1-1-2018

Generalized Catalan Numbers And Objects: X; Y Equivalence Classes And Polyominoes

Emily S. Dautenhahn
University of Dayton

Hannah E. Pieper
University of Dayton

Elec. Proc. Undergrad. Math. Day
5 (2018), 25–35.

GENERALIZED CATALAN NUMBERS AND OBJECTS: X, Y EQUIVALENCE CLASSES AND POLYOMINOES

EMILY S. DAUTENHAHN AND HANNAH E. PIEPER

Communicated by Jonathan Brown

Abstract. There are numerous sets of combinatorial objects that are counted by the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, and many mathematicians have constructed bijections between these sets. The Catalan numbers can be generalized using the parameter k to obtain the family of sequences known as the k-Catalan numbers, $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$. Here we present two new generalizations of Catalan sets to k-Catalan sets, k-generalized x, y equivalence classes and k-generalized polyominoes, including a novel bijection between them.

Keywords: *Catalan numbers, polyominoes*

MSC (2010): Primary 05A10

1. Introduction

1.1. **Catalan Numbers.** The Catalan Numbers are a well known combinatorial sequence that can be characterized in many equivalent ways. For example, Catalan numbers are described in terms of triangulating polygons. Given $n \in \mathbb{N}$, the number of ways to insert $n-1$ noncrossing diagonals into the interior of a regular, convex $(n+2)$-gon such that the interior of the polygon is divided into triangles, is the nth Catalan number. This interpretation was first described by Euler [4], and can be represented by the following recursive formula,

$$C_n = \frac{4n-2}{n+1} C_{n-1}, \ C_0 = 1.$$

Equivalently, Catalan characterized this sequence as the number of ways to group $n+1$ x's and n sets of parentheses such that they represent a non-associative binary operation. Catalan represented this with the closed formula [4],

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

which generates the same sequence as that recursively defined by Euler.

1.2. **Generalized Catalan Numbers.** Gould [1] developed the following generalization of Catalan numbers,

$$C_n^{k,r} = \frac{r}{kn+r}\binom{kn+r}{n}.$$

If we fix $r = 1$, this equation can be written more in a more useful way as,

$$C_n^k = \frac{1}{kn+1}\binom{kn+1}{n},$$

a family of sequences which we will refer to as the k-Catalan numbers.

From this generalization, the standard Catalan numbers can be obtained by substituting $r = 1$ and $k = 2$. Just as the Catalan numbers can be generalized with these two parameters, the combinatorial objects that are counted by the Catalan numbers can be generalized as well. In particular, looking at $k = 3$ can reveal how to adapt standard Catalan objects to also be counted by the k-Catalan numbers for any value of k.

Some previous work has been done on these generalizations, as can be found in [2, 3]. Here, we present two new generalizations of previously existing Catalan objects to the k-Catalan numbers, k-generalized x, y equivalence classes and k-generalized polyominoes. We also present justification for why these sets are counted by the k-Catalan numbers. Part of this justification consists of a novel bijection between these two sets.

2. k - Generalized x, y Equivalence Classes

One pre-existing Catalan set, labeled #158 by Stanley [4], corresponds to cyclic equivalence classes of $n + 1$ 1's and n 0's. These objects easily generalize to a related k-Catalan set.

Definition 2.1. A *k-generalized cyclic equivalence class* is a cyclic equivalence class consisting of $(k-1)n + 1$ x's and n y's. In other words, given a sequence with the appropriate number of x's and y's we define the elements in its equivalence class by all cyclic shifts of the given sequence. If we assign each x the value of $\frac{1}{k-1}$ and each y the value of -1, then we choose as the equivalence class representative the sequence with solely positive partial sums. We define the set of these equivalence classes for a given k and n by E_n^k.

Before proceeding, we look at an example to gain a better understanding of these objects.

