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Coding Strategies, the Choquet Game and Domain Representability

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ABSTRACT. We prove that if the NONEMPTY player has a winning strategy in the strong Choquet game on a regular space X then NONEMPTY has a winning coding strategy in that game (a strategy that only depends on the previous 2 moves). We also prove that any regular domain representable space is generalized subcompact.

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1. INTRODUCTION AND MOTIVATION

To motivate completeness properties studied in this paper, we begin by sketching the proof that complete metric spaces are Baire spaces (that is, they satisfy the conclusion of the Baire Category Theorem). Let X be a complete metric space and let $\{U_n : n \in \omega\}$ be a decreasing sequence of dense open subsets of X. By induction, define nonempty open sets V_n satisfying $V_n \subseteq U_n$, $\operatorname{diam}(V_n) < 2^{-n}$, and, if $n > 0$, $\operatorname{cl} V_n \subseteq V_{n-1}$. Then, because X is complete, $\bigcap_{n\in\omega} V_n$ is not empty, and by construction, $\bigcap_{n\in\omega} V_n$ is contained in $\bigcap_{n\in\omega} U_n$.

Let us make some observations that will illustrate key concepts related to generalized completeness. Here we assume, as we do throughout the paper, that all spaces are regular. First, the proof above applies unchanged to show that compact spaces, in fact countably compact spaces, are Baire spaces. Second, the proof does not mention points, just open sets. Third, as the indices, n, get bigger, the open sets, V_n, get smaller. The main tool in the proof is the sequence $\{V_n : n \in \omega\}$, which is "strongly decreasing" ($\operatorname{cl} V_{n+1} \subseteq V_n$). A more general concept is that of a regular filter base. A family $\mathcal{F}$ of open sets is called a **regular filter base** if for all U, $W \in \mathcal{F}$, there is $V \in \mathcal{F}$ such that $\operatorname{cl} V \subseteq U \cap W$. So, in a compact space, all regular filter bases have nonempty intersection, in a countably compact space, all countable filter bases have nonempty intersection, and in a completely metrizable space, certain countable regular filter bases have nonempty intersection.

We continue with the historical development of completeness, postponing some definitions until Section 2. Čech complete spaces are those (completely regular) spaces that are G_δ subspaces of their Stone-Čech compactification. The class of Čech complete spaces is a nice subclass of the class of Baire spaces which includes all completely metrizable spaces and all locally compact spaces. In the 1950's, Bourbaki showed that arbitrary products of completely metrizable spaces are Baire. The proof is a straightforward modification of the proof that complete metric spaces are Baire: Let $\prod_{i\in I} X_i$ be a product of complete metric spaces. Define V_n as in the original proof, but additionally require V_n to be of the form

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$\prod_{i \in I}(V_n)_i$ such that if $(V_n)_i \neq X_i$, then $\mathrm{diam}(V_n)_i < 2^{-n}$. This result suggested finding a nice class of Baire spaces including the completely metrizable spaces and the locally compact spaces which is closed under arbitrary products. In the 1960's, three such classes were introduced. Oxtoby [Ox61] defined pseudocomplete spaces, a point-free analog of the complete metric spaces, Choquet [Ch69] introduced strongly α-favorable spaces by introducing points into the Banach-Mazur game, and DeGroot [dG63] introduced subcompact spaces. A space X is **subcompact** if there is a base $\mathcal{B}$ for X such that every regular filter base from $\mathcal{B}$ has nonempty intersection.

Also in the 1960's, Scott suggested using continuous directed complete posets, briefly called **domains**, as models for the lambda calculus. Researchers noted that the space of maximal points of a domain equipped with the Scott topology enjoys completeness properties similar to those discussed in the previous paragraph. For example, Martin [Ma03] showed that the space of maximal points is Choquet complete; hence if it is metrizable, it is completely metrizable. Bennett and Lutzer [BL06] called these spaces domain representable, proved that subcompact spaces are domain representable, and asked whether the converse holds. Fleissner and the author [FY] modified Debs' space to give a counterexample, and introduced properties between subcompact and domain representable. In this paper, we use coding to show that domain representable spaces are generalized subcompact. We use the same tools to show that if NONEMPTY has a winning strategy in the Choquet game, NONEMPTY in fact has a coding winning strategy.[1]

2. Definitions and Preliminaries

2.1. **Completeness properties.** Definition 2.1 gives a simplified characterization of domain representability. As shown in [FY13], a space X is represented by a triple $(Q, \ll, B)$ if and only if X is homeomorphic to the space of maximal points of a domain.

Definition 2.1. [FY13] We say that a triple $(Q, \ll, B)$ **represents** a T_1 space X if

(1) $\{B(q) : q \in Q\} \subseteq \tau^*(X)$ is a base for the topology on X,
(2) $\ll$ is a transitive, antisymmetric relation on Q,
(3) for all p, q in Q, $p \ll q$ implies $B(q) \subseteq B(p)$,
(4) For all $x \in X$, $\{q \in Q : x \in B(q)\}$ is upward directed by $\ll$, and
(5) if $D \subseteq Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(p) : p \in D\} \neq \varnothing$.

Roughly speaking, if $(Q, \ll, B)$ represents X, then X is homeomorphic to the space of maximal points of the domain generated by $(Q, \ll)$. The details of how a domain is generated, and even the precise definition of what a domain is will not be needed in this paper.

Given a space, X, represented by a triple, $(Q, \ll, B)$, there is no obvious reason to assume that the map B is one-to-one. (We will show how the flexibility of allowing $\mathcal{B}$ to be not one-to-one is useful in discussions after Questions 6.5 and 6.6.) If B were one-to-one, however, we could define $B(p) \ll B(q)$ if and only if $p \ll q$, making the index set Q is unnecessary. The order reversal between (3) above, and (G3) in the next definition, comes from the fact that Q is an index set, but $\mathcal{B}$ is a family of open sets.

[1]The author would like to thank William Fleissner for suggestions and historical information

Definition 2.2. [FY] We say that a T_1, regular space (X, τ) is **generalized subcompact** if there are $\mathcal{B}$ and $\prec$ satisfying

(G1) $B \subseteq \tau^*(X)$ is a base for τ,
(G2) $\prec$ is a transitive, antisymmetric relation on $\mathcal{B}$,
(G3) $B \prec B'$ implies $B \subseteq B'$,
(G4) if $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is downward directed by $\prec$, and
(G5) if $\mathcal{F} \subseteq \mathcal{B}$ and $(\mathcal{F}, \prec)$ is downward directed, then $\bigcap \mathcal{F} \neq \varnothing$.

We call such a base $\mathcal{B}$, **a GSC base** for X.

We write $B \prec_{cl} B'$ if and only if $\operatorname{cl} B \subseteq B'$. In Definition 2.2, if $\prec$ is replaced with $\prec_{cl}$, then items (G1) through (G5) characterize DeGroot's [dG63] subcompact property for regular spaces. The example in Section 5 of [FY] is generalized subcompact (hence domain representable) but not subcompact. It is a modification of Debs' example, [De85], of a space X on which NONEMPTY has a winning strategy in $BM(X)$ but no stationary winning strategy. (See Section 2.3 for definitions).

