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David Constantine  
*Wesleyan University*

Joanna Furno  
*Indiana University - Purdue University, Indianapolis, jfurno@iupui.edu*

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Liouville Numbers and One-sided Ergodic Hilbert Transformations

Dave Constantine \(^1\) Joanna Furno \(^2\)

\(^1\)Wesleyan University
\(^2\)Indiana University-Purdue University, Indianapolis

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\[ \sum_{n=0}^{\infty} f \circ T^n(x) b_n \]

- \((X, \mathcal{B}, \mu)\) is a probability measure space.
- \(T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)\) is invertible, measure-preserving, (For all \(B \in \mathcal{B}\), \(\mu(T^{-1}B) = \mu(B)\).) ergodic. (For all \(B \in \mathcal{B}\), if \(T^{-1}B = B\), then \(\mu(B)\) is 0 or 1.)
- \(\sum b_n\) is a positive, divergent series.

One-sided ergodic Hilbert transform when \(b_n = \frac{1}{n}\).
Birkhoff Ergodic Theorem

One-sided EHT: \[ \sum_{n=0}^{\infty} \frac{1}{n} f \circ T^n(x) = \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{k} f \circ T^k(x) \]

Theorem (Birkhoff Ergodic Theorem)

Suppose
- \((X, \mathcal{B}, \mu)\) is a probability measure space,
- \(T\) is ergodic and measure-preserving on \((X, \mathcal{B}, \mu)\), and
- \(f \in L^1(\mu)\).

Then
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int f \, d\mu \]

almost everywhere.
Theorem (Constantine, F)

Let $f = 2\chi_U - 1$, where $U$ is finite union of intervals with 
$m(U) = 1/2$. Then there are irrational $\alpha$ such that 
$\sum f \circ R^n_\alpha(x)/n$ diverges for all points $x \in S^1$. Such $\alpha$ can be provided explicitly 
in terms of the continued fraction expansion.
For an irrational $\alpha \in (0, 1)$, let $\alpha = [a_1 a_2 a_3 \ldots]$ be the continued fraction expansion.

Denote the $n$th convergent by $\frac{p_n}{q_n}$.

Then $q_0 = 1$, $q_1 = a_1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for all $n \geq 2$. 
Fix $n$. For $k \in [1, a_{n+1} - 1]$, let

$$s(\sigma_k) = \sum_{i=q_{n-1}+(k-1)q_{n+1}}^{q_{n-1}+kq_n} f \circ R^i_\alpha(x).$$

**Lemma (Constantine, F)**

Let $C = \{k \in [1, a_{n+1} - 1] : s(\sigma_k) \neq s(\sigma_{k+1})\}$. i.e. $C$ is the set of $k$ at which $s(\sigma_k)$ changes. Then $|C| \leq 2B$, where $B$ is the number of intervals in the definition of $f$. 
Let $c_m$ be a sequence, and let $s_n = \sum_{m=1}^{n} c_m$.

Using summation by parts:

$$\sum_{m=1}^{N} \frac{c_m}{m+k} \cdot \frac{1}{m + k} = \frac{s_N}{N + k + 1} - \sum_{m=1}^{N} \frac{s_m}{m + k} \left( \frac{1}{m + k + 1} - \frac{1}{m + k} \right)$$

$$= \frac{s_N}{N + k + 1} + \sum_{m=1}^{N} \frac{s_m}{m + k} \cdot \frac{1}{m + k(m + k + 1)}.$$
An irrational real number $\alpha$ is Liouville if for all $k \geq 1$, there exists a rational number $\frac{p}{q}$ such that

$$\left| \alpha - \frac{p}{q} \right| < q^{-(k+1)}.$$
Let $f = 2\chi_U - 1$, where $U$ is finite union of intervals with $m(U) = 1/2$.

**Theorem (Kakutani, Petersen 1981)**

If $\alpha$ is not a Liouville number, then the ergodic Hilbert transform of $f$ converges at all points. Hence the set of $\alpha$ for which the EHT of $f$ diverges for any $x$ has Hausdorff dimension zero.

**Theorem (Constantine, F)**

There exist Liouville numbers $\alpha$ for which $\sum f \circ R_n^\alpha(x)/n$ converges for all $x \in S^1$. The set of such $\alpha$ is dense.
Let $f$ be any mean zero function on $S^1$. Let $[a, b]$ be any interval of length $q_n$. Then for any $x \in S^1$,

$$\left| \sum_{k \in [a, b]} f \circ R^k_\alpha(x) \right| < \text{Var}(f).$$

**Corollary**

Let $f = 2\cdot \chi_U - 1$, where $U$ is the union of $B$ intervals and $m(U) = \frac{1}{2}$. Then, for any interval $[a, b]$ of length $q_n$ and any $x \in S^1$,

$$\left| \sum_{k \in [a, b]} f \circ R^k_\alpha(x) \right| < 4B.$$
Strategy

- Pick near-alternating subsequences.
- Bound the first term in the subsequences (to bound the sum).
- Use the continued fraction expansion:
  - If not Liouville, then there exists $k > 1$ such that $q_{n+1} < q_n^k$ for all $n$.
  - Taking $a_{n+1} = q_n^{n-1}$ for all (large enough) $n$ implies Liouville and $q_{n+1} \leq 2q_n^n$. 

Constantine, Furno

Liouville Numbers and EHTs
Open Questions

Can you find $f_1 = 2\chi_U - 1$ and $f_2 = 2\chi_V - 1$, two mean-zero indicator functions on finite unions of interval, and a (Liouville) number $\alpha$ such that the EHT of $f_1$ diverges for all $x$ but the EHT of $f_2$ converges for all $x$?

Do there exist a mean-zero indicator function $f$ and a (Liouville) $\alpha$ such that the EHT of $f$ diverges for some $x$ and converges for other $x$?
Thanks!