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FROM SUBCOMPACT TO DOMAIN REPRESENTABLE

WILLIAM FLEISSNER AND LYNNE YENGULALP

ABSTRACT. We introduce the property generalized subcompactness and prove that subcompactness implies generalized subcompactness and that generalized subcompactness implies domain representability. We develop a simplified characterization of domain representability. We present an extension X of Debs' space and prove that X is generalized subcompact but α does not have a stationary winning strategy in the Banach-Mazur game on X . A fortiori, domain representability does not imply subcompactness. We investigate whether G_δ subspaces of subcompact (generalized subcompact, domain representable) spaces are subcompact (generalized subcompact, domain representable). We show that Čech complete generalized ordered spaces are subcompact. We show that the union of two domain representable subspaces is domain representable, and that a locally domain representable space is domain representable.

Domain representable subcompact α -favorable Choquet complete
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1. INTRODUCTION

There are two streams of research leading to the results of this paper. The first stream starts when de Groot 1963 introduces subcompact spaces [dG63]. He notes that the important and useful Baire Category Theorem is implied by two very different properties – complete metrizability and local compactness. He argues that there should be a concept unifying these properties. Čech completeness is such a concept, but it is too restrictive. For example, \mathbb{R}^X is not Čech complete, where \mathbb{R} is the real line, X is an uncountable index set, and \mathbb{R}^X has the usual finite support topology.

A second stream of research starts with continuous directed complete partial orders, often called domains. Domains have been extensively studied since they were introduced by Scott as a model for the λ -calculus, see [AJ94] for definitions, history, etc. In 2003 Martin [Ma03] asserts that the question “Which classical spaces have domain theoretic

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models?” (known as *the model problem* in certain circles) stands on its own as an alluring foundational issue in need of resolution. He made significant progress, proving that a metric space may be realized as the set of maximal elements in a domain if and only if it is completely metrizable by showing more generally that the space of maximal elements in a domain is always complete in a sense first introduced by Choquet [Ch69].

These two streams merge in recent work by Bennett and Lutzer, especially the survey paper [BLqa], where they ask several questions which we answer, fully or partially (– to be specific, Questions 3.4, 3.6, and 5.2, 5.4, and 7.7(c)).

In Section 2, we present the terminology needed to define subcompactness. We discuss the G_δ problem, and present two generalizations of subcompactness, the first of which we call generalized subcompactness. In Section 3, we present the terminology needed to define domain representability and show the second generalization of subcompactness is equivalent to domain representability. We give criteria on a domain P which imply that $\max P$ is generalized subcompact. In Section 4 we define the Banach-Mazur and Choquet games along with related completeness properties and summarize their relationships to the other properties of this paper. In Section 5 we present an example of a domain representable space X in which α does not have a stationary winning strategy in the Banach-Mazur game, hence X is not subcompact. In Section 6 we discuss a situation where domain representable spaces are generalized subcompact. We return to the properties of domain representable spaces and generalized subcompact spaces in Section 7 where we discuss basic constructions and in Section 8 where we discuss spaces with G_δ diagonals. In Section 9 we consider G_δ subspaces. In particular, we prove that Čech complete generalized ordered spaces are subcompact. Working with subcompact spaces, we observed that modifying a subcompact base often ruins the subcompact property. We make “subcompact bases are fragile” precise in Section 10. We finish with a list of questions suggested by the results in this paper.

2. SUBCOMPACTNESS AND GENERALIZATIONS

When X is a topological space, we let $\tau(X)$ denote the topology on the set X , and we let $\tau^*(X)$ denote the family of nonempty elements of $\tau(X)$. When A is a subset of a topological space X , we denote the closure of A in X by $\text{cl}_X A$ and the interior of A in X by $\text{int}_X A$.

When discussing open filter bases and completeness properties, we often say $\mathcal{B} \subseteq \tau^*(X)$ is a base for X instead of $\mathcal{B} \cup \{\emptyset\}$ is a base for X .

Definition 2.1. An **upward directed set** is a nonempty set P together with a reflexive and transitive binary relation (usually \ll or \prec) with the additional property that every pair of elements has an upper bound. Downward directed is defined analogously. Let us define \prec_{cl} on $\tau^*(X)$ via $V \prec_{\text{cl}} U$ iff $\text{cl } V \subseteq U$. An **open filter base** on a space X is a nonempty subset \mathcal{F} of $\tau^*(X)$ such that (\mathcal{F}, \subseteq) is downward directed. A **regular open filter base** on a space X is a nonempty subset \mathcal{F} of $\tau^*(X)$ such that $(\mathcal{F}, \prec_{\text{cl}})$ is downward directed. In this example, $U \prec_{\text{cl}} U$ iff U is clopen.