2.2. **Concepts with and without points.** The proof of Theorem 5.5 uses point free tools, and the proof of Theorem 5.8 uses the analogous concepts with points. We now introduce the basic definitions in pairs: the first without points, the second with points.

Definition 2.3.

(1) A collection, $\mathcal{B}$, of non-empty open subsets of a space X is called a π**-base** for the topology on X if for every $U \in \tau(X)$ there is $B \in \mathcal{B}$ with $B \subseteq U$.
(2) A collection, $\mathcal{B}$, of (non-empty) open subsets of a space X is called a **base** for the topology on X if for every $U \in \tau(X)$ and every $x \in U$ there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Definition 2.4.

(1) The π**-weight**, denoted $\pi w(X)$, of a space X is the minimum cardinality of a π-base for X.
(2) The **weight**, denoted $w(X)$, of a space X is the minimum cardinality of a base for X.

Definition 2.5.

(1) An open subset U of a space X is called π**-weight homogeneous** if every non-empty open subset V of U has the same π-weight as U.
(2) An open subset U of a space X is called **weight homogeneous** if every non-empty open subset V of U has the same weight as U.

Theorem 5.5 involves the (point-free) Banach-Mazur game, and in the proof, we create π-bases for π-homogeneous subspaces of X. Constructions in the proofs of Theorems 4.2 and 5.8 are analogous, but involve concepts "with points". Weight homogeneity is the most obvious analog of π-weight homogeneity, but it is not the most useful in this context. Instead, we need an analog that has weight *and* cardinality homogeneity. Here is the precise definition.

Definition 2.6. For a subspace Z of a space X, a collection $\mathcal{B} \subseteq \tau^*(X)$ is said to be **an outer base of** Z (in X) if for all $y \in Z$ and for all $U \in \tau^*(X)$ with $y \in U$ there is $B \in \mathcal{B}$ with $y \in B \subseteq U$. We write $w(Z, X)$ for the minimum cardinality of an outer base of Z in X. Note that $w(Z, X)$ may be different than (greater than) the weight of Z considered as a subspace of X. Let $\lambda = |Z|$ and $\kappa = w(Z, X)$. We say that Z is **WC homogeneous** if for every $V \in \tau^*(X)$ that meets Z, $w(V \cap Z, X) = \kappa$ and $|V \cap Z| = \lambda$. The letters WC stand for weight and cardinality. If Z is WC homogeneous and $\mu = \max\{\lambda, \kappa\}$ we say that Z **is μ-WC homogeneous**.

Although it does not appear in our theorems, Oxtoby's pseudocomplete is relevant to the topic. Pseudocompleteness is a point-free generalization of complete metrizability. Suppose X is a metric space and let $\mathcal{B}_n$ be the family of basic open balls of radius at most 2^{-n}. The metric is complete if and only if every regular filter base with elements from *every* $\mathcal{B}_n$ has non-empty intersection. A space, X, is **pseudocomplete** if there is a sequence, $\{\mathcal{B}_n : n \in \omega\}$, of π-bases for X such that every regular filter base with elements from every $\mathcal{B}_n$ has non-empty intersection.

The point-free analog of regularity is quasi-regularity: a space is **quasi-regular** if for every open subset U of X, there is a non-empty open subset V of X such that $\operatorname{cl} V \subseteq U$. Every quasi-regular pseudocomplete space satisfies the Baire category theorem via the proof in the first paragraph of the introduction.

2.3. **Topological games and winning strategies.** Both the Banach-Mazur game, denoted $BM(X)$, and the strong Choquet game, denoted $Ch(X)$, are two player infinite games played on a space X. The first player in these games is called β, Player I, or EMPTY and the second player is called α, Player II or NONEMPTY. More information about topological games can be found in Telgarsky's 1987 paper [Te87].

In the **Banach-Mazur game**, player EMPTY starts the first round by playing a non-empty open subset U_0 of X and then NONEMPTY responds with a non-empty open subset $V_0 \subseteq U_0$. In the second round, player EMPTY plays a non-empty open set U_1 with $U_1 \subseteq V_0$ and player NONEMPTY with a non-empty open subset $V_1 \subseteq U_1$. They continue in this manner for infinitely many rounds, producing a decreasing nested sequence $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$ of non-empty open subsets of X. If $\bigcap_{i \in \omega} U_i$ is nonempty, then NONEMPTY wins. Otherwise EMPTY wins.

The **strong Choquet game**, [Ch69], is similar to the Banach Mazur game, but the EMPTY player gets an advantage of selecting a point in addition to an open set. In the first round, player EMPTY starts by selecting a point, x_1 and an open set U_1 containing x_1 and then player NONEMPTY responds with an open set V_1 such that $x_1 \in V_1 \subseteq U_1$. In the second round, player EMPTY selects a point x_2 and an open set U_2 such that $x_2 \in U_2 \subseteq V_1$ and NONEMPTY responds with an open set V_2 such that $x_2 \in V_2 \subseteq U_2$. Continuing in this way, the players play infinitely many rounds generating the sequence $(U_1, x_1), V_1, (U_2, x_2), V_2 \ldots$. If $\bigcap_{i \in \omega} U_i$ is nonempty, then NONEMPTY wins. Otherwise EMPTY wins.

Note that in the Choquet game, NONEMPTY must choose an open set V that contains a point x and is contained in an open set U. Therefore, when studying the collection of possible moves for NONEMPTY in the Choquet game, one must consider a base for the topology. On the other hand, in the Banach Mazur game, NONEMPTY needs only to choose V a subset of U and hence he may play from a π-base.

A **strategy** for a player in $Ch(X)$ or $BM(X)$ is a rule for choosing what to play on each round given the full information of moves up until that point. A **winning strategy** for a player is a strategy that produces a win for that player in any game when playing according to that strategy. A **stationary** strategy is a strategy that only depends on the opponents previous move. A **coding strategy** for a player is a strategy that depends only on the opponent's previous move and the player's own previous move.

The following theorem, 4.3 from [FY], summarizes some results about strategies in the strong Choquet game played on complete spaces.

Theorem 2.7. *Let X be a space.*

(1) *If X is subcompact, then NONEMPTY has a stationary winning strategy in $Ch(X)$.*
(2) *If X is generalized subcompact, then NONEMPTY has a coding winning strategy in $Ch(X)$.*
(3) [Ma03] *If X is domain representable, then NONEMPTY has a winning strategy in $Ch(X)$.*

A space X satisfies the Baire Category Theoem if and only if EMPTY does not have a winning strategy in $BM(X)$ – again showing that the Baire Category Theorem is a point-free notion.

Results in the area of topological games are often expressed using the following terms.

Definition 2.8.

(1) X is called **weakly α-favorable** if NONEMPTY has a winning strategy in $BM(X)$.
(2) X is called **α-favorable** if NONEMPTY has a stationary winning strategy in $BM(X)$.
(3) X is called **Choquet complete** if NONEMPTY has a winning strategy in $Ch(X)$.
(4) X is called **strongly α-favorable** if NONEMPTY has a stationary winning strategy in $Ch(X)$.

