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Spaces with No S or L Subspaces

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Spaces with no S or L Subspaces

We consider spaces that contain neither an S-space nor an L-space.

Outline of talk

1. History
2. ESLC spaces
3. Spoiling an ESLC product
4. Questions

Reference

J. Hart & K. Kunen, Spaces with no S or L Subspaces, preprint. . .

See my home page

http://www.uwosh.edu/faculty_staff/hartj/

All spaces are T_3 (Hausdorff and regular).

History

A space X is

hereditarily separable (*HS*) iff all subspaces of X are separable, and

hereditarily Lindelöf (*HL*) iff all subspaces of X are Lindelöf.

Also, X is *strongly* HS/HL iff X^n is HS/HL for all $n \in \omega$.

Then, X is an *S-space* iff X is HS but not HL,

and X is a *strong S-space* iff in addition X is strongly HS.

Similarly, X is an *L-space* iff X is HL but not HS,

and X is a *strong L-space* iff in addition X is strongly HL.

In **ZFC**, a strong S-space exists iff a strong L-space exists [Zenor 1980].

Strong S-spaces are refuted by **MA**(\aleph_1) [Kunen 1976],

but exist under **CH** [Kunen 1975].

L-spaces exist in **ZFC** [J. T. Moore 2006].

S-spaces are consistent with **MA**(\aleph_1) [Szentmiklóssy 1983]

but are refuted by **PFA** [Todorčević 1981].

First S- or L-spaces were from a Suslin line:

L-space: Kurepa (1935) S-space: M.E. Rudin (1972)

Con(\exists a Suslin line): Tennenbaum, Jech, Jensen (1967,'68)

S- or L-spaces using Cohen forcing:

S-space: Hajnal and Juhász (1971-72); strong L-space: Roitman (1979)

For more background:

Roitman, Basic S and L, *Handbook of set-theoretic topology* (1984)

OR

Juhász, A survey of S- and L-spaces, *Topology, Vol. II*

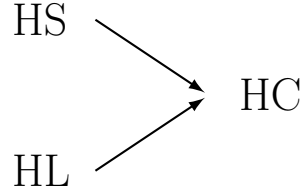
(Proc. Fourth Colloq., Budapest, 1978)

OR

Juhász, *Topology and its Applications*, 158 (2011) 2460–2462

No S or L subspaces:

Def. The space X is *ESLC* iff every subspace of X is either both HS and HL or neither HS nor HL.



The space X is **HC** (hereditarily ccc)
iff every subspace of X has the ccc (countable chain condition)
iff X has countable spread
iff X has no uncountable discrete subspaces.

HC $\not\rightarrow$ HS and HC $\not\rightarrow$ HL: S-space \oplus L-space is HC

X is HC iff X has no discrete ω_1 -sequences.

X is HS iff X has no left separated ω_1 -sequences.

X is HL iff X has no right separated ω_1 -sequences.

A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is

discrete provided that each $x_\alpha \notin \text{cl}(\{x_\xi : \xi \neq \alpha\})$, and

left separated provided that each $x_\alpha \notin \text{cl}(\{x_\xi : \xi < \alpha\})$, and

right separated provided that each $x_\alpha \notin \text{cl}(\{x_\xi : \xi > \alpha\})$.

X is ESLC iff HS \leftrightarrow HL \leftrightarrow HC holds for all subspaces of X .

Pf: \Leftarrow : by def of ESLC. \Rightarrow : Suppose X is ESLC and $Y \subseteq X$.

If Y is HS, then Y is also HC and by ESLC is HL.

If Y is not HS, apply ESLC to get a discrete ω_1 -sequence in Y :

By not HS, Y has a left separated $\langle x_\alpha : \alpha < \omega_1 \rangle$.

By ESLC, this left separated $\{x_\alpha : \alpha < \omega_1\}$ is also not HL,

and hence has a right separated $\langle x_{\alpha\beta} : \beta < \omega_1 \rangle$.

ESLC examples:

0. *HS+HL spaces:*

Sorgenfrey line is HS + HL

PFA(\mathcal{S})[\mathcal{S}] \rightarrow each compact ccc T_5 space is HS + HL [Todorćević]

T_5 = hereditarily normal

page 39: http://www.math.toronto.edu/~stevo/todorcevic_chain_cond.pdf

MA + \neg **CH** \rightarrow each separable hereditarily supercompact space

is HS + HL [Banach, Kosztołowicz, Turek, 2014]

A space X is hereditarily supercompact if every *closed* subspace of X is supercompact; it is supercompact if it has a subbase \mathcal{S} so that each cover of X by elements of \mathcal{S} has a 2-element subcover.