De Groot [dG63] calls a completely regular space X subcompact if it has an open base \mathcal{B} with the property that every regular open filter base from \mathcal{B} has nonempty intersection. Here is a redundant list of items characterizing T_1 , regular, subcompact spaces.

Definition 2.2. We say that a T_1 , regular space (X, τ) is **subcompact** when there is \mathcal{B} satisfying

- (1) $\mathcal{B} \subseteq \tau^*(X)$ is a base for τ ,
- (2) \prec_{cl} is a transitive, antisymmetric relation on \mathcal{B} ,
- (3) $B \prec_{\text{cl}} B'$ implies $B \subseteq B'$,
- (4) if $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is downward directed by \prec_{cl} , and
- (5) if $\mathcal{F} \subseteq \mathcal{B}$ and $(\mathcal{F}, \prec_{\text{cl}})$ is downward directed, then $\bigcap \mathcal{F} \neq \emptyset$.

In this situation, we say that the base \mathcal{B} is **subcompact**.

We say that X is **countably subcompact** if (1)-(4) and (5) $_{\omega_1}$ hold.

- (5) $_{\omega_1}$ if $\mathcal{F} \subseteq \mathcal{B}$, \mathcal{F} is countable and $(\mathcal{F}, \prec_{\text{cl}})$ is downward directed, then $\bigcap \mathcal{F} \neq \emptyset$.

The Baire Category Theorem involves the intersection of countably many dense open sets. The completely metrizable spaces are exactly those spaces which are G_δ in every metrizable extension. Analogously, the Čech complete spaces are exactly those spaces which are G_δ in every completely regular extension. In this context, it is natural to ask about G_δ subspaces of subcompact spaces. Bennett and Lutzer write [BLqa]

It is surprising that, after almost 50 years, open questions still remain about de Groot's subcompactness and the other Amsterdam properties. The most fundamental is Question 3.1. Suppose X is subcompact and Y is

a (dense) G_δ -subset of X . Is Y subcompact? In particular, must every Čech-complete space be subcompact?

We present a naive attempt towards proving that a G_δ subspace of a subcompact space is subcompact. Let \mathcal{B} be a subcompact base for a space X , let $\{U_n : n \in \omega\}$ be a sequence of open subsets of X such that $U_{n+1} \subseteq U_n$ for all n , and let $Y = \bigcap_{n \in \omega} U_n$. Define $B \prec_Y B'$ iff $B \prec_{cl} B'$ and $\forall n((\forall m < n \ B' \subseteq U_m) \rightarrow B \subseteq U_n)$. With this definition, \prec_Y -directed sets have non-empty intersection on Y . One flaw is that the definition of subcompact does not allow us to define the relation on the base; instead it requires specifically that \prec_{cl} -directed sets have non-empty intersection. We obtain the notion generalized subcompactness by allowing a binary relation \prec similar to, but not necessarily equal to \prec_{cl} .

Definition 2.3. We say that a T_1 , regular space (X, τ) is **generalized subcompact** if there are \mathcal{B} and \prec satisfying

- (1) $B \subseteq \tau^*(X)$ is a base for τ ,
- (2) \prec is a transitive, antisymmetric relation on \mathcal{B} ,
- (3) $B \prec B'$ implies $B \subseteq B'$,
- (4) if $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is downward directed by \prec , and
- (5) if $\mathcal{F} \subseteq \mathcal{B}$ and (\mathcal{F}, \prec) is downward directed, then $\bigcap \mathcal{F} \neq \emptyset$.

We say that that X is countably generalized subcompact if (1)-(4) and (5) $_{\omega_1}$ hold.

- (5) $_{\omega_1}$ if $\mathcal{F} \subseteq \mathcal{B}$, \mathcal{F} is countable and (\mathcal{F}, \prec) is downward directed, then $\bigcap \mathcal{F} \neq \emptyset$.