3. Partition Lemmas

Theorem 5.8 concerns winning strategies for the NONEMPTY player in the strong Choquet game. As a warm-up, we prove an analogous result, Theorem 5.5, for the Banach-Mazur game, originally proven by Galvin and Telgarsky ([GT86], Theorem 7) and independently by Debs ([De85], Theorem 4). We continue, in this section, to organize lemmas leading to the two results in pairs. The first (point-free) in the pair leads to the Banach-Mazur game result and is followed by the corresponding item (with points) leading to the Choquet game result. In more detail, 3.1 and 3.3 lead to Theorem 5.5 and 3.2 and 3.4 lead to Theorem 5.8. The preliminary definitions and lemmas partition the space into homogeneous pieces and then define a special base (or π-base) serving as the collection of possible plays for NONEMPTY.

In Section 4, we use the tools leading to Theorem 5.8 from this section for a different purpose. We show that a domain representable space X is generalized subcompact by constructing a base, $\mathcal{B}$, for X out of the family of plays by NONEMPTY and defining the GSC relation $\prec$ on $\mathcal{B}$ using the triple $(Q, \ll, B)$.

Lemmas 3.1 and 3.3 are implied by Lemma 6.4 in [FY].

Lemma 3.1. *Let X be a space. There is a pairwise disjoint family $\mathcal{O}$ of π-weight homogeneous open subsets such that $\bigcup \mathcal{O}$ is dense in X.*

Proof. By recursion on η we will define an increasing sequence of open sets Y_η and pairwise disjoint open sets O_η. Set $Y_0 = \emptyset$. If Y_η has been defined, let O_η be an element of $\{B \in \tau^*(X) : B \cap Y_\eta = \emptyset\}$ that is minimum with respect to the cardinal $\pi w(O_\eta)$. Observe that O_η is π-weight homogeneous. Indeed, if V is a non-empty open subset of O_η, then $\pi w(V) \leq \pi w(O_\eta)$ since πw is monotone. Conversely, since O_η was chosen to be minimal with respect to $\pi w(O_\eta)$, it must be that $\pi w(V) = \pi w(O_\eta)$. Set $Y_{\eta+1} = Y_\eta \cup O_\eta$. If ξ is a limit ordinal, set $Y_\xi = \bigcup_{\eta<\xi} Y_\eta$. This recursion stops only when $\{B \in \tau^*(X) : B \cap Y_\nu = \emptyset\}$ is empty, which means that $Y_\nu = \bigcup_{\eta<\nu} O_\eta$ is dense in X. Set $\mathcal{O} = \{O_\eta : \eta < \nu\}$. $\square$

Lemma 3.2. *Let X be a space. There is a pairwise disjoint family $\mathcal{Z} = \{Z_\eta : \eta \in \nu\}$ of WC homogeneous sets such that $X = \bigcup \mathcal{Z}$ and $Y_\xi = \bigcup_{\eta<\xi} Z_\eta$ is open in X for each $\xi \leq \nu$.*

Proof. By recursion on η we will define an increasing sequence of open sets Y_η and pairwise disjoint sets Z_η. Set $Y_0 = \emptyset$. Suppose Y_η has been defined. Set $\hat{Z}_\eta = X \setminus Y_\eta$. Let U be any open subset of X meeting $\hat{Z}_\eta$ minimal with respect to the cardinals $w(U \cap \hat{Z}_\eta, X)$ and $|U \cap \hat{Z}_\eta|$. In other words, the pair $(w(U \cap \hat{Z}_\eta, X), |U \cap \hat{Z}_\eta|)$ is minimal in class of ordered pairs of cardinals with the product order. Then, $Z_\eta = U \cap \hat{Z}_\eta$ is μ_η-WC homogeneous where $\mu_\eta = \max\{\lambda_\eta, \kappa_\eta\}$ and $\lambda_\eta = |Z_\eta|$ and $\kappa_\eta = w(Z_\eta, X)$. Indeed, if V is a non-empty open subset of X that meets Z_η, then $w(V \cap Z_\eta, X) \leq w(Z_\eta, X)$ and $|V \cap Z_\eta| \leq |Z_\eta|$ since $w(\,\cdot\,, X)$ and cardinality are monotone. Conversely, since U was chosen to be minimal with respect to $w(U \cap \hat{Z}_\eta, X)$ and $|U \cap \hat{Z}_\eta|$, it must be that $w(V \cap Z_\eta, X) = w(V \cap U \cap \hat{Z}_\eta, X) = w(U \cap \hat{Z}_\eta, X) = w(Z_\eta, X)$ and $|V \cap Z_\eta| = |U \cap Z_\eta|$. Set $Y_{\eta+1} = Y_\eta \cup Z_\eta = Y_\eta \cup U$. If ξ is a limit, set $Y_\xi = \bigcup\{Z_\eta : \eta < \xi\}$. This recursion stops only when $X \setminus Y_\nu = \emptyset$, which means that $X = Y_\nu = \bigcup_{\eta<\nu} Z_\eta$. $\square$

Lemma 3.3. *Let O be π-weight homogeneous with $\pi w(O) = \mu$ infinite. There is a pairwise disjoint collection $\{\mathcal{B}_\alpha : \alpha < \mu\}$ such that each $\mathcal{B}_\alpha$ is a π-base for O.*

Proof. Fix any π-base, $\mathcal{B} = \{B_\gamma : \gamma < \mu\}$, for O of cardinality μ. Note that $|\{B \in \mathcal{B} : B \subseteq B_\gamma\}| = \mu$ for each $\gamma \in \mu$. Since $|\mu \times \mu| = \mu$, there is a bijection $\eta \mapsto (\gamma_\eta, \alpha_\eta)$ from μ onto $\mu \times \mu$. We define B^η by recursion on $\eta \in \mu$.

First, let $B^0 \in \mathcal{B}$ be such that $B^0 \subseteq B_{\gamma_0}$. Suppose B^β has been defined for each $\beta < \eta$. Choose $B^\eta \in \mathcal{B} \setminus \{B^\beta : \beta < \eta\}$ such that $B^\eta \subseteq B_{\gamma_\eta}$. Let $\mathcal{B}_\alpha = \{B^\eta : \alpha_\eta = \alpha\}$. We now verify that $\mathcal{B}_\alpha$ is a π-base for O for each $\alpha \in \mu$. Let $\alpha \in \mu$ and $\gamma \in \mu$. Since $\{B_\gamma : \gamma < \mu\}$ is a π-base for O, it suffices to find $B \in \mathcal{B}_\alpha$ with $B \subseteq B_\gamma$. There is $\eta \in \mu$ with $(\gamma_\eta, \alpha_\eta) = (\gamma, \alpha)$. So, by definition, $B^\eta \subseteq B_{\gamma_\eta} = B_\gamma$ and $B^\eta \in \mathcal{B}_\alpha$ since $\alpha_\eta = \alpha$. $\square$

Lemma 3.4. *Suppose $Z \neq \emptyset$ is a WC homogeneous subspace of an open subset O of a space X. Let $\mu = \max\{w(Z, X), |Z|\}$. There is a collection $\{\mathcal{A}_\alpha : \alpha \in \mu\}$, such that for each $\alpha \in \mu$*

