. The closed orbits are **simple**: \( c \), any period \( \ell \), \( x \in c \),

\[
\det(\text{id} - \phi_{\ell}^*: T_x\mathcal{F} \rightarrow T_x\mathcal{F}) \neq 0,
\]

\( \iff \epsilon_{\ell}(c) = \text{sign det.} \)

2. The fixed points are **simple**: \( p \),

\[
\det(\text{id} - \phi_{\star}^t : T_pM \rightarrow T_pM) \neq 0 \quad \forall t \neq 0,
\]

\( \iff \epsilon_{p} = \text{sign det.} \)

\( \iff \phi_{\star}^t = e^{\kappa_p t} \text{ on } N_p\mathcal{F} := T_pM/T_p\mathcal{F}, \ \ \kappa_p \neq 0. \)

\( \iff M^0 \text{ is a finite union of compact leaves.} \)

3. \( \phi^t \pitchfork \mathcal{F} \text{ on } M^1 := M \setminus M^0. \)
Hypotheses

1. The closed orbits are simple: \( c, \) any period \( \ell, \ x \in c, \)

\[
\text{det}(\text{id} - \phi_{\ast}^{\ell} : T_x \mathcal{F} \to T_x \mathcal{F}) \neq 0 ,
\]

\( \xrightarrow{\text{}} \epsilon_{\ell}(c) = \text{sign det.} \)

2. The fixed points are simple: \( p, \)

\[
\text{det}(\text{id} - \phi_{\ast}^{t} : T_p M \to T_p M) \neq 0 \ \ \forall t \neq 0 ,
\]

\( \xrightarrow{\text{}} \epsilon_p = \text{sign det.} \)

\( \xrightarrow{\text{}} \phi_{\ast}^{t} = e^{\nu_p t} \text{ on } N_p \mathcal{F} := T_p M / T_p \mathcal{F}, \ \nu_p \neq 0. \)

\( \xrightarrow{\text{}} M^0 \text{ is a finite union of compact leaves.} \)

3. \( \phi^{t} \cap \mathcal{F} \text{ on } M^1 := M \setminus M^0. \)
Hypotheses

1. The closed orbits are simple: $c$, any period $\ell$, $x \in c$,

$$\det(\text{id} - \phi^\ell_* : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0 ,$$

$\implies \epsilon_\ell(c) = \text{sign det.}$

2. The fixed points are simple: $p$,

$$\det(\text{id} - \phi^t_* : T_pM \to T_pM) \neq 0 \quad \forall t \neq 0 ,$$

$\implies \epsilon_p = \text{sign det.}$

$\implies \phi^t_* = e^{\kappa_p t}$ on $N_p\mathcal{F} := T_pM / T_p\mathcal{F}$, $\kappa_p \neq 0$.

$\implies M^0$ is a finite union of compact leaves.

3. $\phi^t \pitchfork \mathcal{F}$ on $M^1 := M \setminus M^0$. 
Hypotheses

1. The closed orbits are simple: \( c, \) any period \( \ell, \) \( x \in c, \)
   \[
   \det(\text{id} - \phi_\ell^*: T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0 ,
   \]
   \( \leadsto \epsilon_\ell(c) = \text{sign det.} \)

2. The fixed points are simple: \( p, \)
   \[
   \det(\text{id} - \phi_t^* : T_pM \to T_pM) \neq 0 \quad \forall t \neq 0 ,
   \]
   \( \leadsto \epsilon_p = \text{sign det.} \)
   \( \leadsto \phi_t^* = e^{\kappa_p t} \) on \( N_p\mathcal{F} := T_pM/T_p\mathcal{F}, \) \( \kappa_p \neq 0. \)
   \( \leadsto M^0 \) is a finite union of compact leaves.

3. \( \phi_t \pitchfork \mathcal{F} \) on \( M^1 := M \setminus M^0. \)
Hypotheses

1. The closed orbits are simple: \( c, \) any period \( \ell, \ x \in c, \)
   \[ \det (\text{id} - \phi^\ell_* : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \]
   \( \Leftrightarrow \epsilon_\ell(c) = \text{sign det}. \)

2. The fixed points are simple: \( p, \)
   \[ \det (\text{id} - \phi^t_* : T_pM \to T_pM) \neq 0 \quad \forall t \neq 0, \]
   \( \Leftrightarrow \epsilon_p = \text{sign det}. \)
   \( \Leftrightarrow \phi^t_* = e^{\chi_p t} \) on \( N_p\mathcal{F} := T_pM / T_p\mathcal{F}, \ \chi_p \neq 0. \)
   \( \Leftrightarrow M^0 \) is a finite union of compact leaves.

3. \( \phi^t \cap \mathcal{F} \) on \( M^1 := M \setminus M^0. \)
Example
Example
Define:

- a “leafwise cohomology” $H^i$, $\rightsquigarrow \phi^* = (\phi^t^*)$ on $H^i$,
- a “distributional trace” $\text{Tr}(\phi^*|_{H^i}) \in C^{-\infty}(\mathbb{R})$,
- “Leftschetz distribution”

$L(\phi) := \text{Tr}^s(\phi^*) := \sum_i (-1)^i \text{Tr}(\phi^*|_{H^i}) \in C^{-\infty}(\mathbb{R})$.

Prove a trace formula: on $\mathbb{R}^+$,

$L(\phi) = \sum_c \ell(c) \sum_{k=1}^{\infty} \epsilon_k \delta_\ell(c) \delta_k(c) + \sum_p \frac{\epsilon_p}{|1 - e^{\epsilon_p t}|}$,

c runs in the closed orbits and $p$ in the fixed points of $\phi$,
$\ell(c)$ minimal positive period of $c$. 
The problem of the trace formula
Guillemin-Sternberg, C. Deninger

Define:
- a “leafwise cohomology” $H^i$, $\mapsto \phi^* = (\phi^{t*})$ on $H^i$,
- a “distributional trace” $\text{Tr}((\phi^*)_{\mid H^i}) \in C^{-\infty}(\mathbb{R})$,
- “Leftschetz distribution”
  $$L(\phi) := \text{Tr}^s(\phi^*) := \sum_i (-1)^i \text{Tr}(\phi^*_{\mid H^i}) \in C^{-\infty}(\mathbb{R}).$$