A second flaw in this naive attempt is that the relation \prec_Y is on $\tau^*(X)$, rather than on $\tau^*(Y)$. If Y is dense in X , we can repair this flaw – see Theorem 9.2. If we ask about not necessarily dense G_δ subspaces of X , then restriction map from $\tau^*(X) \rightarrow \tau^*(Y)$ defined by $B \mapsto B \cap Y$ is irreparably not one-to-one. Definition 2.4 allows a multi-valued indexing of the subcompact base of Y . (Why the order reversal? \prec is a relation on open sets, while \ll is a relation on the indices of the open sets).

In Section 3, we will show that the spaces that satisfy Definition 2.4 are exactly the domain representable spaces.

Definition 2.4. [FY13] We say that a triple (Q, \ll, B) **represents** a T_1 space (X, τ) if

- (1) $\{B(q) : q \in Q\} \subseteq \tau^*(X)$ is a base for τ ,
- (2) \ll is a transitive, antisymmetric relation on Q ,

- (3) for all p, q in Q , $p \ll q$ implies $B(q) \subseteq B(p)$,
- (4) For all $x \in X$, $\{q \in Q : x \in B(q)\}$ is upward directed by \ll , and
- (5) if $D \subseteq Q$ and (D, \ll) is upward directed, then $\bigcap \{B(p) : p \in D\} \neq \emptyset$.

Later, we will consider spaces in which (1)-(4) and $(5)_{\omega_1}$ hold.

- $(5)_{\omega_1}$ if $D \in [Q]^{<\kappa}$ and (D, \ll) is countable and upward directed, then $\bigcap \{B(p) : p \in D\} \neq \emptyset$.

It is clear that subcompact spaces are generalized subcompact, and it is easy to verify that generalized subcompact spaces have triples representing them. The converse holds if the map B is one-to-one.

Lemma 2.5. *If X is generalized subcompact, then there is a triple (Q, \ll, B) which represents X . Moreover, B is one-to-one.*

Proof. Let \mathcal{B} and \prec be as in Definition 2.3. Define Q to be \mathcal{B} and let B be the identity map on \mathcal{B} . For $V, V' \in \mathcal{B} = Q$, let $V \ll V'$ if and only if $V' \prec V$. \square

Lemma 2.6. *If the triple (Q, \ll, B) represents a space X and the map B is one-to-one, then X is generalized subcompact.*

Proof. Set $\mathcal{B} = \{B(q) : q \in Q\}$ and $B(q) \prec B(q')$ iff $q' \ll q$. Because B is one-to-one, the definition of \prec is unambiguous. \square

3. DOMAINS

In the introduction, we mentioned continuous, directed complete, partially ordered sets. We begin this section by defining these concepts.

Definition 3.1. We say that a partially ordered set (P, \sqsubseteq) is **directed complete**, or that (P, \sqsubseteq) is a **dcpo**, when every upward directed subset D of P has a least upper bound, denoted $\bigsqcup D$. One writes that $a \ll b$ (often spoken, “ a is **approximates** b ” or “ a is **way below** b ”) if for each directed set $D \subseteq P$ having $b \sqsubseteq \bigsqcup D$, some $d \in D$ has $a \sqsubseteq d$. Note that \ll is transitive and antisymmetric. For each $a \in P$ define $\downarrow(a) = \{b \in P : b \ll a\}$. A dcpo P is said to be **continuous** if $\downarrow(a)$ is directed and has $a = \sup(\downarrow(a))$ for each $a \in P$. A continuous dcpo is often called a **domain**. A domain with the additional property that every bounded doubleton has a least upper bound is called a **Scott domain**. A subset Q of P is called **basis** for P if for every element p of P , the set $\downarrow(p) \cap Q$ contains a directed set with supremum p . For example, a dcpo P is a basis for itself.

Given a domain (P, \sqsubseteq) with basis Q , we let $\uparrow(a) = \{c \in P : a \ll c\}$ for each $a \in P$. Then the collection $\{\uparrow(a) : a \in Q\}$ is a

base for what is called the **Scott topology** on P , and the collection $\{\uparrow(a) \cap \max(P) : a \in Q\}$ is a base for the subspace topology on the set $\max(P)$ consisting of all maximal elements of P . When a space X is homeomorphic to the space $\max(P)$ for a continuous dcpo, Martin [Ma03] writes that X has a model, while Bennett and Lutzer [BLqa] write that X is **domain representable**.

We record a few useful results.