1. *metrizable spaces:* Every metric space is either second countable or has an uncountable discrete subspace.

2. *separately continuously semi-metrizable spaces:*

(semi-metrizable = symmetrizable and first countable)

Every HC separately continuously semi-metrizable space is also HG (more later ...).

3. *countable products of monarch butterfly spaces*

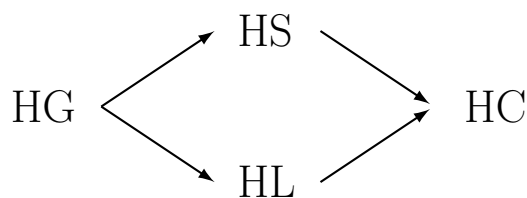
Special case: $\prod_{n \in \omega} X_n$ with each X_n a subspace of the Sorgenfrey line.

In **ZFC**, such products cannot contain an S- or L-space. It's consistent that these products can be HS + HL, or neither HS nor HL.

In contrast to #3:

Theorem (**CH** or V [one Cohen real]) There are X, Y that are stHG (so stHS and stHL) such that $X \times Y$ contains strong S- and L-spaces.

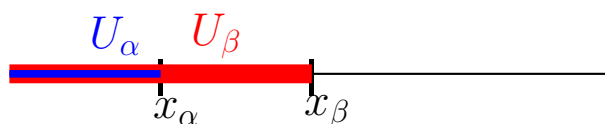
The fourth property:



The space X is **HG** iff there are no bad sequences in X .

A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in X is *bad* iff there are open $U_\alpha \ni x_\alpha$ for $\alpha < \omega_1$ such that for all $\{\alpha, \beta\} \in [\omega_1]^2$, $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$.

HS + HL $\not\rightarrow$ HG : the Sorgenfrey line is not HG.



To see Sorgenfrey line E is HL: Suppose $Y \subseteq E$.

Recall that if \mathcal{U} is a cover of Y by basic clopen $U_x = (x - \varepsilon_x, x] \cap Y$, then $|Y \setminus \bigcup \{U_x \setminus \{x\} : x \in Y\}| \leq \aleph_0$ (because \mathbb{R} is ccc).

Also, letting $V_x = U_x \setminus \{x\}$,

$\exists \mathcal{V} \in [\{V_x : x \in Y\}]^{\leq \aleph_0}$ with $\bigcup \mathcal{V} = \bigcup \{V_x : x \in Y\}$ (\mathbb{R} is HL).

Recap: All four properties start

“if $\{x_\alpha : \alpha < \omega_1\} \subseteq X$ and $\forall \alpha < \omega_1 \ x_\alpha \in U_\alpha \stackrel{\text{open}}{\subseteq} X$ ”

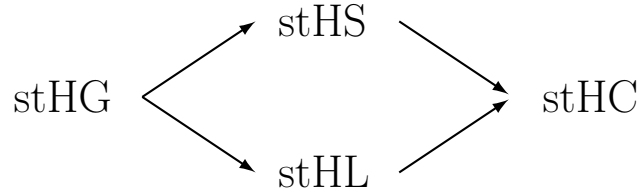
and then conclude that X is:

- | | | |
|--------|--|-----------------------------|
| HS iff | $\exists \alpha < \beta [x_\alpha \in U_\beta]$ | no left separated sequence |
| HL iff | $\exists \alpha < \beta [x_\beta \in U_\alpha]$ | no right separated sequence |
| HC iff | $\exists \alpha \neq \beta [x_\beta \in U_\alpha]$ | no discrete sequence |
| HG iff | $\exists \alpha \neq \beta [x_\beta \in U_\alpha \ \& \ x_\alpha \in U_\beta]$ | no bad sequence |

The four strong properties:

Def. X is *strongly* \mathcal{P} iff X^n is \mathcal{P} for all $n \in \omega$.

Ex: X is *strongly* HG/HC iff X^n is HG/HC for all $n \in \omega$.