(1) *$\mathcal{A}_\alpha \subseteq \tau^*(O) \subseteq \tau^*(X)$ is an outer base of Z in X,*
(2) *$|\mathcal{A}_\alpha| = \mu$,*
(3) *if $A \in \mathcal{A}_\alpha$ then $A \cap Z \neq \emptyset$,*
(4) *if $\beta \neq \alpha$ then $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$.*

Proof. Let λ be such that if $W \in \tau^*(X)$ and $W \cap Z \neq \emptyset$ then $|W \cap Z| = \lambda$. So, we have in particular (letting $W = X$) that $|Z| = \lambda$. If $\mu = \lambda = 1 = |Z|$, then we set $\mathcal{A}_0 = \{Z\}$ and we are done. Let κ be such that if $W \in \tau^*(X)$ and $W \cap Z \neq \emptyset$ then $w(W \cap Z, X) = \kappa$. If $|Z| = \lambda = 1 < \omega \leq \kappa = \mu$, it is straightforward to split one neighborhood base, $\mathcal{A}$, of x in X of size μ into μ pairwise disjoint neighborhood bases, $\mathcal{A}_\alpha$, each of size μ. Otherwise, Z is not a singleton and κ is infinite.

Claim For each $x \in Z$ and each neighborhood $V \in \tau^*(O)$ of x, there are μ distinct members, $\hat{V}$, of $\tau^*(O)$ with $x \in \hat{V} \subseteq V$. First, suppose $|V \cap Z| = \lambda \geq \kappa = w(Z, X)$. Then $\{V \setminus \{y\} : y \neq x\}$ is a collection of size μ of such sets, $\hat{V}$. If, on the other hand, $\lambda < \kappa$, then fix $y_0 \in V \cap Z$ with $y_0 \neq x$ and $V_0 \in \tau^*(O)$ with $x \in V_0 \subseteq \operatorname{cl} V_0 \subseteq V \setminus \{y_0\}$. Then, $V' = V \setminus \operatorname{cl} V_0$ is a non-empty open subset of X that meets Z, so $w(V', X) = \kappa$. Fix $\mathcal{W}$, a collection of $\mu = \kappa$ distinct non-empty open subsets of V'. Then $\{V_0 \cup W : W \in \mathcal{W}\}$ is a collection of μ distinct sets $\hat{V}$ with $x \in \hat{V} \subseteq V \subseteq O$. The Claim is proven.

Fix an outer base $\mathcal{A} \subseteq \tau^*(O)$ for Z in X of cardinality μ such that $|\{A : x \in A\}| = \mu$ for all $x \in Z$. Since $|Z| \leq \mu$, we may list the pairs (x, A) from $Z \times \mathcal{A}$ in a sequence $\{(x_\gamma, A_\gamma) : \gamma \in \mu\}$. Following the proof of Lemma 3.3, fix a bijection $\eta \mapsto (\gamma_\eta, \alpha_\eta)$ from μ onto $\mu \times \mu$. We define A^η by recursion on $\eta \in \mu$.

Let $A^0 \in \mathcal{A}$ be such that $x_{\gamma_0} \in A^0 \subseteq A_{\eta_0}$. Suppose that A^β has been defined for $\beta < \eta$. Choose $A^\eta \in \mathcal{A} \setminus \{A^\beta : \beta < \eta\}$ such that $x_{\gamma_\eta} \in A^\eta \subseteq A_{\gamma_\eta}$. Let $\mathcal{A}_\alpha = \{A^\eta : \alpha_\eta = \alpha\}$.

Items (1) through (4) of the Lemma are clear from the construction.

□

4. Domain Representability

We follow the proof of Proposition 6.1 in [FY] to create, from a partitioned collection of outer bases, an outer GSC base for a subspace Z of a domain representable space X.

Lemma 4.1. *Suppose Z is a non-empty subset of an open subset, O, of domain representable space X, μ is an infinite cardinal and for each $\alpha \in \mu$*

(1) *$\mathcal{A}_\alpha \subseteq \tau^*(O) \subseteq \tau^*(X)$ is an outer base of Z in X,*
(2) *$|\mathcal{A}_\alpha| = \mu$,*
(3) *if $A \in \mathcal{A}_\alpha$ then $A \cap Z \neq \emptyset$,*
(4) *if $\beta \neq \alpha$ then $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$.*

Then there is a collection $\mathcal{B} \subseteq \tau^*(O) \subseteq \tau^*(X)$ and a relation $\prec$ on $\mathcal{B}$ with the following properties.

(g1) *$\mathcal{B}$ is an outer base of Z in X,*
(g2) *$\prec$ is a transitive, antisymmetric relation on $\mathcal{B}$,*

(g3) $B \prec B'$ *implies* $B \subseteq B'$,
(g4) *for each* $y \in Z$, *the collection* $\{B \in \mathcal{B} : y \in B\}$ *is downward directed by* $\prec$, *and*
(g5) *if* $\mathcal{F} \subseteq \mathcal{B}$ *is downward directed by* $\prec$, *then* $\bigcap \mathcal{F} \neq \emptyset$.

Proof. Let $(Q, \ll, B)$ represent X. Without loss of generality, suppose Q has a $\ll$-least element 0_Q and that $B(0_Q) = X$. Also, fix a well-ordering, $\leq$, of Q. Set $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha \in \mu\}$ and let $h : \mathcal{A}^2 \to \mu\backslash\{0\}$ be a one-to-one function. Note that $\mathcal{A} \subseteq \tau^*(O) \subseteq \tau^*(X)$. We will define several things by induction on $n \in \omega$. Set $\mathcal{B}_0 = \mathcal{A}_0$. For each $A \in \mathcal{B}_0$, set $p(A) = 0_Q$, and $\mathrm{pred}(A) = \varnothing$.

Suppose for each $m < n$, that $\mathcal{B}_m$ has been defined, as well as $\mathrm{pred}(A)$ and $p(A)$ for every $A \in \bigcup_{m<n} \mathcal{B}_m$. Let T_n be the set of triples $t = (x, A', A'')$ satsfying $A' \in \mathcal{B}_{n-1}$, $A'' \in \bigcup_{m<n} \mathcal{B}_m$, and $x \in A' \cap A'' \cap Z$. Find $q_t \in Q$ such that $p(A')$, $p(A'') \ll q_t$, $x \in B(q_t) \subseteq A' \cap A''$, and q_t is the $\leq$-least member of Q with these properties. Find $A_t \in \mathcal{A}_{h(A',A'')}$ satisfying $x \in A_t \subseteq B(q_t)$. Set $p(A_t) = q_t$ and $\mathrm{pred}(A_t) = \{A', A''\} \cup \mathrm{pred}(A') \cup \mathrm{pred}(A'')$. The map $t \mapsto A_t$ is not necessarily one-to-one, but the assignment $A_t \mapsto q_t$ is well defined because of the well-ordering.[2] Also the function h ensures that $\mathrm{pred}(A_t)$ is well defined. Set $\mathcal{B}_n = \{A_t : t \in T_n\}$. Set $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$, and define $B \prec B'$ iff $B' \in \mathrm{pred}(B)$.