Prove a trace formula:
$$L(\phi) = \sum_c \ell(c) \sum_{k=1}^{\infty} \varepsilon_k \delta_{k\ell(c)}(c) + \sum_p \frac{\varepsilon_p}{1 - e^{\varepsilon_p t}},$$
c runs in the closed orbits and $p$ in the fixed points of $\phi$,
$\ell(c)$ minimal positive period of $c$. 
The problem of the trace formula
Guillemin-Sternberg, C. Deninger

Define:
- a "leafwise cohomology" $H^i$, $\sim \phi^* = (\phi^t)$ on $H^i$,
- a "distributional trace" $\mathrm{Tr}(\phi^*|_{H^i}) \in C_-^\infty(\mathbb{R})$,
- "Leftschetz distribution"

$L(\phi) := \mathrm{Tr}^s(\phi^*) := \sum_{i} (-1)^i \mathrm{Tr}(\phi^*|_{H^i}) \in C_-^\infty(\mathbb{R})$.

Prove a trace formula: on $\mathbb{R}^+$,

$$L(\phi) = \sum_c \ell(c) \sum_{k=1}^\infty \epsilon_{k\ell}(c) (c) \delta_{k\ell}(c) + \sum_p \frac{\epsilon_p}{|1 - e^{\epsilon_p t}|},$$

$c$ runs in the closed orbits and $p$ in the fixed points of $\phi$,
$\ell(c)$ minimal positive period of $c$. 
Motivation

- **Guillemin-Sternberg**: Quantization.
- **C. Deninger**: Arithmetic Geometry (Berlin, ICM, 1998).
- Deninger’s program needs a version for **foliated spaces**. Arithmetic foliated spaces?
Guillemin-Sternberg: Quantization.


Deninger’s program needs a version for foliated spaces. Arithmetic foliated spaces?
Motivation

- **Guillemin-Sternberg**: Quantization.
- **C. Deninger**: Arithmetic Geometry (Berlin, ICM, 1998).
- Deninger’s program needs a version for *foliated spaces*. Arithmetic foliated spaces?
Non-singular foliated flows

- \( \phi \) has no fixed point. \( \Leftrightarrow \) infinitesimal generator \( X \neq 0 \).
- \( \Leftrightarrow \) a Riemannian metric on \( M \) so that \( |X| = 1 \) and \( X \perp F \).
- \( \Leftrightarrow F \) is defined by local Riemannian submersions: a bundle-like metric, a Riemannian foliation.
- Leafwise complex: \( C^\infty(M; \Lambda F) \), \( \Lambda F := \bigwedge T^* F \), \( d_F \) defined by the de Rham diff. operator on the leaves.
- \( \Leftrightarrow \) Reduced leafwise cohomology: \( \overline{H}(F) = \ker d_F / \text{im} d_F \).
- \( \Leftrightarrow \phi^t : \overline{H}^i(F) \rightarrow \overline{H}^i(F) \). Trace?
- \( \overline{H}^i(F) \) may be of infinite dimension.
Non-singular foliated flows

- $\phi$ has no fixed point. $\implies$ infinitesimal generator $X \neq 0$.
- $\implies$ a Riemannian metric on $M$ so that $|X| = 1$ and $X \perp \mathcal{F}$.
- $\implies$ $\mathcal{F}$ is defined by local Riemannian submersions: a bundle-like metric, a Riemannian foliation.

- Leafwise complex: $C^\infty(M; \wedge \mathcal{F})$, $\wedge \mathcal{F} := \bigwedge T^* \mathcal{F}$, $d_{\mathcal{F}}$ defined by the de Rham diff. operator on the leaves.
- $\implies$ Reduced leafwise cohomology: $\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}}/\text{im } d_{\mathcal{F}}$.
- $\implies$ $\phi^t : \overline{H}^i(\mathcal{F}) \rightarrow \overline{H}^i(\mathcal{F})$. Trace?
- $\overline{H}^i(\mathcal{F})$ may be of infinite dimension.
Non-singular foliated flows

- Φ has no fixed point. ↦ infinitesimal generator X ≠ 0.
- ↦ a Riemannian metric on M so that |X| = 1 and X ⊥ F.
- F is defined by local Riemannian submersions: a bundle-like metric, a Riemannian foliation.

Leafwise complex: \( C^\infty(M; \Lambda F) \), \( \Lambda F := \bigwedge T^*F \), d\(_{\mathcal{F}}\) defined by the de Rham diff. operator on the leaves.

- Reduced leafwise cohomology: \( \bar{H}(\mathcal{F}) = \ker d_{\mathcal{F}}/\operatorname{im} d_{\mathcal{F}} \).
- \( \phi^t \) : \( \bar{H}^i(\mathcal{F}) \to \bar{H}^i(\mathcal{F}) \). Trace?

\( \bar{H}^i(\mathcal{F}) \) may be of infinite dimension.
The trace formula
Case of non-singular foliated flows
General case

Non-singular foliated flows

- $\phi$ has no fixed point. $\Rightarrow$ infinitesimal generator $X \neq 0$.
- $\Rightarrow$ a Riemannian metric on $M$ so that $|X| = 1$ and $X \perp \mathcal{F}$.
  $\Rightarrow$ $\mathcal{F}$ is defined by local Riemannian submersions:
  a bundle-like metric, a Riemannian foliation.

Leafwise complex: $C^\infty(M; \wedge \mathcal{F})$, $\wedge \mathcal{F} := \wedge T^* \mathcal{F}$,
$d_\mathcal{F}$ defined by the de Rham diff. operator on the leaves.

- $\Rightarrow$ Reduced leafwise cohomology: $\overline{H}(\mathcal{F}) = \text{ker } d_\mathcal{F} / \text{im } d_\mathcal{F}$.
- $\Rightarrow \phi^t_* : \overline{H}^i(\mathcal{F}) \rightarrow \overline{H}^i(\mathcal{F})$. Trace?
  $\overline{H}^i(\mathcal{F})$ may be of infinite dimension.
Non-singular foliated flows

- \( \phi \) has no fixed point. \( \leadsto \) infinitesimal generator \( X \neq 0 \).

- \( \leadsto \) a Riemannian metric on \( M \) so that \( |X| = 1 \) and \( X \perp \mathcal{F} \).