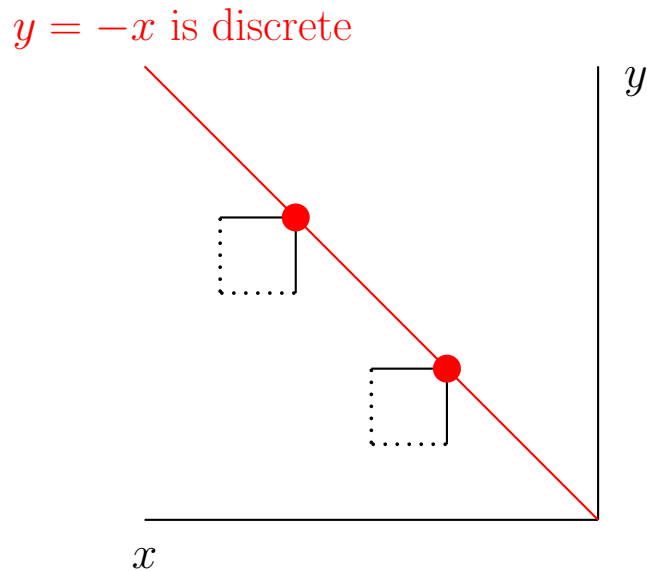


CH: None of the four \rightarrow reverses:

- stHC $\not\rightarrow$ stHS: strong L-space
- stHC $\not\rightarrow$ stHL: strong S-space
- stHS + stHL $\not\rightarrow$ stHG: Sorgenfrey $X \in [\mathbb{R}]^{\aleph_1}$
that is n -entangled $\forall n \in \omega$

Without n -entangled:

(Sorgenfrey line) \times (Sorgenfrey line) is not HC.



Separately continuous semi-metrizable spaces

Def. For any space X , let $d : X \times X \rightarrow \mathbb{R}$, and $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The function d is a *scsymmetric* for X iff d satisfies the following four conditions:

1. $d(x, y) \geq 0$, and $d(x, y) = 0 \leftrightarrow x = y$.
2. $d(x, y) = d(y, x)$.
3. For each $x \in X$, $\{B(x, \varepsilon) : \varepsilon > 0\}$ is a local base at x .
4. For each x , the map $y \mapsto d(x, y)$ is continuous.

Then X is *scsymmetrizable* iff there exists a scsymmetric for X .

Note: each $B(x, \varepsilon)$ is open by #4.

symmetric = #1 & #2; semi-metric = #1 & #2 & #3

Davis, Gruenhage, Nyikos (1978): \exists a symmetrizable space in which some closed set is not a G_δ .

Our scsymmetrizable space: Every closed set is a G_δ .

Lemma. For X scsymmetrizable, the four properties (HG, HS, HL, HC) are equivalent. So every scsymmetrizable X is ESLC.

Proof: HC \rightarrow HG:

Suppose $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a bad sequence with open $U_\alpha \ni x_\alpha$ for $\alpha < \omega_1$ such that $\forall \{\alpha, \beta\} \in [\omega_1]^2$ $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$.

By #3, each $U_\alpha \supseteq B(x_\alpha, 2^{-n_\alpha})$ for some $n_\alpha \in \omega$.

Passing to a subsequence, $n_\alpha = n \in \omega$ for all α .

Now $d(x_\alpha, x_\beta) \geq 2^{-n}$, so $x_\alpha \notin B(x_\beta, 2^{-n})$, for all $\{\alpha, \beta\} \in [\omega_1]^2$.