Example 2.1. Let $k = 3$ and $n = 3$. Then we have a sequences of x's and y's where we have assigned $x = \frac{1}{2}$ and $y = -1$. Consider the sequence

$$xxxyxxxyxy.$$

Let s_m denote the partial sum up to the m-th term of the sequence. For the above sequence, we see that

$$\begin{aligned} s_1 =&.5 \quad s_2 = 1 \quad s_3 = 1.5 \quad s_4 = .5 \quad s_5 = 1 \\ s_6 =&1.5 \quad s_7 = 2 \quad s_8 = 1 \quad s_9 = 1.5 \quad s_{10} = .5, \end{aligned}$$

so it has strictly positive partial sums. Therefore, it is an equivalence class representative. However, the cyclic shift,

$$xxyxxxyxyx,$$

which is in the same equivalence class as the original sequence, does not meet the conditions to be an equivalence class representative, as $s_3 = \frac{1}{2} + \frac{1}{2} - 1 = 0$, which is not positive.

Theorem 2.1. *$|E_n^k| = C_n^k$ for all $n \geq 0$ and $k \geq 2$.*

Proof. In large part, this proof is based off of the proof for the pre-existing Catalan set, as presented in [4]. We begin by noting that $(k-1)n+1$ and n must be relatively prime, as $(k-1)n+1$ differs from a multiple of n by 1. Therefore, there must be exactly $(k-1)n+1+n = kn+1$ elements in each equivalence class. Furthermore, as the number of sequences is determined by the placement of the y's and the total number of elements in each sequence is $kn+1$, the total possible number of sequences is $\binom{kn+1}{n}$. Hence, the total number of equivalence classes is $\frac{1}{kn+1}\binom{kn+1}{n}$, which we recognize as C_n^k. ∎

Although this proves that these equivalence classes are counted by the k-Catalan numbers, we have not yet justified our use of the phrase "equivalence class representative" earlier, and the existence and uniqueness of this representative will be crucial in constructing bijections between these equivalence classes and other k-Catalan objects. The following theorem provides the necessary justification.

Theorem 2.2. *Let $\alpha = a_1a_2\dots a_{kn+1} \in E_n^k$. Then all $kn+1$ cyclic shifts of α are distinct, exactly one of which has solely positive partial sums when assigning values $x = \frac{1}{k-1}$ and $y = -1$.*

Proof. This proof closely follows the case for a similar Catalan object as outlined in [4]. As already mentioned, $(k-1)n+1$ and n are relatively prime, so all of the cyclic shifts will be distinct. We prove the second result by induction on n.

When $n = 0$, regardless of the value of k, there is only one possible sequence, x, which has a positive partial sum.

Suppose that the claim is true for $n \geq 0$, and consider a sequence of $(k-1)(n+1)+1 = (k-1)n+k$ x's and $n+1$ y's, $\gamma = g_1g_2\dots g_{(k(n+1)+1}$, where $\gamma \in E_{n+1}^k$. If we divide the number of x's by the number of y's, we have

$$\frac{(k-1)n+k}{n+1} = \frac{k(n+1)-n}{n+1} = k - \frac{n}{n+1} > k-1.$$

Therefore, even if all of the y's were spread as far apart as possible in the sequence, making it possible that there are few x's between them, by the pigeonhole principle, there must still be a point at which we have a subseqence of $k-1$ x's followed by a y. Thus, there must exist a j such that $g_j = g_{j+1} = g_{j+(k-2)} = x$ and $g_{j+(k-1)} = y$. This subsequence may start near the end of γ and wrap back around to the beginning, so we may have $l = k(n+1)+1$ and $l+1 = 1$ for some l.

Remove $g_jg_{j+1}\dots g_{j+(k-1)}$ from γ to obtain the sequence $\beta = b_1b_2\dots b_{kn+1}$, which has precisely $(k-1)n+1$ x's and n y's. By the inductive hypothesis, there is a exactly one cyclic shift of $\beta, \beta' = b_lb_{l+1}\dots b_{l-1}$ which has strictly positive partial sums.

Suppose that $b_l = g_h$. Then let $\gamma' = g_hg_{h+1}\dots g_{h-1}$. As every partial sum of β' was positive, every partial sum of γ' up until g_j will be positive. Furthermore, the terms $g_jg_{j+1}\dots g_{j+(k-2)}$ will be $k-1$ x's and will add a total of 1 to the partial sum, and as each of these terms is positive, the successive partial sums will remain positive. Then $g_{j+(k-1)}$ is a y and will subtract 1 from the partial sum, and the partial sum will still be positive as the sum up to g_j was positive. After $g_{j+(k-1)}$, the partial sums will return to agreeing with those of β' and will therefore be positive.