Note that like the domain representable relation $\ll$, the relation $\prec$ is not necessarily reflexive, and, in fact, in many cases it would be asymmetric.

Since $\mathcal{B}_0 = \mathcal{A}_0 \subseteq \mathcal{B}$ and $\mathcal{A}_0$ is an outer base of Z in X, we have that $\mathcal{B}$ is an outer base of Z in X. It is clear that $\mathcal{B} \subseteq \tau^*(O)$ since $\mathcal{B} \subseteq \mathcal{A}$. Items (g1) through (g4) are clear from construction. Towards (g5), suppose that $\mathcal{F} \subseteq \mathcal{B}$ is downward directed by $\prec$. If $\mathcal{F}$ has a minimal element A, then $\bigcap \mathcal{F} = A \neq \varnothing$. Otherwise $\mathcal{F}$ and $\{B(p(A)) : A \in \mathcal{F}\}$ are entwined and $\{p(A) : A \in \mathcal{F}\}$ is upward directed, hence $\bigcap \mathcal{F} = \bigcap\{B(p(A)) : A \in \mathcal{F}\} \neq \varnothing$. □

Theorem 4.2. *Suppose that a regular space* X *has a representation* $(Q, \ll, B)$. *Then* X *has a GSC base.*

Proof. Apply Lemma 3.2 to get $\mathcal{Z} = \{Z_\eta : \eta \in \nu\}$, a pairwise disjoint family of WC homogeneous subsets of X such that $X = \bigcup \mathcal{Z}$ and $Y_\xi = \bigcup_{\eta<\xi} Z_\eta$ is open in X for each $\xi \leq \nu$. For each $\eta \in \nu$, let $\mu_\eta = \max\{w(Z_\eta, X), |Z_\eta|\}$ and apply Lemma 3.4 with $Z = Z_\eta$, $O = Y_{\eta+1}$ and $X = X$ to get $\mathcal{A}^\eta_\xi$ for each $\xi < \mu_\eta$.

For each $\eta \in \nu$, apply Lemma 4.1 with $X = X$, $O = Y_{\eta+1}$, $Z = Z_\eta$, $\mu = \mu_\eta$ and $\mathcal{A}_\alpha = \mathcal{A}^\eta_\alpha$ to get $\mathcal{B}_\eta \subseteq \tau^*(Y_{\eta+1}) \subseteq \tau^*(X)$, an outer base of Z_η in $Y_{\eta+1}$ (hence also in X), and a GSC relation $\prec_\eta$ on $\mathcal{B}_\eta$. Note that $B \in \mathcal{B}_\eta$ implies $B \cap Z_\eta \neq \emptyset$ and for $\gamma < \eta$, $B' \in \mathcal{B}_\gamma$ implies $B \subseteq Y_{\gamma+1} \subseteq X \setminus Z_\eta$. So, we have that $\mathcal{B}_\eta \cap \mathcal{B}_\gamma = \emptyset$ for all $\eta \neq \gamma$. Set $\mathcal{B} = \bigcup_{\eta\in\nu} \mathcal{B}_\eta$ and define $B_1 \prec B_2$ if

(1) there is $\eta \in \nu$ with $B_1, B_2 \in \mathcal{B}_\eta$ and $B_1 \prec_\eta B_2$, or
(2) there are $\gamma, \eta \in \nu$ with $\gamma < \eta$, $B_1 \in \mathcal{B}_\gamma$, $B_2 \in \mathcal{B}_\eta$ and $B_1 \subseteq B_2$.

For a given pair B_1, B_2, at most one of the above holds since $\mathcal{B}_\eta \cap \mathcal{B}_\gamma = \emptyset$ for $\eta \neq \gamma$.

The relations $\prec_\eta$ and $\subseteq$ are antisymmetric and transitive. Also, that $\prec$ is transitive follows from the observations: ($B_1 \prec B_2 \subseteq B_3 \implies B_1 \subseteq B_3$) and ($B_1 \subseteq B_2 \prec_\eta B_3 \implies$

[2] This construction repairs an error in the proof of Proposition 6.1 from [FY].

$B_1 \subseteq B_3$). We verify that $(\mathcal{B}, \prec)$ is a GSC base for X.

(G1): Since $\mathcal{B}_\eta$ is an outer base of Z_η and $X = \bigcup_{\eta\in\nu} Z_\eta$, we have the $\mathcal{B} = \bigcup_{\eta\in\nu} \mathcal{B}_\eta$ is a base for X.
(G2): That $\prec$ is antisymmetric follows from the fact that $B_1 \prec B_2$ implies $B_1 \subseteq B_2$. Set $\eta(B) = \min\{\eta : B \in \mathcal{B}_\eta\}$. Suppose $B_1 \prec B_2 \prec B_3$. Then $\beta(B_1) \leq \beta(B_2) \leq \beta(B_3)$. Then, if $\beta = \eta(B_1) = \eta(B_2) = \eta(B_3)$ then $B_1 \prec_\eta B_3$ and hence $B_1 \prec B_3$. Otherwise, $\eta(B_1) < \eta(B_3)$, in which case $B_1 \subseteq B_3$ and so $B_1 \prec B_3$.
(G3): Follows from the fact that $B_1 \prec_\eta B_2$ implies $B_1 \subseteq B_2$ for all $\eta \in \nu$.
(G4): Fix $x \in X$, let $\beta = \min\{\eta : x \in Z_\eta\}$ and suppose $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$. Then $B_1 \cap B_2 \cap Y_\beta$ is an open set in X containing x. So, since $\mathcal{B}_\beta$ is an outer base of Z_β in X, there is $B_3 \in \mathcal{B}_\beta$ with $B_3 \subseteq B_1 \cap B_2$. If either of $\beta(B_1)$ or $\beta(B_2)$ is equal to β, then we can further assume that $B_3 \prec_\beta B_1$ or $B_3 \prec_\beta B_2$. If they are greater, then automatically $B_3 \prec B_1$ and $B_3 \prec B_2$.
(G5): Suppose $\mathcal{F} \subseteq \mathcal{B}$ is downward directed by $\prec$. As noted before, if $B_1 \prec B_2$ then $\beta(B_1) \leq \beta(B_2)$. If $\mathcal{F}$ has a $\prec$-minimum element, B, then $\bigcap \mathcal{F} = B \neq \emptyset$. If not, let β be the minimum of the set $\{\eta \in \nu : \exists B_1, B_2 \in \mathcal{F}, B_1 \prec_\eta B_2\}$. Then $\hat{\mathcal{F}} = \{B \in \mathcal{F} : B \in \mathcal{B}_\beta\}$ is a $\prec_\beta$ downward directed set in $\mathcal{B}_\beta$. Hence $\bigcap \mathcal{F} \supseteq \bigcap \hat{\mathcal{F}} \neq \emptyset$.

□

Corollary 4.3. *A regular space is homeomorphic to the maximal points of a domain if and only if it has a GSC base.*

We rephrase Theorem 4.2 in the language of domain theory. (For undefined terms see see [AJ94].) If X is domain representable, then there are a domain P, a basis Q for P such that $q \mapsto \Uparrow q \cap \max P$ from Q to $\tau^*(\max P)$ is one-to-one, and a homeomorphism $\phi : X \to \max P$. This answers Question 11.3 of [FY]. Examination of the proof of Theorem 4.2 shows that the cardinality of the basis Q is $\max\{|X|, w(X)\}$. Because $|Q|$ is bounded by cardinal functions on X, the domain P is a candidate for the "best" domain representing X, answering Question 11.10 of [FY].