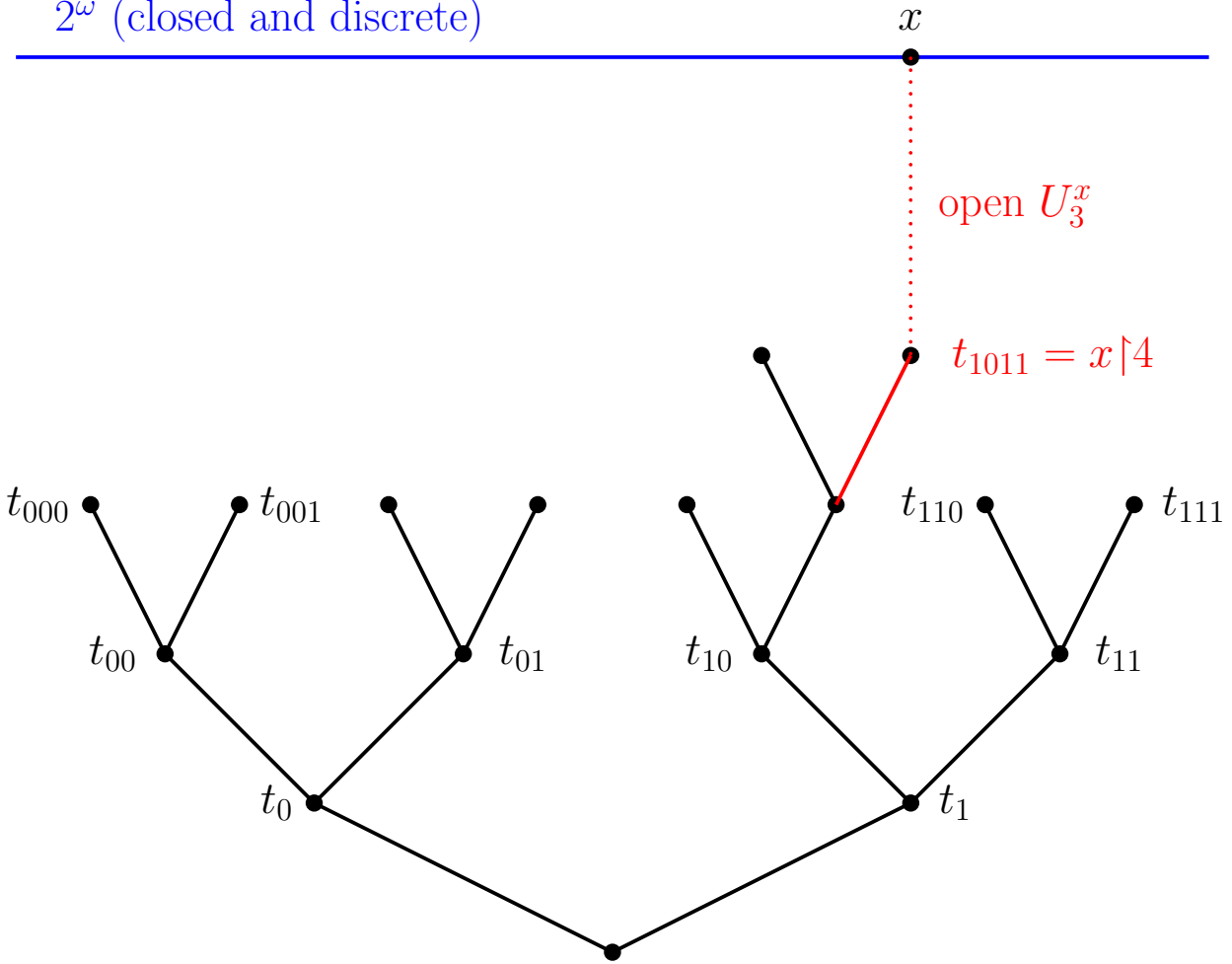
By #4, the sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is discrete.

Sorgenfrey line is ESLC, but is *not* scsymmetrizable (fails #3) ...

Simple scsymmetric example:

The Cantor Tree Space $X = 2^\omega \cup 2^{<\omega}$

2^ω (closed and discrete)



For $x, y \in X$ with $x \not\subseteq y$:

$$d(x, y) = d(y, x) = 2$$

For $s, t \in 2^{<\omega}$ and $s \subseteq t$:

$$d(s, t) = d(t, s) = |2^{-\text{lh}(s)} - 2^{-\text{lh}(t)}|$$

For $x \in 2^\omega$:

$$d(x, x \upharpoonright n) = d(x \upharpoonright n, x) = 2^{-n}$$

d is a *scsymmetric* with basic nbds:

$$2^{<\omega}: U_s = \{s\} \quad 2^\omega: U_n^x := \{x \upharpoonright \nu : n \leq \nu \leq \omega\} = B(x, 2^{-n+1})$$

d is not a metric: X separable, but 2^ω discrete $\rightarrow X$ not HC

For $Y \subseteq X$:

the *five* properties HC, HS, HL, HG, and *countable* are equivalent.

Preserving products

Theorem Every countable product of monarch spaces is ESLC.

A *butterfly* or *bow-tie*: Alexandroff and Niemytzki [1938] (Engelking 3.1.I).
Burke and van Douwen [1980]

Another version:

Def. A *monarch* space is a set X with a butterfly refinement of some separable metric on X . A *butterfly refinement* of a separable metric space (X, \mathcal{T}) is a topology $\widehat{\mathcal{T}}$ on X with base $\{U_x^n : x \in X \ \& \ n \in \omega\}$ satisfying:

1. $x \in U_x^n$,
2. $U_x^n \setminus \{x\}$ is \mathcal{T} open,
3. $\text{diam}(U_x^n) \searrow_n 0$, and
4. $\text{cl}(U_x^{n+1}, \mathcal{T}) \subseteq U_x^n$.

Example: $X \subseteq \mathbb{R}$ with the Sorgenfrey topology is a monarch space:

$$U_x^n = X \cap (x - 2^{-n}, x]$$

Trivial example: $\widehat{\mathcal{T}} = \mathcal{T}$

Another trivial example: $\widehat{\mathcal{T}}$ is discrete

Spoiling ESLC products

Theorem (CH or V[one Cohen real]) There are X, Y that are strongly HG (hence ESLC) such that $X \times Y$ contains strong L- and S-spaces.