It remains to be shown that γ' is the unique cyclic shift of γ with solely positive partial sums. The cyclic shift that begins with g_j will have the partial sum $g_j + g_{j+1} + \cdots + g_{j+(k-1)} = 0$, which is a partial sum that is not positive. Any cyclic shift that begins with g_{j+i} where $1 < i \leq k-1$ will have a partial sum $g_{j+i} + g_{j+i+1} + \cdots + g_{j+(k-1)} < 0$, and will therefore have a partial sum that is not positive. Any other cyclic shift of γ must begin with a term that corresponds to b_r, where $r \neq l$, and hence that shift will contain partial sums of the cyclic shift of β, $b_r b_{r+1} \ldots b_{r-1}$. By the induction hypothesis, β' is the unique cyclic shift of β such that the partial sums are solely positive, so at least one of the partial sums must be non-positive.

Therefore, by induction, γ' is the unique cyclic shift of γ such that all of the partial sums are solely positive. Hence, such cyclic shifts can serve as representatives for each equivalence class. ∎

3. k - GENERALIZED POLYOMINOES

In this section, we present a generalization of a pre-existing Catalan set of polyominoes, labelled #204 by Stanley [4]. Since polyominoes are harder to count directly, we will place them in bijection with k-generalized x, y equivalence classes in order to demonstrate that they are also counted by the k-Catalan numbers.

Definition 3.1. A *Catalan polyominoe* is a horizontally convex polyominoe of width $n+1$ such that each row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row. Each block in this polyominoe has a height and width of 1 unit.

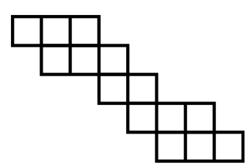

FIGURE 1. A Catalan polyominoe

Definition 3.2. A *k-generalized polyominoe* is a horizontally convex polyominoe of width $n+1$ such that each row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row. Each block in this polyominoe has a height and width of unit 1, with the possible exception of the end of every row but the last one. These rows can end in up to $(k-2)$ blocks of width $\frac{1}{k-1}$ and height 1. Additionally, subsequent rows must overlap with the previous row by at least 1 block.

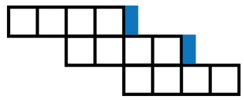

Figure 2. A 3-generalized polyominoe

Theorem 3.1. *There exists a bijection, $\phi : P_n^k \to E_n^k$, where P_n^k denotes the set of k-generalized polyominoes and E_n^k is the set of k-generalized x, y equivalence classes defined in the previous section.*

Proof. We form this bijection as follows:

(1) Given a polyominoe, $\alpha \in P_n^k$, construct the following sequences,
- $a_1 + 1, a_2 + 1, \ldots, a_l + 1$
- $b_1 + 1, b_2 + 1, \ldots, b_{l-1} + 1$

where $a_i + 1$ is the length of row i, and $b_i + 1$ is the length of the overlap between row i and row $(i+1)$.

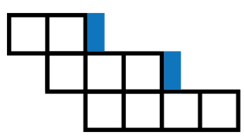

Figure 3. 3-generalized polyominoe for $n = 5$

row i	a_i	b_i
1	1.5	.5
2	2.5	1.5
3	3	–

Table 3.1. Row Widths and Overlaps

row i	$(a_i - b_{i-1})(k-1)$	$a_i - b_i$
1	$(1.5-0)(3-1)=3$	$1.5-.5=1$
2	4	1
3	3	3