- \( \leadsto \) \( \mathcal{F} \) is defined by local Riemannian submersions: a bundle-like metric, a Riemannian foliation.

- Leafwise complex: \( C^\infty(M; \Lambda \mathcal{F}) \), \( \Lambda \mathcal{F} := \bigwedge T^* \mathcal{F} \), \( d_{\mathcal{F}} \) defined by the de Rham diff. operator on the leaves.

- \( \leadsto \) Reduced leafwise cohomology: \( \overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \text{im} d_{\mathcal{F}} \).

- \( \leadsto \phi^* : \overline{H}^i(\mathcal{F}) \rightarrow \overline{H}^i(\mathcal{F}) \). Trace?

\( \overline{H}^i(\mathcal{F}) \) may be of infinite dimension.
The trace formula
Case of non-singular foliated flows
General case

Leafwise Hodge isomorphism
for any Riemannian foliation on a closed manifold, of arbitrary codimension

- $\delta_F$ on $C^\infty(M; \Lambda F)$ defined by the adjoint of $d_F$ on the leaves.
- Leafwise Laplacian $\Delta_F = d_F \delta_F + \delta_F d_F$.
- Bundle-like metric $\Rightarrow \Delta_F$ is symmetric in $L^2(M; \Lambda F)$.
- $\mathcal{H} = \ker \Delta_F$ in $C^\infty(M; \Lambda F)$, $L^2\mathcal{H} = \ker \Delta_F$ in $L^2(M; \Lambda F)$
- $\Pi : L^2(M; \Lambda F) \to L^2\mathcal{H}$ the orthogonal projection.
- $\exists$ a restriction $\Pi : C^\infty(M; \Lambda F) \to \mathcal{H}$ inducing $\overline{H}(F) \cong \mathcal{H}$.

(J.A., Y. Kordyukov, 2001)
Leafwise Hodge isomorphism
for any Riemannian foliation on a closed manifold, of arbitrary codimension

- $\delta_\mathcal{F}$ on $C^\infty(M; \Lambda\mathcal{F})$ defined by the adjoint of $d_\mathcal{F}$ on the leaves.
- Leafwise Laplacian $\Delta_\mathcal{F} = d_\mathcal{F}\delta_\mathcal{F} + \delta_\mathcal{F}d_\mathcal{F}$.
- Bundle-like metric $\Rightarrow \Delta_\mathcal{F}$ is symmetric in $L^2(M; \Lambda\mathcal{F})$.
- $\mathcal{H} = \ker \Delta_\mathcal{F}$ in $C^\infty(M; \Lambda\mathcal{F})$, $L^2\mathcal{H} = \ker \Delta_\mathcal{F}$ in $L^2(M; \Lambda\mathcal{F})$
- $\Pi : L^2(M; \Lambda\mathcal{F}) \to L^2\mathcal{H}$ the orthogonal projection.
- $\exists$ a restriction $\Pi : C^\infty(M; \Lambda\mathcal{F}) \to \mathcal{H}$ inducing $\overline{\mathcal{H}}(\mathcal{F}) \cong \mathcal{H}$.

(J.A., Y. Kordyukov, 2001)
Leafwise Hodge isomorphism
for any Riemannian foliation on a closed manifold, of arbitrary codimension

- $\delta_{\mathcal{F}}$ on $C^\infty(M; \Lambda\mathcal{F})$ defined by the adjoint of $d_{\mathcal{F}}$ on the leaves.
  \[ \leadsto \text{Leafwise Laplacian} \quad \Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}. \]

- Bundle-like metric $\Rightarrow \Delta_{\mathcal{F}}$ is symmetric in $L^2(M; \Lambda\mathcal{F})$.

- $\mathcal{H} = \ker \Delta_{\mathcal{F}}$ in $C^\infty(M; \Lambda\mathcal{F})$, $L^2\mathcal{H} = \ker \Delta_{\mathcal{F}}$ in $L^2(M; \Lambda\mathcal{F})$
  \[ \Pi : L^2(M; \Lambda\mathcal{F}) \to L^2\mathcal{H} \] the orthogonal projection.

- $\exists$ a restriction $\Pi : C^\infty(M; \Lambda\mathcal{F}) \to \mathcal{H}$ inducing $\overline{\mathcal{H}}(\mathcal{F}) \cong \mathcal{H}$.
Leafwise Hodge isomorphism
for any Riemannian foliation on a closed manifold, of arbitrary codimension

- $\delta_\mathcal{F}$ on $C^\infty(M; \Lambda \mathcal{F})$ defined by the adjoint of $d_\mathcal{F}$ on the leaves.
  $\leadsto$ Leafwise Laplacian $\Delta_\mathcal{F} = d_\mathcal{F}\delta_\mathcal{F} + \delta_\mathcal{F}d_\mathcal{F}$.

- Bundle-like metric $\Rightarrow$ $\Delta_\mathcal{F}$ is symmetric in $L^2(M; \Lambda \mathcal{F})$.

- $\mathcal{H} = \ker \Delta_\mathcal{F}$ in $C^\infty(M; \Lambda \mathcal{F})$, $L^2\mathcal{H} = \ker \Delta_\mathcal{F}$ in $L^2(M; \Lambda \mathcal{F})$.
  $\Pi : L^2(M; \Lambda \mathcal{F}) \to L^2\mathcal{H}$ the orthogonal projection.

- $\exists$ a restriction $\Pi : C^\infty(M; \Lambda \mathcal{F}) \to \mathcal{H}$ inducing $\overline{\mathcal{H}}(\mathcal{F}) \cong \mathcal{H}$.
∀ f ∈ 𝒞_c^∞(ℝ), the operator

\[ A_f = \int_{ℝ} \phi^t \cdot f(t) \, dt \circ \Pi. \]

is smoothing (\(\sim\) of trace class) \((\phi^t \circ \Pi\) is not.)

