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# Liouville Numbers and One-sided Ergodic Hilbert Transformations

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# Liouville Numbers and One-sided Ergodic Hilbert Transformations

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# Object of Study

$$\sum_{n=0}^{\infty} f \circ T^n(x) b_n$$

- $(X, \mathcal{B}, \mu)$  is a probability measure space.
- $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is
  - invertible,
  - measure-preserving, (For all  $B \in \mathcal{B}$ ,  $\mu(T^{-1}B) = \mu(B)$ .)
  - ergodic. (For all  $B \in \mathcal{B}$ , if  $T^{-1}B = B$ , then  $\mu(B)$  is 0 or 1.)
- $\sum b_n$  is a positive, divergent series.

One-sided ergodic Hilbert transform when  $b_n = \frac{1}{n}$ .

# Birkhoff Ergodic Theorem

$$\text{One-sided EHT: } \sum_{n=0}^{\infty} \frac{1}{n} f \circ T^n(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{k} f \circ T^k(x)$$

## Theorem (Birkhoff Ergodic Theorem)

*Suppose*

- $(X, \mathcal{B}, \mu)$  is a probability measure space,
- $T$  is ergodic and measure-preserving on  $(X, \mathcal{B}, \mu)$ , and
- $f \in L^1(\mu)$ .

*Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int f \, d\mu$$

*almost everywhere.*

# Divergence Theorem - Basic Version

## Theorem (Constantine, F)

*Let  $f = 2\chi_U - 1$ , where  $U$  is finite union of intervals with  $m(U) = 1/2$ . Then there are irrational  $\alpha$  such that  $\sum f \circ R_\alpha^n(x)/n$  diverges for all points  $x \in S^1$ . Such  $\alpha$  can be provided explicitly in terms of the continued fraction expansion.*

# Continued Fraction Expansions

For an irrational  $\alpha \in (0, 1)$ , let  $\alpha = [a_1 a_2 a_3 \dots]$  be the continued fraction expansion.

Denote the  $n$ th convergent by  $\frac{p_n}{q_n}$ .

Then  $q_0 = 1$ ,  $q_1 = a_1$ , and  $q_n = a_n q_{n-1} + q_{n-2}$  for all  $n \geq 2$ .

# Bounded Number of Sign Changes

Fix  $n$ . For  $k \in [1, a_{n+1} - 1]$ , let

$$s(\sigma_k) = \sum_{i=q_{n-1}+(k-1)q_{n+1}}^{q_{n-1}+kq_n} f \circ R_\alpha^i(x).$$

## Lemma (Constantine, F)

Let  $C = \{k \in [1, a_{n+1} - 1] : s(\sigma_k) \neq s(\sigma_{k+1})\}$ . i.e.  $C$  is the set of  $k$  at which  $s(\sigma_k)$  changes. Then  $|C| \leq 2B$ , where  $B$  is the number of intervals in the definition of  $f$ .

# Summation by Parts

Let  $c_m$  be a sequence, and let  $s_n = \sum_{m=1}^n c_m$ .

Using summation by parts:

$$\begin{aligned}\sum_{m=1}^N c_m \frac{1}{m+k} &= \frac{s_N}{N+k+1} - \sum_{m=1}^N s_m \left( \frac{1}{m+k+1} - \frac{1}{m+k} \right) \\ &= \frac{s_N}{N+k+1} + \sum_{m=1}^N \frac{s_m}{m+k} \frac{1}{m+k+1}.\end{aligned}$$



# Liouville numbers

An irrational real number  $\alpha$  is Liouville if for all  $k \geq 1$ , there exists a rational number  $\frac{p}{q}$  such that

$$\left| \alpha - \frac{p}{q} \right| < q^{-(k+1)}.$$

# Convergence Theorems

Let  $f = 2\chi_U - 1$ , where  $U$  is finite union of intervals with  $m(U) = 1/2$ .

## Theorem (Kakutani, Petersen 1981)

*If  $\alpha$  is not a Liouville number, then the ergodic Hilbert transform of  $f$  converges at all points. Hence the set of  $\alpha$  for which the EHT of  $f$  diverges for any  $x$  has Hausdorff dimension zero.*

## Theorem (Constantine, F)

*There exist Liouville numbers  $\alpha$  for which  $\sum f \circ R_\alpha^n(x)/n$  converges for all  $x \in S^1$ . The set of such  $\alpha$  is dense.*

# Denjoy-Koksma Lemma

## Lemma (Denjoy-Koksma Lemma)

Let  $f$  be any mean zero function on  $S^1$ . Let  $[a, b]$  be any interval of length  $q_n$ . Then for any  $x \in S^1$ ,

$$\left| \sum_{k \in [a, b]} f \circ R_\alpha^k(x) \right| < \text{Var}(f).$$

## Corollary

Let  $f = 2\chi_U - 1$ , where  $U$  is the union of  $B$  intervals and  $m(U) = \frac{1}{2}$ . Then, for any interval  $[a, b]$  of length  $q_n$  and any  $x \in S^1$ ,

$$\left| \sum_{k \in [a, b]} f \circ R_\alpha^k(x) \right| < 4B.$$

# Strategy

- Pick near-alternating subsequences.
- Bound the first term in the subsequences (to bound the sum).
- Use the continued fraction expansion:
  - If not Liouville, then there exists  $k > 1$  such that  $q_{n+1} < q_n^k$  for all  $n$ .
  - Taking  $a_{n+1} = q_n^{n-1}$  for all (large enough)  $n$  implies Liouville and  $q_{n+1} \leq 2q_n^n$ .

# Open Questions

- Can you find  $f_1 = 2\chi_U - 1$  and  $f_2 = 2\chi_V - 1$ , two mean-zero indicator functions on finite unions of interval, and a (Liouville) number  $\alpha$  such that the EHT of  $f_1$  diverges for all  $x$  but the EHT of  $f_2$  converges for all  $x$ ?
- Do there exist a mean-zero indicator function  $f$  and a (Liouville)  $\alpha$  such that the EHT of  $f$  diverges for some  $x$  and converges for other  $x$ ?

Thanks!