TABLE 3.2. x & y Subsequences Corresponding to Figure 3

(2) We now create an x, y equivalence class, with partial sums that are locally maximal at values $a_1, a_2, \ldots, a_l$ and partial sums that are local minima at values $b_1, b_2, \ldots b_{l-1}$.
(3) Now that we have these lists of values, we can construct a sequence, $\beta \in E_n^k$. Heuristically speaking, the a_i's will aid us in determining the number of consecutive x terms in the sequence, and the b_i's will help determine the number of consecutive y terms.
(4) Compute $a_1(k-1)+1$. This will be how many x terms β begins with. While a_1 may not be an integer, its fractional part will be a multiple of $\frac{1}{k-1}$. Therefore, the value we compute for the number of x's in our sequence will be an integer, as each x is assigned a value of $\frac{1}{k-1}$, and this justification holds for all arithmetic done in these steps.
(5) Now compute $a_1 - b_1$. This is how many consecutive y's will follow the initial sequence of x terms created in the previous step.
(6) Compute $(a_2 - b_1)(k-1)$. This is the number of consecutive x terms following the subsequence of y's we constructed in the previous step.
(7) Continue this way for each a_i and a_i, b_i pair until you reach a_l. Note that in the following table, we treat undefined b_i values as zeros.

(8) After adding $(a_l - b_{l-1})(k-1)$ x terms followed by $a_l - b_l = a_l$ y terms (as b_l is treated as zero) to the end of the sequence, we are done.

Thus, we see that the output from Figure 3 is $xxxxyxxxxyxxxyyy$.

Now we will show that our output is a valid equivalence class representative. Since each polyominoe has a total width of $(n+1)$, we will add

$$((a_1) + (a_2 - b_1) + \cdots + (a_l - b_{l-1}))\,(k-1) + 1$$

x terms in the construction of our sequence. However, since we are subtracting off the overlap we incur from each row, we are actually just computing the width of the polyominoe. Therefore, this sum simplifies to

$$((a_1) + (a_2 - b_1) + \cdots + (a_l - b_{l-1}))\,(k-1) + 1 = n(k-1) + 1.$$

Therefore, our output sequence has the appropriate number of x's.

Additionally, we will add

$$(a_1 - b_1) + (a_2 - b_2) + \cdots + (a_{l-1} - b_{l-1}) + a_l$$

y terms to our sequence. However, we can reorder and regroup terms to obtain

$$(a_1) + (a_2 - b_1) + (a_3 - b_2) + \cdots + (a_{l-1} - b_{l-2}) + (a_l - b_{l-1}).$$

This is just the sum from above, in which we are subtracting off the width of the overlap from the total width of each row; which simplifies to n. So we add n y terms to our sequence as well.

First, note that since the output has $(k-1)n+1$ x terms and n y terms, it has the correct number of x and y terms to be a k-generalized cyclic equivalence class. In order to see that all of the partial sums will be positive, consider the jth local minimum. The jth local minimum is given by the formula,

$$\begin{aligned}
\frac{1}{k-1}(a_1(k-1)+1)-(a_1-b_1)+\sum_{i=2}^{j}\frac{1}{k-1}(a_i-b_{i-1})(k-1)-(a_i-b_i)\\
&= a_1+\frac{1}{k-1}-(a_1-b_1)+\sum_{i=2}^{j}(a_i-b_{i-1})-(a_i-b_i)\\
&= \frac{1}{k-1}+b_j\\
&> 0.
\end{aligned}$$

It then follows that since the local minimums of β are greater than 0, so are the rest of the partial sums.

Now we will show that ϕ is injective. Consider the following equivalence class generated using the above algorithm,

$$\beta = \underbrace{x\dots x}_{a_1(k-1)+1}\ \underbrace{y\dots y}_{a_1-b_1}\ \underbrace{x\dots x}_{(a_2-b_1)(k-1)}\ \dots \underbrace{y\dots y}_{a_l},$$

where from a given polyominoe the width of the first row is a_1, the width of the overlap between the first and the second row is b_1 and so on. Additionally, assume that this same equivalence class can be generated from a different polyominoe, with the width of the first row being a_1', the first overlap being b_1' and so on. Since the number of consecutive x's in β is directly determined by the widths of the rows in the input polyominoe, it is clear that $a_i = a_i'$. Similarly, since the number of consecutive y's in β is determined by the width of the row overlaps, it is clear that $b_i = b_i'$. Therefore, the two input polyominoes are actually the same element and we see that ϕ is injective.

We will prove ϕ is surjective by providing the construction for the inverse of the function outlined above.

