6-2017

Liouville Numbers and One-sided Ergodic Hilbert Transformations

David Constantine
Wesleyan University

Joanna Furno
Indiana University - Purdue University, Indianapolis, jfurno@iupui.edu

Liouville Numbers and One-sided Ergodic Hilbert Transformations

Dave Constantine [1] Joanna Furno [2]

[1] Wesleyan University

[2] Indiana University-Purdue University, Indianapolis

June 28, 2017

Object of Study

$$\sum_{n=0}^{\infty} f \circ T^n(x) b_n$$

- $(X, \mathcal{B}, \mu)$ is a probability measure space.
- $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is
  - invertible,
  - measure-preserving, (For all $B \in \mathcal{B}$, $\mu(T^{-1}B) = \mu(B)$.)
  - ergodic. (For all $B \in \mathcal{B}$, if $T^{-1}B = B$, then $\mu(B)$ is 0 or 1.)
- $\sum b_n$ is a positive, divergent series.

One-sided ergodic Hilbert transform when $b_n = \dfrac{1}{n}$.

Birkhoff Ergodic Theorem

One-sided EHT: $\sum_{n=0}^{\infty} \frac{1}{n} f \circ T^n(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{k} f \circ T^k(x)$

Theorem (Birkhoff Ergodic Theorem)

Suppose

- $(X, \mathcal{B}, \mu)$ *is a probability measure space,*
- T *is ergodic and measure-preserving on* $(X, \mathcal{B}, \mu)$*, and*
- $f \in L^1(\mu)$.

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int f \, d\mu$$

almost everywhere.

Divergence Theorem - Basic Version

Theorem (Constantine, F)

Let $f = 2\chi_U - 1$, where U is finite union of intervals with $m(U) = 1/2$. Then there are irrational α such that $\sum f \circ R_\alpha^n(x)/n$ diverges for all points $x \in S^1$. Such α can be provided explicitly in terms of the continued fraction expansion.

Continued Fraction Expansions

For an irrational $\alpha \in (0, 1)$, let $\alpha = [a_1 a_2 a_3 \ldots]$ be the continued fraction expansion.

Denote the nth convergent by $\frac{p_n}{q_n}$.

Then $q_0 = 1$, $q_1 = a_1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for all $n \geqslant 2$.

Bounded Number of Sign Changes

Fix n. For $k \in [1, a_{n+1} - 1]$, let

$$s(\sigma_k) = \sum_{i=q_{n-1}+(k-1)q_n+1}^{q_{n-1}+kq_n} f \circ R_\alpha^i(x).$$

Lemma (Constantine, F)

Let $C = \{k \in [1, a_{n+1} - 1] : s(\sigma_k) \neq s(\sigma_{k+1})\}$. i.e. C is the set of k at which $s(\sigma_k)$ changes. Then $|C| \leqslant 2B$, where B is the number of intervals in the definition of f.

Summation by Parts

Let c_m be a sequence, and let $s_n = \sum_{m=1}^{n} c_m$.

Using summation by parts:

$$\sum_{m=1}^{N} c_m \frac{1}{m+k} = \frac{s_N}{N+k+1} - \sum_{m=1}^{N} s_m \left(\frac{1}{m+k+1} - \frac{1}{m+k} \right)$$
$$= \frac{s_N}{N+k+1} + \sum_{m=1}^{N} \frac{s_m}{m+k} \frac{1}{m+k+1}.$$

Liouville numbers

An irrational real number α is Liouville if for all $k \geqslant 1$, there exists a rational number $\frac{p}{q}$ such that

$$\left| \alpha - \frac{p}{q} \right| < q^{-(k+1)}.$$

Convergence Theorems

Let $f = 2\chi_U - 1$, where U is finite union of intervals with $m(U) = 1/2$.

Theorem (Kakutani, Petersen 1981)

If α is not a Liouville number, then the ergodic Hilbert transform of f converges at all points. Hence the set of α for which the EHT of f diverges for any x has Hausdorff dimension zero.

Theorem (Constantine, F)

There exist Liouville numbers α for which $\sum f \circ R_\alpha^n(x)/n$ converges for all $x \in S^1$. The set of such α is dense.

Denjoy-Koksma Lemma

Lemma (Denjoy-Koksma Lemma)

Let f be any mean zero function on S^1. Let $[a, b]$ be any interval of length q_n. Then for any $x \in S^1$,

$$\left| \sum_{k \in [a,b]} f \circ R_\alpha^k(x) \right| < Var(f).$$

Corollary

Let $f = 2\chi_U - 1$, where U is the union of B intervals and $m(U) = \frac{1}{2}$. Then, for any interval $[a, b]$ of length q_n and any $x \in S^1$,

$$\left| \sum_{k \in [a,b]} f \circ R_\alpha^k(x) \right| < 4B.$$

Strategy

- Pick near-alternating subsequences.
- Bound the first term in the subsequences (to bound the sum).
- Use the continued fraction expansion:
    - If not Liouville, then there exists $k > 1$ such that $q_{n+1} < q_n^k$ for all n.
    - Taking $a_{n+1} = q_n^{n-1}$ for all (large enough) n implies Liouville and $q_{n+1} \leqslant 2q_n^n$.

Open Questions

- Can you find $f_1 = 2\chi_U - 1$ and $f_2 = 2\chi_V - 1$, two mean-zero indicator functions on finite unions of interval, and a (Liouville) number α such that the EHT of f_1 diverges for all x but the EHT of f_2 converges for all x?
- Do there exist a mean-zero indicator function f and a (Liouville) α such that the EHT of f diverges for some x and converges for other x?

Thanks!