V[one Cohen real] *Pf*: Adapt **strong L-space** to get stHG spaces:

Our strongly HG spaces will be $\mathcal{F}^\Psi = \{f_\beta^\Psi : \beta \in \omega_1\} \subseteq \omega^{\omega_1}$,

where each $f_\beta^\Psi : \omega_1 \rightarrow 2$ by $f_\beta^\Psi(\alpha) = \Psi(\alpha, \beta)$,

and $\Psi : \omega_1 \times \omega_1 \rightarrow 2$ is defined from Cohen reals.

Instead of forcing with $\text{Fn}(\omega, 2)$ to add a Cohen real $x : \omega \rightarrow 2$,

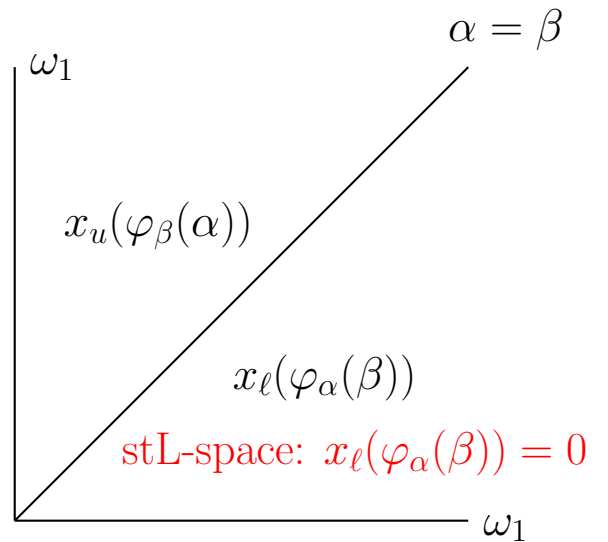
use $\text{Fn}(\omega, 2) \times \text{Fn}(\omega, 2)$ to add Cohen reals $x_u : \omega \rightarrow 2, x_\ell : \omega \rightarrow 2$.

In V , choose $\varphi_\beta : \beta \xrightarrow{1-1} \omega$ for $\beta < \omega_1$

with $\alpha \neq \beta \rightarrow |\text{ran}(\varphi_\alpha) \cap \text{ran}(\varphi_\beta)| < \aleph_0$.

Define $\Psi : \omega_1 \times \omega_1 \rightarrow 2$ by

$$\Psi(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ x_u(\varphi_\beta(\alpha)) & \text{if } \alpha < \beta, \\ x_\ell(\varphi_\alpha(\beta)) & \text{if } \alpha > \beta. \end{cases}$$



Now get $X \neq Y$ from x_u^i, x_ℓ^i for $i = 0, 1$, as above, with $x_\ell^0 = x_\ell^1$.

So $X = \mathcal{F}^{\Psi_0}$ and $Y = \mathcal{F}^{\Psi_1}$, where $\Psi_i : \omega_1 \times \omega_1 \rightarrow 2$. Let $f_\beta^i = f_\beta^{\Psi_i}$.

Then $\Delta = \{(f_\beta^0, f_\beta^1) : \beta < \omega_1\} \subseteq X \times Y$. Note Δ is constructed from

$\widehat{\Psi} : \omega_1 \times \omega_1 \rightarrow 2 \times 2$ by $\widehat{\Psi}(\alpha, \beta) = (\Psi_0(\alpha, \beta), \Psi_1(\alpha, \beta))$.

Δ is a strong L-space because $\widehat{\Psi}(\text{lower right}) \subsetneq \widehat{\Psi}(\text{upper left})$:

$$x_\ell^0 = x_\ell^1 \text{ generic} \rightarrow \widehat{\Psi}(\text{lower right}) = \{(0, 0), (1, 1)\} \cong 2$$

$$\text{and } x_u^i \text{ generic} \rightarrow \widehat{\Psi}(\text{upper left}) = \{(0, 0), (1, 1), (1, 0), (0, 1)\} \cong 4.$$

Questions

1. Is HG equivalent to “hereditarily \mathcal{P} ” for some known property?
2. Is $\text{HG} \rightarrow \text{strongly HG consistent?}$ is it provable from $\text{MA}(\aleph_1)$?
3. $?? \times \text{ESLC} \rightarrow \text{ESLC}$

Example: Does $?? = \text{scsymmetric work?}$

Theorem $\text{metric} \times \text{ESLC} \rightarrow \text{ESLC}$

Pf: Use the fact that in a metric space, every uncountable set has an uncountable subset that is either discrete or second countable.