- \(L(\phi) = (f \mapsto Tr^s A_f) \in 𝒞^{-\infty}(ℝ).\)
- On \(ℝ^+,\)

\[ L(\phi) = \sum_c \ell(c) \sum_{k=0}^{\infty} \epsilon_c(k\ell(c)) \delta_{k\ell(c)}. \]
For non-singular foliated flows \( (J.A., Y. Kordyukov, 2002) \):

- For all \( f \in C_c^\infty(\mathbb{R}) \), the operator
  \[
  A_f = \int_{\mathbb{R}} \phi^* \cdot f(t) \, dt \circ \Pi.
  \]
  is smoothing (\( \sim \) of trace class) \( (\phi^* \circ \Pi \text{ is not}) \).

- \( L(\phi) = (f \mapsto \text{Tr}^s A_f) \in C^{-\infty}(\mathbb{R}) \).

- On \( \mathbb{R}^+ \),
  \[
  L(\phi) = \sum_c \ell(c) \sum_{k=0}^{\infty} \epsilon_c(k\ell(c)) \delta_{k\ell(c)}.
  \]
The trace formula
Case of non-singular foliated flows
General case

\textbf{Leftschetz trace formula}
for non-singular foliated flows \ (J.A., Y. Kordyukov, 2002)

\begin{itemize}
  \item \( \forall f \in C_c^\infty(\mathbb{R}) \), the operator
    \[ A_f = \int_{\mathbb{R}} \phi_t^* \cdot f(t) \, dt \circ \Pi. \]
  \item is smoothing (\( \sim \) of trace class) \ (\( \phi_t^* \circ \Pi \) is not.)
  \item \( L(\phi) = (f \mapsto \text{Tr}^s A_f) \in C^{-\infty}(\mathbb{R}). \)
  \item On \( \mathbb{R}^+ \),
    \[ L(\phi) = \sum_c \ell(c) \sum_{k=0}^{\infty} \epsilon_c(k\ell(c)) \delta_{k\ell(c)}. \]
\end{itemize}
Recall: $M = M^0 \sqcup M^1$, $M^0 = (\text{finite})$ union of (compact) leaves with fixed points.

- $\mathcal{F}$ is not Riemannian,
- $\mathcal{F}^1 := \mathcal{F}|_{M^1}$ is Riemannian,
- $\mathcal{F}$ is a transversely affine foliation almost without holonomy.

- The Schwartz kernel of $A_f$ is not smooth at $M^0$.
- $\sim (C^\infty(M; \Lambda \mathcal{F}), d_{\mathcal{F}})$ doesn’t work,
- $\sim$ another leafwise complex,
- $\sim$ elements of $C^{-\infty}(M; \Lambda \mathcal{F})$ with “nice” singularities at $M^0$. 

J.A. Álvarez López  A trace formula for foliated flows
Recall: $M = M^0 \sqcup M^1$, 
$M^0 = \text{(finite) union of (compact) leaves with fixed points.}$

$\mathcal{F}$ is not Riemannian, 
$\mathcal{F}^1 := \mathcal{F}|_{M^1}$ is Riemannian, 
$\mathcal{F}$ is a transversely affine foliation almost without holonomy.

The Schwartz kernel of $A_f$ is not smooth at $M^0$.

$\sim (C^\infty(M; \Lambda \mathcal{F}), d_\mathcal{F})$ doesn’t work, 
$\sim$ another leafwise complex, 
$\sim$ elements of $C^{-\infty}(M; \Lambda \mathcal{F})$ with “nice” singularities at $M^0$. 
Difficulties

- Recall: \( M = M^0 \sqcup M^1 \),  
  \( M^0 = \) (finite) union of (compact) leaves with fixed points.

- \( \mathcal{F} \) is not Riemannian,
  \( \mathcal{F}^1 := \mathcal{F}|_{M^1} \) is Riemannian,
  \( \mathcal{F} \) is a transversely affine foliation almost without holonomy.

- The Schwartz kernel of \( A_f \) is not smooth at \( M^0 \).

  - \( \rightsquigarrow (C^\infty(M; \Lambda \mathcal{F}), d_\mathcal{F}) \) doesn’t work,
  - \( \rightsquigarrow \) another leafwise complex,
  - \( \rightsquigarrow \) elements of \( C^{-\infty}(M; \Lambda \mathcal{F}) \) with “nice” singularities at \( M^0 \).
The trace formula
Case of non-singular foliated flows
General case

Difficulties

- Recall: $M = M^0 \sqcup M^1$,
  $M^0$ = (finite) union of (compact) leaves with fixed points.
- $\mathcal{F}$ is not Riemannian,
  $\mathcal{F}^1 := \mathcal{F}|_{M^1}$ is Riemannian,
  $\mathcal{F}$ is a transversely affine foliation almost without holonomy.
- The Schwartz kernel of $A_f$ is not smooth at $M^0$.
- $\rightsquigarrow (\mathcal{C}^\infty(M; \Lambda \mathcal{F}), dF)$ doesn’t work,
  $\rightsquigarrow$ another leafwise complex,
  $\rightsquigarrow$ elements of $\mathcal{C}^{-\infty}(M; \Lambda \mathcal{F})$ with “nice” singularities at $M^0$. 
\( \mathfrak{x}(M, \mathcal{F}) = \{ \text{infinitesimal transformations of } (M, \mathcal{F}) \} = \{ \text{infinitesimal generators of foliated flows} \}. \)

\( \mathfrak{x}(M, \mathcal{F}) \) generates the \( C^\infty(M) \)-module
\( \mathfrak{x}(M, M^0) = \{ Y \in \mathfrak{x}(M) \mid Y \text{ is tangent to } M^0 \}. \)

\( \mathfrak{x}(M, M^0) \rightarrow \text{Diff}(M, M^0; \wedge \mathcal{F}), \)
\( d\mathcal{F} \in \text{Diff}(M, M^0; \wedge \mathcal{F}). \)

\( H^s(M; \wedge \mathcal{F}) \) Sobolev space of order \( s \).
The trace formula
Case of non-singular foliated flows
General case

Distributional leafwise forms conormal to $M^0$

\[ \mathfrak{X}(M, \mathcal{F}) = \{ \text{infinitesimal transformations of } (M, \mathcal{F}) \} = \{ \text{infinitesimal generators of foliated flows} \}. \]

\[ \mathfrak{X}(M, \mathcal{F}) \text{ generates the } C^\infty(M)\text{-module } \mathfrak{X}(M, M^0) = \{ Y \in \mathfrak{X}(M) \mid Y \text{ is tangent to } M^0 \}. \]

\[ \mathfrak{X}(M, M^0) \cong \text{Diff}(M, M^0; \Lambda \mathcal{F}), \]
\[ d\mathcal{F} \in \text{Diff}(M, M^0; \Lambda \mathcal{F}). \]

\[ H^s(M; \Lambda \mathcal{F}) \text{ Sobolev space of order } s. \]
Distributional leafwise forms conormal to $M^0$

- $\mathfrak{X}(M,\mathcal{F}) = \{\text{infinitesimal transformations of } (M,\mathcal{F})\}$
  $= \{\text{infinitesimal generators of foliated flows}\}$.