5. Choquet Game

In [Ma03], Martin showed that if X is domain representable, then X is Choquet complete (NONEMPTY has a winning strategy in $Ch(X)$). It follows from Theorem 4.2 in this paper and Theorem 4.3 in [FY] that if X is domain representable then, in fact, NONEMPTY has a coding winning strategy in $Ch(X)$, answering a question of Bennet and Lutzer ([BLqa], 5.2 (a)). The goal of this section is to show something more. We show that if NONEMPTY has a winning strategy in $Ch(X)$, then NONEMPTY has a coding winning strategy in $Ch(X)$. We prove this result, Theorem 5.8, in two Lemmas, 5.6 and 5.7. First, we give an alternative proof (also broken into two Lemmas, 5.3 and 5.4) of a theorem of Galvin and Telgarsky (independently Debs) about the Banach-Mazur game to introduce some key ideas of the proof of Theorem 5.8.

Definition 5.1. The modification of the Banach-Mazur game on X in which the EMPTY player is required to play open sets from a specified π-base, $\mathcal{B}$, is denoted $BM_{\mathcal{B}}(X)$.

Suppose $Z \subseteq O \subseteq X$. The modification of the Choquet game on X in which the EMPTY player is required to play open subsets of O and points from Z is denoted $Ch(Z, O)$. A further modification requires EMPTY to play sets from a specified outerbase, $\mathcal{A} \subseteq \tau^*(O)$, for Z in X and is denoted $Ch_{\mathcal{A}}(Z, O)$.

The proof of Lemma 5.2 is straightforward and therefore omitted.

Lemma 5.2. *NONEMPTY has a winning coding strategy in $BM(X)$ (in $Ch(Z, O)$) if and only if NONEMPTY has a winning coding strategy in $BM_{\mathcal{B}}(X)$ (in $Ch_{\mathcal{A}}(Z, O)$).*

Lemmas 5.3 and 5.6 extend local winning strategies for NONEMPTY to global winning strategies. Although the lemmas are phrased in terms of coding strategies, the arguments are general enough that they apply to any type (stationary, Markov, n-tactics, etc.) of winning strategy.

Lemma 5.3. *Suppose that $\{O_\eta : \eta < \nu\}$ is a pairwise disjoint collection of open subsets of X whose union is dense in X. Then, if NONEMPTY has a coding winning strategy in $BM(O_\eta)$ for each $\eta \in \nu$, then NONEMPTY has a coding winning strategy in $BM(X)$.*

Proof. Suppose that $\{O_\eta : \eta < \nu\}$ is a pairwise disjoint collection of open subsets of X whose union is dense in X and that NONEMPTY has a coding winning strategy, σ_η, in $B(O_\eta)$ for each $\eta < \nu$. We define a coding winning strategy σ for NONEMPTY on X as follows. Let U be non-empty open in X. Since $\bigcup\{O_\eta : \eta < \nu\}$ is dense in X, we can choose $\eta < \nu$ such that $U \cap O_\eta \neq \emptyset$ and set $\sigma(U) = U \cap O_\eta$. For $U \subseteq V \subseteq O_\eta$, let $\sigma(V, U) = \sigma_\eta(V, U)$. The strategy σ is winning, since after NONEMPTY's first move, σ completely coincides with σ_η for some $\eta < \nu$.

□

Lemma 5.4. *Let O be a π-weight homogeneous open subset of X. If NONEMPTY has a winning strategy in $BM(O)$, then NONEMPTY has a coding winning strategy in $BM(O)$.*

Proof. By Lemma 3.3, we can define $\{\mathcal{B}_\alpha : \alpha < \mu\}$ a pairwise disjoint collection of pi-bases for O with $\mu = |\mathcal{B}_\alpha| = \pi w(O)$. Set $\mathcal{B} = \bigcup_{\alpha<\mu} \mathcal{B}_\alpha$. For each $B \in \mathcal{B}$, let $\alpha(B)$ be the unique α with $B \in \mathcal{B}_\alpha$.

Fix a winning strategy ρ for NONEMPTY in BM(O). By Lemma 5.2, it suffices to prove that NONEMPTY has a winning coding strategy, σ, in BM$_{\mathcal{B}}(O)$. For any open singleton, $\{x\}$, we must define $\sigma(\{x\})$ and $\sigma(\{x\}, \{x\})$ to be $\{x\}$. If $\{x\}$ is ever played, then all other plays must be $\{x\}$ and NONEMPTY wins the game.

We now describe a coding strategy, σ, for NONEMPTY in the case $\mu \geq \omega$. Let $\mathcal{J}$ be the collection of finite sequences from $\mathcal{B}$. We have that $|\mathcal{J}| = \mu$, so we may fix a bijection $h : \mathcal{J} \to \mu$.

For $U \in \mathcal{B}$, let $\hat{V} = \rho(U)$ and choose $V \in \mathcal{B}_{h(U)}$ with $V \subseteq \hat{V} \subseteq U$. Set $\sigma(U) = V$. For $W \in \mathcal{B}$ consider $J' = h^{-1}(\alpha(W))$. If J' is in the domain of ρ, then $J' = (U_0, V_0, \dots, U_n)$ for some $n \in \omega$ and we set $\hat{V} = \rho(J')$. Then, suppose $J = (U_0, V_0, \dots, U_n, \hat{V}, U)$ is also in the domain of ρ and choose V in $\mathcal{B}_{h(J)}$ with $V \subseteq \rho(J) \subseteq U$. Set $\sigma(W, U) = V$. If either J or J' is not in the domain of ρ, define $\sigma(W, U) = U$.

We now verify that σ is a winning coding strategy for NONEMPTY. Suppose EMPTY plays $U_0 \in \mathcal{B}$. Then $V_0 = \sigma(U_0) \in \mathcal{B}_{h(U_0)}$ with $U_0 \supseteq \rho(U_0) \supseteq V_0$. Then, EMPTY plays $U_1 \in \mathcal{B}$ with $U_0 \supseteq \rho(U_0) \supseteq V_0 \supseteq U_1$. Then NONEMPTY plays $V_1 = \sigma(V_0, U_1) \in \mathcal{B}_{h(U_0,\rho(U_0),U_1)}$ with

$$U_0 \supseteq \rho(U_0) \supseteq V_0 \supseteq U_1 \supseteq \rho(U_0, \rho(U_0), U_1) \supseteq V_1.$$

Continuing in this way, we get the sequence

$$U_0 \supseteq \rho(U_0) \supseteq V_0 \supseteq U_1 \supseteq \rho(U_0, \rho(U_0), U_1) \supseteq V_1 \cdots \supseteq U_n \supseteq \rho(U_0, \ldots, U_n) \supseteq V_n \supseteq \ldots.$$

Then,

$$U_0 \supseteq \rho(U_0) \supseteq U_1 \supseteq \rho(U_0, \rho(U_0), U_1) \supseteq V_1 \cdots \supseteq U_n \supseteq \rho(U_0, \ldots, U_n) \supseteq \ldots$$

is an entwined play of the game in which NONEMPTY uses the strategy ρ. Since ρ is a winning strategy for NONEMPTY, $\bigcap_{n\in\omega} U_n \neq \emptyset$. Hence σ is a winning coding strategy for NONEMPTY.

