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Properties of Weak Domain Representable Spaces

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What is a weak domain representable space?

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A space X is weak domain representable if and only if there is a weak domain D such that $\max D$, with the topology it inherits from the weakly way below topology, is homeomorphic to X.

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A weak domain is an ordered set with a relation $\ll_w$ defined as follows. For every $p, q \in D$, $p \ll_w q$ if and only if for every directed subset A of D with $\sup A = q$ there is $r \in A$ such that $p \leq r$.

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2. D is exact $[\forall p \in D(\downarrow_w p$ is directed with supremum $p)]$
3. $\ll_w$ is weakly increasing $[\forall p, q, r, s \in D(p \ll_w q \leq r \ll_w s \implies p \ll_w s)]$

What is the weakly way below topology?

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The weakly way below topology on a weak domain D is the topology generated by $\{\Uparrow_w p : p \in D\}$.

How does Weak Domain Representability Compare to Domain Representability?

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Domain Representable $\Longrightarrow$ Weak Domain Representable

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Domain Representable $\implies$ Weak Domain Representable

Weak Domain Representable $\not\Rightarrow$ Domain Representable

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Weak Domain Representable $\not\Rightarrow$ Domain Representable

Every LOTS is an open dense subset of a weak domain representable space [Mashburn, 2007].

Is every space Weak Domain Representable?

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THEOREM

Let X be a T_1 countable topological space and let N be the set of non-isolated elements of X. If X is weak domain representable, then N is nowhere dense in X.

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Therefore the rationals with either the usual or the cofinite topology are not weak domain representable.

Is every uncountable space weak domain representable?

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EXAMPLE

ω_1 with the cocountable topology is not weak domain representable.

Proof

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Let $q \in (\omega_1 \cap \twoheaduparrow_w p) - \{0\}$. There is $p_0 \in \twoheaddownarrow_w q$ such that $p_0 \not\ll 0$.
Set $A_0 = \{p_0\}$.

Proof

Let $\beta \in \omega_1$ with $\beta \geq 1$ and assume that $\forall \alpha \in \beta \exists A_\alpha \subseteq D$ such that

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1. A_α is a countable directed set
2. $\forall p \in A_\alpha (\omega_1 \cap \Uparrow_w p \neq \emptyset)$

Set $A = \cup_{\alpha \in \beta} A_\alpha$. Then $B = \cap \{\omega_1 \cap \Uparrow_w p : p \in A\}$ is a cocountable subset of ω_1.

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$A_{\omega_1} = \cup_{\alpha \in \omega_1} A_\alpha$ is a directed subset of D which has no supremum.

Do open subsets always inherit Weak Domain Representability?

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We have already seen that $\mathbb{Q}$ is an open dense subset of a weak domain representable space and that $\mathbb{Q}$ is not weak domain representable.

Do closed subsets always inherit Weak Domain Representability?

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So $\mathbb{Q}$ is a closed subset of a (weak) domain representable space.

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Let X be a weak domain representable space. If $Y \subseteq X$ such that $|Y| = |X|$ then Y, as a subspace of X, is weak domain representable.

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It is essential that only a finite number of elements of X be declared below a given element of X.

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Let $X = \mathbb{Q} \times (\omega + 1)$. Isolate every element of $\mathbb{Q} \times \omega$ and give each element of $\mathbb{Q} \times \{\omega\}$ its usual product neighborhood base. X is Baire and the set of non-isolated points of X is $\mathbb{Q} \times \{\omega\}$ which is nowhere dense. But $\mathbb{Q} \times \{\omega\}$ is not weak domain representable and therefore X cannot be weak domain representable.

Will the product of Weak Domains be a Weak Domain?

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If $\{D_\alpha : \alpha \in \kappa\}$ *is a collection of nonempty weak domains, each of which has a least element, then* $\Pi_{\alpha\in\kappa} D_\alpha$ *is a weak domain.*

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Of course, if a weak domain does not already have a least element, one can be added without changing the set of maximal elements. The result is similar to one for domains which can be found in *Continuous Lattices and Domains* by Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott.

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If $\{X_\alpha : \alpha \in \kappa\}$ is a collection of weak domain representable spaces then $\Pi_{\alpha\in\kappa} X_\alpha$ is weak domain representable.

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$I \times I$ is a complete metric space and therefore (weak) domain representable. $\mathbb{Q} \times I$ is a subspace having the same cardinality, so it is weak domain representable.
Önal and Vural recently showed that retracts of domain representable spaces are domain representable, so factors of a domain representable product must all be domain representable.

Will at least one of the factors of a product that is Weak Domain Representable also be Weak Domain Representable?

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Yes, if at least one of the factors has the same cardinality as the product.

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Let X be a set with a topology τ and let $S \subseteq X$. Define τ^S to be the topology generated by $\tau \cup \{\{p\} : p \in S\}$.

THEOREM

If $\langle X, \tau \rangle$ is weak domain representable and $S \subseteq X$, then $\langle X, \tau^S \rangle$ is weak domain representable.

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A similar result for domain representable spaces was obtained by Bennett and Lutzer in 2006.

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Let $\langle X, \tau \rangle$ be a topological space and $Y \subseteq X$. If $\langle X, \tau \rangle$ is weak domain representable, then so is $\langle X, \sigma \rangle$, where σ is the topology on X generated by $\tau \cup \{Y\}$

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Domain representable spaces do not have this property.

If a subset of a weak domain representable space is declared open, does the space remain weak domain representable?

Question: If $\langle X, \tau \rangle$ is weak domain representable and $\tau \subseteq \sigma$ is $\langle X, \sigma \rangle$ weak domain representable?

The End

Thank you!