- $\mathfrak{X}(M,\mathcal{F})$ generates the $C^\infty(M)$-module
  $\mathfrak{X}(M, M^0) = \{ Y \in \mathfrak{X}(M) \mid Y \text{ is tangent to } M^0 \}$.

- $\mathfrak{X}(M, M^0) \leadsto \text{Diff}(M, M^0; \wedge \mathcal{F})$,
  $d\mathcal{F} \in \text{Diff}(M, M^0; \wedge \mathcal{F})$.

- $H^s(M; \wedge \mathcal{F})$ Sobolev space of order $s$. 

The trace formula
Case of non-singular foliated flows
General case

Distributional leafwise forms conormal to $M^0$

- $\mathfrak{x}(M, \mathcal{F}) = \{\text{infinitesimal transformations of } (M, \mathcal{F})\}$
- $\mathfrak{x}(M, \mathcal{F}) = \{\text{infinitesimal generators of foliated flows}\}$.

- $\mathfrak{x}(M, \mathcal{F})$ generates the $C^\infty(M)$-module
  $\mathfrak{x}(M, M^0) = \{Y \in \mathfrak{x}(M) \mid Y \text{ is tangent to } M^0\}$.

- $\mathfrak{x}(M, M^0) \hookrightarrow \text{Diff}(M, M^0; \Lambda \mathcal{F})$, $d\mathcal{F} \in \text{Diff}(M, M^0; \Lambda \mathcal{F})$.

- $H^s(M; \Lambda \mathcal{F})$ Sobolev space of order $s$. 
Distributional leafwise forms conormal to $M^0$: 

$$I^{[s]}(M, M^0; \Lambda \mathcal{F}) = \{ \alpha \in H^s(M; \Lambda \mathcal{F}) \mid \text{Diff}(M, M^0; \Lambda \mathcal{F}) \cdot \alpha \subset H^s(M; \Lambda \mathcal{F}) \} ,$$

$$I(M, M^0; \Lambda \mathcal{F}) = \bigcup_s I^{[s]}(M, M^0; \Lambda \mathcal{F}) .$$

- $I(M, M^0; \Lambda \mathcal{F}), \ d_{\mathcal{F}}$ 
  \equiv \text{the continuous extension of } d_{\mathcal{F}} \text{ to } C^{-\infty}(M; \Lambda \mathcal{F}).$
- $\rightsquigarrow \overline{H}(I(M, M^0; \Lambda \mathcal{F}), d_{\mathcal{F}}).$
Distributional leafwise forms conormal to $M^0$ (contd.):

- Distributional leafwise forms conormal to $M^0$:

$$I^{[s]}(M, M^0; \Lambda \mathcal{F}) = \{ \alpha \in H^s(M; \Lambda \mathcal{F}) \mid \text{Diff}(M, M^0; \Lambda \mathcal{F}) \cdot \alpha \subset H^s(M; \Lambda \mathcal{F}) \},$$

$$I(M, M^0; \Lambda \mathcal{F}) = \bigcup_s I^{[s]}(M, M^0; \Lambda \mathcal{F}).$$

- $I(M, M^0; \Lambda \mathcal{F}), \ d_{\mathcal{F}}$

  $\equiv$ the continuous extension of $d_{\mathcal{F}}$ to $C^{-\infty}(M; \Lambda \mathcal{F})$.

- $\leadsto \overline{H}(I(M, M^0; \Lambda \mathcal{F}), d_{\mathcal{F}}).$
Distributional leafwise forms conormal to $M^0$ (contd.)

- Distributional leafwise forms conormal to $M^0$:

$$I^{[s]}(M, M^0; \Lambda\mathcal{F}) = \{ \alpha \in H^s(M; \Lambda\mathcal{F}) \mid \text{Diff}(M, M^0; \Lambda\mathcal{F}) \cdot \alpha \subset H^s(M; \Lambda\mathcal{F}) \} ,$$

$$I(M, M^0; \Lambda\mathcal{F}) = \bigcup_s I^{[s]}(M, M^0; \Lambda\mathcal{F}) .$$

- $I(M, M^0; \Lambda\mathcal{F})$, $d_{\mathcal{F}}$

  $d_{\mathcal{F}} \equiv$ the continuous extension of $d_{\mathcal{F}}$ to $C^{-\infty}(M; \Lambda\mathcal{F})$.

- $\sim \overline{H}(I(M, M^0; \Lambda\mathcal{F}), d_{\mathcal{F}}).$
The trace formula
Case of non-singular foliated flows
General case

Canonical short exact sequence

\( \alpha \in I(M, M^0; \Lambda \mathcal{F}) \mapsto \exists \alpha|_{M^1} \in C^\infty(M^1; \Lambda \mathcal{F}^1). \)

\( \mapsto \) a canonical short exact sequence

\[ 0 \to \{ \alpha \in I(M, M^0; \Lambda \mathcal{F}) \mid \text{supp} \, \alpha \subset M^0 \} \to I(M, M^0; \Lambda \mathcal{F}) \to \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda \mathcal{F}) \} \to 0. \]

\( \exists \) a non-canonical continuous section of complexes

\( I(M, M^0; \Lambda \mathcal{F}) \leftarrow \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda \mathcal{F}) \}. \)

\( \mapsto \) direct sum decomposition of \( \overline{H}(I(M, M^0; \Lambda \mathcal{F}), d\mathcal{F}). \)

\( \mapsto \) define \( L(\phi) \) on both terms of the direct sum, and study the corresponding trace formulae.
The trace formula
Case of non-singular foliated flows
General case