(1) Begin with an x, y equivalence class representative.
    Example input: $xxxyxxxxxyyxxxyy$ $\quad k=3$, $n=5$
(2) Remove the first x from the sequence. With this alteration, our sequence will now have nonnegative partial sums. This is an important step since the values of the local minimas will correspond to the overlap between rows in the polyominoes. Reading the sequence from left to right, find and record the values of the partial sums.

Example: $xxyxxxxxyyxxxyy$
Partial Sums:

$$\begin{array}{lllll} s_1 = .5, & s_2 = 1, & s_3 = 0, & s_4 = .5, & s_5 = 1 \\ s_6 = 1.5, & s_7 = 2, & s_8 = 2.5, & s_9 = 1.5, & s_{10} = .5 \\ s_{11} = 1, & s_{12} = 1.5, & s_{13} = 2, & s_{14} = 1, & s_{15} = 0 \end{array}$$

(3) Call the value of the first local maximum a_1, the second local maximum a_2, and so on until you reach the last local maximum, a_l. Call the value of the first local minimum b_1, the second local minimum b_2, and so on until you reach the last local minimum, b_{l-1}.

Note that each time we encounter a y in the sequence, we subtract 1 from our partial sum. This decrease is precisely what makes the previous partial sum a local maximum. Thus, we can associate each local maximum to the end of a subsequence of consecutive x's within our sequence. Since each x has been assigned a value of $\frac{1}{k-1}$, this means that $(a_i - b_{i-1})(k-1)$ will give the number of consecutive x's that immediately follow the $(i-1)$st minimum. Furthermore, it follows that the sum of all such differences will yield the total number of x's in the sequence, which we know to be $(k-1)n$.

i	a_i	b_i
1	1	0
2	2.5	.5
3	2	–

TABLE 3.3. Local Maxima and Minima from Example Sequence

(4) The value of $a_1 + 1$ will be the width of the first row. If this is not an integer, draw a_1 with as many blocks of unit height and width as possible. Then, the remainder of the row will have width $c(\frac{1}{k-1})$ with $c < (k-1)$. Add c blocks of height 1 and width $\frac{1}{k-1}$ to the right of the last whole block.

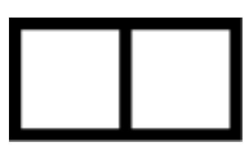

FIGURE 4. Row 1 from Example Input

(5) Similarly $a_2 + 1$ will be the length of the second row. However, $b_2 + 1$ gives the width of the blocks that the second row overlaps with the first row. So draw the second row the same way the first row was drawn, but begin drawing the second row $b_2 + 1$ units to the left of the end of the first row.

Recall that in step (2) we removed the first x in the input sequence so that our partial sums were nonnegative and could take on a value of 0. Since the overlap between rows is determined by the value of a minima $+1$, this forces the overlap to be greater than or equal to 1 block.

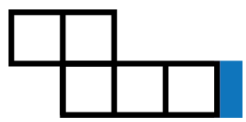

Figure 5. Rows 1 and 2 from Example Input

(6) Continue this way until you have drawn l rows.

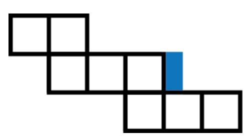

Figure 6. Output 3-generalized polyominoe from Example Input

We know that we get a k-generalized polyominoe because each partial sum that is a local minimum in the sequence must be smaller than the previous local maximum in the sequence (otherwise it would not be a local minimum) and they must differ at least by 1, due to the value assigned to y's in the sequence. Therefore, there will be at least one block in the previous row that is not overlapped by the second row, so next row will begin strictly to the right of the previous row.

Additionally, since any local maximum that comes after a local minimum must have a greater value, any row will end strictly to the right of the previous row. Partial blocks will only occur at the end of rows due to our construction. The difference between the last local maximum and the end of the sequence must be an integer value, since the rest of the sequence must be constructed from y's that have been assigned a value of -1. Therefore, the last row will have no partial blocks.