□

Theorem 5.5. *[Galvin and Telgarksy, 86] Suppose that NONEMPTY has a winning strategy in the Banach Mazur game BM(X). Then NONEMPTY has a coding winning strategy in BM(X).*

Proof. By Lemma 3.1 there is a pairwise disjoint family $\mathcal{O} = \{O_\eta : \eta < \nu\}$ of π-weight homogeneous open subsets such that $\bigcup \mathcal{O}$ is dense in X. Since O_η is open, a winning strategy, ρ, for NONEMPTY in BM(X) gives winning strategies for NONEMPTY in BM(O_η) for each $\eta < \nu$. By Lemma 5.4, NONEMPTY therefore has coding winning strategies in BM(O_η) for each $\eta < \nu$. Finally, by Lemma 5.3, NONEMPTY has a coding winning strategy in BM(X).

□

To motivate the proof of Theorem 5.8 involving the Choquet game, we recall the proof that NONEMPTY has a stationary winning strategy in $Ch(X)$ if X is scattered. A space X is scattered if every non-empty subset of X has a (relatively) isolated point. The η^{th} Cantor-Bendixon level X_η is defined inductively to be the set of isolated points of $X \setminus \bigcup_{\beta<\eta} X_\beta$. Suppose X is scattered and for each $x \in X$, let $\eta(x)$ be the unique η with $x \in X_\eta$. For $x \in X$ let $N(x)$ be an open neighborhood of x in X such that $N(x) \subseteq \bigcup_{\beta\leq\eta(x)} X_\beta$, and $N(x) \cap X_{\eta(x)} = \{x\}$. Then, a stationary winning strategy for NONEMPTY in the Choquet game is to play $V \cap N(x)$ in response to EMPTY playing V and $x \in V$. Since $N(x)$ consists only of points from an equal or lesser Cantor-Bendixon level, EMPTY's next point cannot come from a higher level. Therefore, EMPTY must eventually play the same point repeatedly, giving NONEMPTY the win.

In the proof of Theorem 5.8, X is partitioned into levels X_η and the players play from a base of sets B that are contained in initial unions $\bigcup_{\beta\leq\eta} X_\beta$. Like the scattered space proof, we arrange that plays only consist of points from the same level as the previous round or from a lower level. The game travels downward through the ordinal levels and therefore, after some round, plays must come from a constant level.

Lemma 5.6. *Suppose $X = \bigcup_{\eta\in\nu} Z_\eta$ where $\{Z_\eta : \eta \in \nu\}$ is pairwise disjoint and $Y_\xi = \bigcup_{\eta\in\xi} Z_\eta$ is open for each $\xi \leq \nu$. If NONEMPTY has a coding winning strategy* *in $Ch(Z_\xi, Y_{\xi+1})$ for each $\xi \leq \nu$, then NONEMPTY has a coding winning strategy in $Ch(X)$.*

Proof. Let σ_ξ be a coding winning strategy for NONEMPTY in $Ch(Z_\xi, Y_{\xi+1})$ for each $\xi \leq \nu$. Let $\xi(x)$ be the unique ξ for which $x \in Z_\xi$.

(1) For $x \in U \in \tau^*(X)$, set $\sigma(U,x) = \sigma_{\xi(x)}(U \cap Y_{\xi(x)+1}, x)$.
(2) For $x \in U \subseteq V$,
  (a) if $U \subseteq V \subseteq Y_{\xi(x)+1}$, then let $\sigma(V,U,x) = \sigma_{\xi(x)}(V,U,x)$
  (b) otherwise, let $\sigma(V,U,x) = \sigma_{\xi(x)}(U \cap Y_{\xi(x)+1}, x)$.

Consider a play of the game where NONEMPTY plays according to σ. The play is

$$U_0, x_0, V_0, U_1, x_1, V_1, \ldots, U_i, x_i, V_i, \ldots$$

where $V_0 = \sigma(U_0, x_0)$ and $V_i = \sigma(V_{i-1}, U_i, x_i)$ for each $i \geq 1$. Since $x_i \in V_i \subseteq Y_{\xi(x_i)+1}$ and $Y_{\xi(x_i)+1} \cap Z_\eta = \emptyset$ for all $\eta > \xi(x_i) + 1$, we have that $\{\xi(x_i) : i \in \omega\}$ is a non-increasing sequence of ordinals and hence eventually a constant value $\bar{\xi}$. Let $N \in \omega$ be least such that $\xi(x_N) = \bar{\xi}$. Then, the sequence

$$U_N, x_N, V_N, U_{N+1}, x_{N+1}, V_{N+1}, \ldots, U_{N+i}, x_{N+i}, V_{N+i}, \ldots$$

is a play of the game where EMPTY's plays are all of the form (U,x) where $x \in Z_{\bar{\xi}}$ and $U \subseteq Y_{\bar{\xi}+1}$ and NONEMPTY plays according to $\sigma_{\bar{\xi}}$. Therefore $\bigcap_{i\in\omega} V_i = \bigcap_{i\in\omega} V_{N+i} \neq \emptyset$. □

Lemma 5.7. *Suppose that X is a regular space for which NONEMPTY has a winning strategy in the Choquet game Ch(X). Let $O \subseteq X$ be open and let $Z \subseteq O$ be WC homogeneous in X. Then, NONEMPTY has a coding winning strategy in Ch(Z,O).*

Proof. Let $\mu = \max\{w(Z,X), |Z|\}$ and apply Lemma 3.4 to Z as a subspace of O to get $\mathcal{A}_\xi$ for each $\xi < \mu$. Set $\mathcal{A} = \bigcup_{\xi\in\mu} \mathcal{A}_\xi$. For $V \in \mathcal{A}$, there is a unique $\alpha(V) \in \mu$ with $V \in \mathcal{A}_{\alpha(V)}$. We will show that NONEMPTY has a coding winning strategy in $Ch_{\mathcal{A}}(Z,O)$. By Lemma 5.2, this will imply NONEMPTY has a coding strategy in $Ch(Z,O)$.