Canonical short exact sequence

\( \alpha \in I(M, M^0; \Lambda F) \leadsto \exists \alpha |_{M^1} \in C^\infty(M^1; \Lambda F^1). \)

\( \leadsto \) a canonical short exact sequence

\[
0 \to \{ \alpha \in I(M, M^0; \Lambda F) \mid \text{supp } \alpha \subset M^0 \} \\
\to I(M, M^0; \Lambda F) \to \{ \alpha |_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \} \to 0.
\]

\( \exists \) a non-canonical continuous section of complexes

\[
I(M, M^0; \Lambda F) \leftarrow \{ \alpha |_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \}.
\]

\( \leadsto \) direct sum decomposition of \( \overline{H}(I(M, M^0; \Lambda F), d_F). \)

\( \leadsto \) define \( L(\phi) \) on both terms of the direct sum, and study the corresponding trace formulae.
The trace formula
Case of non-singular foliated flows
General case

Canonical short exact sequence

- $\alpha \in I(M, M^0; \Lambda F) \leadsto \exists \alpha|_{M^1} \in C^\infty(M^1; \Lambda F^1)$.
- A canonical short exact sequence

\[
0 \rightarrow \{ \alpha \in I(M, M^0; \Lambda F) \mid \text{supp} \alpha \subset M^0 \} \\
\rightarrow I(M, M^0; \Lambda F) \rightarrow \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \} \rightarrow 0.
\]

- \exists a non-canonical continuous section of complexes

\[
I(M, M^0; \Lambda F) \leftarrow \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \}.
\]

- Direct sum decomposition of $\overline{H}(I(M, M^0; \Lambda F), dF)$.
- Define $L(\phi)$ on both terms of the direct sum, and study the corresponding trace formulae.
The trace formula
Case of non-singular foliated flows
General case

Canonical short exact sequence

- \( \alpha \in I(M, M^0; \Lambda F) \leadsto \exists \alpha|_{M^1} \in C^\infty(M^1; \Lambda F^1). \)
- \( \leadsto \) a canonical short exact sequence
  
  \[
  0 \to \{ \alpha \in I(M, M^0; \Lambda F) \mid \text{supp } \alpha \subset M^0 \} \to I(M, M^0; \Lambda F) \to \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \} \to 0 .
  \]

- \( \exists \) a non-canonical continuous section of complexes
  
  \[
  I(M, M^0; \Lambda F) \leftarrow \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \} .
  \]

- \( \leadsto \) direct sum decomposition of \( \overline{H}(I(M, M^0; \Lambda F), d_F). \)
- \( \leadsto \) define \( L(\phi) \) on both terms of the direct sum, and study the corresponding trace formulae.

J.A. Álvarez López
A trace formula for foliated flows
The trace formula
Case of non-singular foliated flows
General case

Canonical short exact sequence

- $\alpha \in I(M, M^0; \Lambda F) \leadsto \exists \alpha|M_1 \in C^\infty(M^1; \Lambda F^1)$.
- $\leadsto$ a canonical short exact sequence

$$0 \rightarrow \{ \alpha \in I(M, M^0; \Lambda F) \mid \text{supp } \alpha \subset M^0 \} \rightarrow I(M, M^0; \Lambda F) \rightarrow \{ \alpha|M_1 \mid \alpha \in I(M, M^0; \Lambda F) \} \rightarrow 0.$$

- $\exists$ a non-canonical continuous section of complexes

$$I(M, M^0; \Lambda F) \leftarrow \{ \alpha|M_1 \mid \alpha \in I(M, M^0; \Lambda F) \}.$$

- $\leadsto$ direct sum decomposition of $\overline{H}(I(M, M^0; \Lambda F), d_F)$.

- $\leadsto$ define $L(\phi)$ on both terms of the direct sum, and study the corresponding trace formulae.
Term supported on $M^0$

- Assume $\mathcal{F}$ transversely oriented $\sim \exists \omega, \eta \in C^\infty(M; \Lambda^1 M)$ such that $T\mathcal{F} = \ker \omega$ and $d\omega = \omega \wedge \eta$.

- $\mathcal{F}$ transversely affine $\iff$ we can assume $d\eta = 0$.

- Using $\delta$-sections at $M^0$ and their transverse derivatives,

\[
\{ \alpha \in I(M, M^0; \wedge \mathcal{F}) \mid \text{supp} \alpha \subset M^0 \} 
\equiv \bigoplus_{k=0}^{\infty} C^\infty(M^0; \Lambda M^0 \otimes \Omega^{-1} NM^0),
\]

\[
d\mathcal{F} \equiv \bigoplus_{k=0}^{\infty} (d_{M^0} + k \eta \wedge).
\]
Term supported on $M^0$

- Assume $\mathcal{F}$ transversely oriented $\leadsto \exists \omega, \eta \in C^\infty(M; \Lambda^1 M)$ such that $T\mathcal{F} = \ker \omega$ and $d\omega = \omega \wedge \eta$.
- $\mathcal{F}$ transversely affine $\iff$ we can assume $d\eta = 0$.
- Using $\delta$-sections at $M^0$ and their transverse derivatives,

$$\{ \alpha \in I(M, M^0; \Lambda \mathcal{F}) \mid \text{supp} \alpha \subset M^0 \} \equiv \bigoplus_{k=0}^{\infty} C^\infty(M^0; \Lambda M^0 \otimes \Omega^{-1} N M^0),$$

$$d\mathcal{F} \equiv \bigoplus_{k=0}^{\infty} (d_{M^0} + k \eta \wedge).$$
Term supported on $M^0$

- Assume $\mathcal{F}$ transversely oriented $\rightsquigarrow \exists \omega, \eta \in C^\infty(M; \wedge^1 M)$ such that $T\mathcal{F} = \ker \omega$ and $d\omega = \omega \wedge \eta$.
- $\mathcal{F}$ transversely affine $\iff$ we can assume $d\eta = 0$.
- Using $\delta$-sections at $M^0$ and their transverse derivatives,

$$\{ \alpha \in I(M, M^0; \wedge \mathcal{F}) \mid \text{supp} \alpha \subset M^0 \}$$

$$\equiv \bigoplus_{k=0}^{\infty} C^\infty(M^0; \wedge M^0 \otimes \Omega^{-1} NM^0) ,$$

$$d\mathcal{F} \equiv \bigoplus_{k=0}^{\infty} (d_{M^0} + k \eta \wedge) .$$
Term supported on $M^0$ (contd.)