It remains to show that the output polyominoe has the correct width, $n+1$. Notice that the total width of the polyominoe is going to be the width of the first row, plus the width of the second row that extends past the first row, and so on for all the rows. This can be formulated as,

$$\begin{aligned}(a_1+1)+&(a_2+1-b_1-1)+(a_3+1-b_2-1)+...+(a_l-b_{l-1})= \\ &=(a_1)+(a_2-b_1)+(a_3-b_2)+\cdots+(a_{l-1}-b_{l-2})+(a_l-b_{l-1})+1 \\ &=\sum_{i=1}^{l} a_i-\sum_{i=1}^{l-1} b_i+1.\end{aligned}$$

Notice that the above sum consists of many differences of local maxima and minima. As already noted in step (3) of the construction, these differences, multiplied by $k-1$, give the length of a string of consecutive x's within our sequence. Therefore, the sum

$$\begin{aligned}&(k-1)(a_1)+(k-1)(a_2-b_1)+\cdots+(k-1)(a_{l-1}-b_{l-2})+(k-1)(a_l-b_{l-1}) \\ &=(k-1)\left(\sum_{i=1}^{l} a_i-\sum_{i=1}^{l-1} b_i\right)\end{aligned}$$

must give the total number of x's in the sequence, since it sums over all differences between maxima and minima. As the sequence consists of an equivalence class representative from E_n^k from which we removed one x, we know that the total number of x's in this sequence must be $(k-1)n$. Therefore, we have

$$(k-1)\left(\sum_{i=1}^{l} a_i-\sum_{i=1}^{l-1} b_i\right)=(k-1)n,$$

and thus

$$\sum_{i=1}^{l} a_i-\sum_{i=1}^{l-1} b_i=n.$$

Hence, we conclude that the width of the output polyominoe is

$$\sum_{i=1}^{l} a_i-\sum_{i=1}^{l-1} b_i+1=n+1,$$

as desired. Therefore, the output is in fact a valid element of P_n^k. Thus, the function is surjective, and along with the argument for injectivity above, this proves that the function is a bijection.

∎

Corollary 3.1. $|P_n^k|=C_n^k$ *for all* $n\geq 0$ *and* $k\geq 2$.

Proof. Via Theorem 3.1, there exists a bijection $\phi: P_n^k\to E_n^k$. For any $n\geq 0$ and $k\geq 2$, these sets will be finite, so we can conclude that they have the same cardinality. Via Theorem 2.1, we know that $|E_n^k|=C_n^k$, and it follows that $|P_n^k|=C_n^k$ as well. Therefore, k-generalized polyominoes are in fact k-Catalan objects. ∎

4. Summary

Thus, here we have seen two generalizations of pre-existing Catalan objects to be k-Catalan objects. The x, y equivalence classes and their corresponding counting argument both generalize relatively easily. However, polyominoes are more challenging to generalize, and the proof that our generalization is counted by the k-Catalan numbers involved a bijection with the equivalence classes. Hence we see that Catalan objects can be generalized to be counted by a much broader family of sequences; for an example of more such generalizations, see Heubach et al. [2]. Thus we see that we can frequently find generalizations of Catalan sets to this more general family, although the ease with which the set translates may vary.

REFERENCES

[1] H. Gould, *Binomial coefficients, the bracket functions, and compositions with relatively prime summands,* The Fibonacci quarterly, **4** (1964) no. 2, 241-259.

[2] Silvia Heubach, Nelson Y. Li, and Toufik Mansour, *A Garden of k-Catalan Structures*, 2008.

[3] Dongseok Kim, *On the (n, k)-th Catalan Numbers*, Communications of the Korean Mathematical Society, Vol. 23, No. 3 (2008), 349-356.

[4] Richard P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

Acknowledgements. This work was partially supported by National Science Foundation grant DMS-1659113, which funds a Research Experiences for Undergraduates program at Grand Valley State University. Additionally, the authors would like to thank their research mentor, Dr. Shelly Smith, for all of the help and guidance she has provided.

Emily S. Dautenhahn, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

E-mail address: emily.dautenhahn@uky.edu

Hannah E. Pieper, Department of Mathematics, Oberlin College, Oberlin, OH 44074, USA

E-mail address: hpieper@oberlin.edu

Received December 28, 2017; revised February 25, 2018.