Fix a winning strategy, ρ, for NONEMPTY in the Choquet game played on X. If $\mu = 1$, then Z is a singleton $\{x\}$. In this case, define $\sigma(\{x\}, x) = \{x\}$ and $\sigma(\{x\}, \{x\}, x) = \{x\}$. Now, suppose that $\mu \geq \omega$. Let $\mathcal{J}$ be the collection of all finite sequences from $Z \cup \mathcal{A}$. Since $\mu = \max\{\mathcal{A}, |Z|\}$, we have that $|\mathcal{J}| = \mu$. Fix a bijection $h : \mathcal{J} \to \mu \setminus \{\emptyset\}$. For $x \in Z$ and $U \in \mathcal{A}$, set $\hat{V} = \rho(U,x)$. Choose $V \in \mathcal{A}_{h(U,x)}$ with $x \in V \subseteq \hat{V}$. Set $\sigma(U,x) = V$. For $W \in \mathcal{A}$, $x \in Z$ and $U \in \mathcal{A}$, consider $J' = h^{-1}(\alpha(W))$. Suppose that $J' = (U_0, x_0, V_0, \ldots, U_n, x_n)$ for some $n \in \omega$ and that J' is in the domain of ρ. Set $\hat{V} = \rho(J')$ and suppose that $J = (U_0, x_0, V_0, \ldots, U_n, x_n, \hat{V}, U, x)$ is also in the domain of ρ. Choose V in $\mathcal{A}_{h(J)}$ with $x \in V \subseteq \rho(J)$. Set $\sigma(W,U,x) = V$. If either of J or J' is not in the domain of ρ, define $\sigma(W,U,x) = U$.

We now verify that σ is a winning coding strategy for the game in which EMPTY plays (U,x) with $x \in Z$ and $U \in \mathcal{A}$. Suppose EMPTY plays (U_0, x_0) with $x_0 \in Z$ and $U_0 \in \mathcal{A}$. Then $V_0 = \sigma(U_0, x_0) \in \mathcal{A}_{h(U_0,x_0)}$ and $x_0 \in V_0 \subseteq \rho(U_0, x_0) \subseteq U_0$. Then, EMPTY plays (U_1, x_1) with $x_1 \in Z$ and $U_1 \in \mathcal{A}$ with $U_1 \subseteq V_0$. Then NONEMPTY plays $V_1 = \sigma(V_0, U_1, x_1) \in \mathcal{A}_{h(J_1)}$ where $J_1 = (U_0, x_0, \rho(U_0, x_0), U_1, x_1)$ and

$$x_1 \in V_1 \subseteq \rho(U_0, x_0, \rho(U_0, x_0), U_1, x_1) \subseteq U_1 \subseteq V_0 \subseteq \rho(U_0, x_0) \subseteq U_0.$$

Continuing in this way, we get the following sequences:

$$\hat{V}_0 = \rho(U_0, x_0),\ \hat{V}_1 = \rho(U_0, x_0, \hat{V}_0, U_1, x_1),\ \hat{V}_2 = \rho(U_0, x_0, \hat{V}_0, U_1, x_1, \hat{V}_1, U_2, x_2), \ldots,$$

$$V_0 = \sigma(U_0, x_0),\ V_1 = \sigma(V_0, U_1, x_1),\ V_2 = \sigma(V_1, U_2, x_2), \ldots,$$

where

$$U_0, x_0, V_0, U_1, x_1, V_1, \ldots, U_n, x_n, V_n, \ldots$$

is a play of the game where NONEMPTY plays according to σ and

$$U_0, x_0, \hat{V}_0, U_1, x_1, \hat{V}_1, \ldots, U_n, x_n, \hat{V}_n, \ldots$$

is an entwined play of the game where NONEMPTY plays according to ρ. Since ρ is a winning strategy for NONEMPTY, $\bigcap_{n\in\omega} U_n \neq \emptyset$. Hence σ is a winning coding strategy for NONEMPTY.

□

Theorem 5.8. *Suppose that X is a regular space for which NONEMPTY has a winning strategy in the Choquet game $Ch(X)$. Then NONEMPTY has a coding winning strategy in $Ch(X)$.*

Proof. Apply Lemma 3.2 to get $\mathcal{Z} = \{Z_\eta : \eta \in \nu\}$, a pairwise disjoint family of WC homogeneous subsets of X such that $X = \bigcup \mathcal{Z}$ and $Y_\eta = \bigcup_{\xi<\eta} Z_\xi$ is open in X for each $\eta \leq \nu$. Since NONEMPTY has a winning strategy in $Ch(X)$, it is immediate that NONEMPTY has a winning strategy in $Ch(Y_{\xi+1}, Z_\xi)$ for each $\xi \leq \nu$. By Lemma 5.7, NONEMPTY has a coding winning strategy in $Ch(Y_{\xi+1}, Z_\xi)$ for each $\xi \leq \nu$. Then, by Lemma 5.6, NONEMPTY has a coding winning strategy in $Ch(X)$.

□

6. QUESTIONS

Tactics are strategies for a player that only depend on the opponent's moves. So, a k-tactic is a strategy depending on the opponent's previous k moves. Debs' example [De85], mentioned previously, is a space X for which NONEMPTY has a winning 2-tactic in $BM(X)$, but has no stationary winning strategy (no winning 1-tactic).

Question 6.1 (Telgarsky)**.** *For each $k \in \omega$, is there a space X for which NONEMPTY has a winning $k+1$-tactic in BM(X), but no winning k-tactic?*

Debs' and independently Galvin and Telgarsky showed that if NONEMPTY has a winning strategy in $BM(X)$, then NONEMPTY has a winning coding strategy in $BM(X)$. The following question remains open, however. One may ask the same question for the strong Choquet game.

Question 6.2. *If NONEMPTY has a winning strategy in BM(X), does NONEMPTY necessarily have a winning 2-tactic in BM(X)?*

An interesting project is to determine which known results for the Banach-Mazur game (many appear in [GT86]) hold for the strong Choquet game. Theorem 5.8 is such a result and the following is an open question.

Question 6.3 (Galvin)**.** *If NONEMPTY has a Markov winning strategy in $Ch(X)$, does NONEMPTY necessarily have a stationary winning strategy in $Ch(X)$?*

The next questions involve the other completeness properties mentioned in this paper.

Question 6.4. *If X is a G_δ subspace of a subcompact space Y, is X subcompact?*

In particular,

Question 6.5. *If X is Čech complete, is X subcompact?*

Bennet and Lutzer proved that G_δ subspaces of domain representable spaces are domain representable. Using Definition 2.1: Suppose $(Q, \ll, B)$ represents X, $\{U_n : n \in \omega\}$ is a decreasing sequence of open subsets of X, and $Y = \bigcap_{n\in\omega} U_n$. Set $Q' = \{(q, n) \in Q : q \in Q$ and $B(q) \subseteq U_n\}$, let $(p, n) \ll' (q, m)$ iff $p \ll q$ and $n < m$, and let $B'((p, n)) = B(p) \cap Y$. Demonstrating that $(Q', \ll', B')$ represents Y is straightforward. This proof illustrates that it can be convenient to use a function B which is not one-to-one.

It is an open question whether if $X \times K$ is subcompact for compact K, then X is subcompact (or equivalently $X \times \{k\}$ is subcompact for some $k \in K$). However, Önul and Vural [OV] showed that continuous retracts of domain representable spaces are domain representable: Suppose $(Q, \ll, B)$ represents Y and $r : Y \to X$ is a continuous retract. They show that $(P, \ll_P, \phi)$ represents X, where $P = \{p \in Q : B(p) \cap X \neq \emptyset\}$, $\phi(p) = B(p) \cap X$, and $p \ll_P q$ iff $r(B(p)) \subseteq B(q)$. Again it is convenient to use a function B which is not one-to-one.

Question 6.6. *Are continuous retracts of subcompact spaces subcompact?*

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