- Novikov complexes on the compact manifold $M^0$ . . .
- contributions of the fixed points.

Expected contributions?
Term supported on $M^0$ (contd.)

- $\rightsquigarrow$ Novikov complexes on the compact manifold $M^0$ \ldots
- $\rightsquigarrow$ contributions of the fixed points.

Expected contributions?
Term supported on $M^1$

- $\mathcal{F}$ almost without holonomy: only the compact leaves in $M^0$ have holonomy.

- "cutting" $M$ through $M^0$, we get a finite number of compact foliated manifolds with boundary $(M_i, \mathcal{F}_i)$, $M^1 \equiv \bigsqcup_i \hat{M}_i$, $\mathcal{F}^1 \equiv \bigsqcup_i \hat{\mathcal{F}}_i$. (Hector)

- $\exists g^1$ appropriate bundle-like metric for $(M^1, \mathcal{F}^1)$ of bounded geometry.

- the Hodge isomorphism

$$
\overline{H}(H^\infty(M^1; \wedge \mathcal{F}^1), d_{\mathcal{F}^1}) \equiv \bigoplus_i \overline{H}(H^\infty(\hat{M}_i; \wedge \hat{\mathcal{F}}_i), d_{\hat{\mathcal{F}}_i})
\cong \bigoplus_i \text{ker } \Delta_{\hat{\mathcal{F}}_i} \quad (\text{in } H^\infty(\hat{M}_i; \wedge \hat{\mathcal{F}}_i)).
$$
Term supported on $M^1$

- $\mathcal{F}$ almost without holonomy: only the compact leaves in $M^0$ have holonomy.
- $\leadsto$ “cutting” $M$ through $M^0$, we get a finite number of compact foliated manifolds with boundary $(M_i, \mathcal{F}_i)$, $M^1 \equiv \bigcup_i \hat{M}_i$, $\mathcal{F}^1 \equiv \bigcup_i \hat{\mathcal{F}}_i$. (Hector)
- $\exists g^1$ appropriate bundle-like metric for $(M^1, \mathcal{F}^1)$ of bounded geometry.
- $\leadsto$ the Hodge isomorphism

$$
\overline{H}(H^\infty(M^1; \Lambda \mathcal{F}^1), d\mathcal{F}^1) \equiv \bigoplus_i \overline{H}(H^\infty(\hat{M}_i; \Lambda \hat{\mathcal{F}}_i), d\hat{\mathcal{F}}_i)
$$

$$
\cong \bigoplus_i \ker \Delta_{\hat{\mathcal{F}}_i} \quad (\text{in } H^\infty(\hat{M}_i; \Lambda \hat{\mathcal{F}}_i)).
$$
Term supported on $M^1$

- $\mathcal{F}$ almost without holonomy: only the compact leaves in $M^0$ have holonomy.

- "cutting" $M$ through $M^0$, we get a finite number of compact foliated manifolds with boundary $(M_i, \mathcal{F}_i)$, $M^1 \equiv \bigsqcup_i \hat{M}_i$, $\mathcal{F}^1 \equiv \bigsqcup_i \hat{\mathcal{F}}_i$. (Hector)

- $\exists g^1$ appropriate bundle-like metric for $(M^1, \mathcal{F}^1)$ of bounded geometry.

- the Hodge isomorphism

$$
\overline{H}(H^\infty(M^1; \wedge\mathcal{F}^1), d_{\mathcal{F}^1}) \equiv \bigoplus_i \overline{H}(H^\infty(\hat{M}_i; \wedge\hat{\mathcal{F}}_i), d_{\hat{\mathcal{F}}_i})
\supseteq \bigoplus_i \ker \Delta_{\hat{\mathcal{F}}_i} \quad \text{(in } H^\infty(\hat{M}_i; \wedge\hat{\mathcal{F}}_i))
$$
Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\tilde{M}_I; \wedge \tilde{F}_I)$.
- Difficulty: $M^1$ is not compact, smoothing operators are not of trace class.
- $g^1$ is a b-metric of the manifolds with boundary $M_i$ (b-calculus, Melrose, 1993)
- $A_f \in \Psi^{-\infty}_b(\tilde{M}_I; \wedge \tilde{F}_I)$.
- $A_f$ has a b-trace $\sim b\text{Tr}(A_f) \sim$ a part of $L(\phi)$.
- Description of this part:
  contribution of the closed orbits + extra term ($b\text{Tr}$ is not a trace: $b\text{Tr}[A, B] \neq 0$).
Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\mathring{M}_i; \Lambda \mathring{\mathcal{F}}_i)$.
- Difficulty: $M^1$ is not compact,
  - smoothing operators are not of trace class.
- $g^1 \equiv$ a b-metric of the manifolds with boundary $M_i$ (b-calculus, Melrose, 1993)
- $A_f \in \Psi_b^{-\infty}(M_i; \Lambda \mathcal{F}_i)$.
- $A_f$ has a b-trace $\rightsquigarrow b\text{Tr}^s(A_f) \rightsquigarrow$ a part of $L(\phi)$.
- Description of this part:
  - contribution of the closed orbits + extra term ($b\text{Tr}$ is not a trace: $b\text{Tr}[A, B] \neq 0$).
The trace formula
Case of non-singular foliated flows
General case

Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\tilde{M}_l; \Lambda\tilde{F}_l)$.
- **Difficulty:** $M^1$ is not compact,
  - smoothing operators are not of trace class.
- $g^1 \equiv a$ b-metric of the manifolds with boundary $M_l$
  (b-calculus, Melrose, 1993)
- $A_f \in \Psi^{-\infty}_b(M_l; \Lambda F_l)$.
- $A_f$ has a b-trace $\sim b\text{Tr}^s(A_f) \sim$ a part of $L(\phi)$.
- Description of this part:
  contribution of the closed orbits + extra term
  ($b\text{Tr}$ is not a trace: $b\text{Tr}[A, B] \neq 0$).
Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\mathcal{M}_l; \Lambda \mathcal{F}_l)$.
- **Difficulty:** $M^1$ is not compact,
  - $\rightsquigarrow$ smoothing operators are not of trace class.
- $g^1 \equiv$ a **b-metric** of the manifolds with boundary $M_l$
  - (b-calculus, Melrose, 1993)
- $A_f \in \Psi^{-\infty}_b(M_l; \Lambda \mathcal{F}_l)$.
- $A_f$ has a b-trace $\rightsquigarrow b\text{Tr}^s(A_f) \rightsquigarrow$ a part of $L(\phi)$.
- **Description of this part:**
  - contribution of the closed orbits + extra term
  - ($b\text{Tr}$ is not a trace: $b\text{Tr}[A, B] \neq 0$).
The trace formula
Case of non-singular foliated flows
General case

Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\hat{M}_i; \Lambda\hat{\mathcal{F}}_i)$.
- Difficulty: $M^1$ is not compact,
  - smoothing operators are not of trace class.
- $g^1 \equiv a b$-metric of the manifolds with boundary $M_i$
  (b-calculus, Melrose, 1993)
- $A_f \in \Psi^{-\infty}_b(M_i; \Lambda\mathcal{F}_i)$.
- $A_f$ has a b-trace $\sim^b \text{Tr}(A_f) \sim$ a part of $L(\phi)$.
- Description of this part:
  - contribution of the closed orbits + extra term
  ($^b\text{Tr}$ is not a trace: $^b\text{Tr}[A, B] \neq 0$).
Term supported on $M^1$ (contd.)

- $A_f$ is defined as above in every $L^2(\hat{M}_l; \Lambda\hat{\mathcal{F}}_l)$.

- Difficulty: $M^1$ is not compact,
  - smoothing operators are not of trace class.

- $g^1 \equiv a$ b-metric of the manifolds with boundary $M_l$ (b-calculus, Melrose, 1993)

- $A_f \in \Psi_b^{-\infty}(M_l; \Lambda\mathcal{F}_l)$.

- $A_f$ has a b-trace $\sim b\text{Tr}^s(A_f) \sim$ a part of $L(\phi)$.

- Description of this part:
  - contribution of the closed orbits + extra term
  - ($b\text{Tr}$ is not a trace: $b\text{Tr}[A, B] \neq 0$).
Term supported on $M^1$ (contd.)

- $\exists \rho \in C^\infty(M^l)$ such that $\partial M_l = \{ \rho = 0 \}$ and $d\rho \neq 0$ on $\partial M_l$, a defining function of $\partial M_l$.
  
  We can also assume $d\rho = \rho \eta$.

- Then

$$\{ \alpha|_{M^1} \mid \alpha \in \mathfrak{l}(M, M^0; \Lambda F) \} = \bigoplus_{l} \bigcup_{m=0}^{\infty} \rho^{-m} H^\infty(\hat{M}_l; \hat{F}_l).$$

- Multiplication by $\rho^m$ defines an isomorphism

$$\left( \rho^{-m} H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l \right) \cong \left( H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l + m \eta \wedge \right).$$

- $\sim$ a Novikov perturbation of $(H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l)$. 
Term supported on $M^1$ (contd.)

- $\exists \rho \in C^\infty(M^l)$ such that $\partial M_l = \{\rho = 0\}$ and $d\rho \neq 0$ on $\partial M_l$, a defining function of $\partial M_l$.
  We can also assume $d\rho = \rho \eta$.

- Then

  $$\{ \alpha |_{M^1} \mid \alpha \in I(M, M^0; \wedge F) \} = \bigoplus_{l} \bigcup_{m=0}^\infty \rho^{-m} H^\infty(\tilde{M}_l; \tilde{F}_l).$$

- Multiplication by $\rho^m$ defines an isomorphism

  $$(\rho^{-m} H^\infty(\tilde{M}_l; \tilde{F}_l), d\tilde{F}_l) \cong (H^\infty(\tilde{M}_l; \tilde{F}_l), d\tilde{F}_l + m \eta \wedge).$$

- $\leadsto$ a Novikov perturbation of $(H^\infty(\tilde{M}_l; \tilde{F}_l), d\tilde{F}_l)$.
Term supported on $M^1$ (contd.)

- $\exists \rho \in C^\infty(M^1)$ such that $\partial M_l = \{\rho = 0\}$ and $d\rho \neq 0$ on $\partial M_l$, a defining function of $\partial M_l$.
- We can also assume $d\rho = \rho \eta$.
- Then
  \[
  \{ \alpha|_{M^1} \mid \alpha \in I(M, M^0; \wedge F) \} = \bigoplus_{l} \bigcup_{m=0}^{\infty} \rho^{-m} H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l).
  \]
- Multiplication by $\rho^m$ defines an isomorphism
  \[
  (\rho^{-m} H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l), d\mathring{\mathcal{F}}_l) \cong (H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l), d\mathring{\mathcal{F}}_l + m\eta \wedge).
  \]
- $\leadsto$ a Novikov perturbation of $(H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l), d\mathring{\mathcal{F}}_l)$. 
The trace formula
Case of non-singular foliated flows
General case

Term supported on $M^1$ (contd.)

- $\exists \rho \in C^\infty(M^l)$ such that $\partial M_l = \{\rho = 0\}$ and $d\rho \neq 0$ on $\partial M_l$, a defining function of $\partial M_l$.
  We can also assume $d\rho = \rho \eta$.

- Then

$$\{ \alpha |_{M^1} \mid \alpha \in I(M, M^0; \Lambda F) \} = \bigoplus_{l=1}^{\infty} \bigcup_{m=0}^{\infty} \rho^{-m} H^\infty(\hat{M}_l; \hat{F}_l).$$

- Multiplication by $\rho^m$ defines an isomorphism

$$(\rho^{-m} H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l) \cong (H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l + m \eta \wedge).$$

- $\sim$ a Novikov perturbation of $(H^\infty(\hat{M}_l; \hat{F}_l), d\hat{F}_l)$. 
We solved the case where $m = 0$ with the above argument using $A_f$.

$\leadsto$ Novikov’s complex versions of the above argument ...
We solved the case where \( m = 0 \) with the above argument using \( A_f \).

\[ \rightsquigarrow \text{Novikov’s complex versions of the above argument } \ldots \]
Thank you very much!