Lemma 3.2. *Let P be a domain and Q a basis for P .*

- (1) *(Increasing property) If $a \sqsubseteq b \ll c \sqsubseteq d$, then $a \ll d$.*
- (2) *(Interpolation property) Let $y \in P$, and let M be a finite subset of P such that $m \ll y$ for each $m \in M$. Then there exists $y' \in Q$ such that $m \ll y' \ll y$ for all $m \in M$.*
- (3) *Let $D \subseteq Q$ be upward directed by \ll . Then $d \ll \bigsqcup D$ for all $d \in D$.*

Preparations completed, we are ready to show that given a domain representable space, X , there is a triple as in Definition 2.4 representing X .

Lemma 3.3. *Suppose that Q is a basis for a domain (P, \sqsubseteq) and $\phi : X \rightarrow \max P$ is a homeomorphism. Define $B : Q \rightarrow \tau^*(X)$ via $B(q) = \{x \in X : q \ll \phi(x)\}$ and define \ll_Q to be the restriction of \ll to Q . Then the triple (Q, \ll_Q, B) represents X .*

Proof. Items (1), (2), and (3) of Definition 2.4 are obvious. Item (4) follows from the interpolation property. Towards item (5), let $D \subseteq Q$ be upward directed by \ll_Q . Let $x \in X$ satisfy $\bigsqcup D \sqsubseteq \phi(x) \in \max(P)$. By items (3) and (1) of Lemma 3.2 we have $d \ll x$ for each $d \in D$. Hence $x \in \bigcap \{B(q) : q \in D\}$. \square

Next we aim for the converse, that spaces represented by triples are domain representable. We follow the proof that subcompact spaces are domain representable [BLHo, Theorem 3.1]. First we review the well known result that the ideal completion of a poset is a domain.

Definition 3.4. Let (Q, \leq) be a poset. An **ideal** J of Q is an upward directed lower set ($q \leq q' \in J$ implies $q \in J$). For each $q \in Q$, the set $\downarrow q$ is a principal ideal. Let $\text{Idl}(Q)$ be the family of all ideals on Q . The pair $(\text{Idl}(Q), \subseteq)$, where \subseteq is the usual subset relation, is called the **ideal completion** of Q .

Proposition 3.5. [AJ94, 2.2.22] *Let (Q, \leq) be a poset. Then the pair $(\text{Idl}(Q), \subseteq)$ is a domain. The map $i : q \mapsto \downarrow q$ is order reversing. The set $i[Q]$ is a basis for $(\text{Idl}(Q), \subseteq)$.*

We are going to follow points of X through the construction of the ideal completion to see that they correspond to maximal elements of a domain.

Theorem 3.6. *Suppose that the triple (Q, \ll_Q, B) represents a space X . For $q, q' \in Q$, set $q \leq q'$ iff $q \ll_Q q'$ or $q = q'$. For $x \in X$, define $N(x) = \{q \in Q : x \in B(q)\}$. Then N is a homeomorphism of X onto $\max \text{Idl}(Q)$. Hence X is domain representable.*

Proof. Our first goal is that $N(x)$ is a maximal ideal of (Q, \leq) and hence a maximal element of $\text{Idl}(Q)$.

Claim. $x \neq y$ implies $N(x) \not\subseteq N(y)$.

By Definition 2.4, $\{B(q) : q \in Q\}$ is a base for a T_1 topology on X , hence there is $q \in Q$ with $x \in B(q)$ and $y \notin B(q)$.

Claim. If $M \in \max(\text{Idl}(Q))$, then $\bigcap \{B(q) : q \in M\} \neq \emptyset$.

If M has a maximal element p , then $\bigcap \{B(q) : q \in M\} = B(p) \neq \emptyset$. If not, then M is \ll_Q -upward directed and by item (5) of Definition 2.4, $\bigcap \{B(p) : p \in M\} \neq \emptyset$.

Claim. $\max(\text{Idl}(Q)) \subseteq \{N(x) : x \in X\}$.

First, note that for any $x \in X$, by (3) and (4) of Definition 2.4, $N(x)$ is an ideal. Let $M \in \max(\text{Idl}(Q))$. If $x \in \bigcap \{B(p) : p \in M\}$, then $M \subseteq N(x) = \{p \in Q : x \in B(p)\}$, and hence $M = N(x)$ because M is maximal.

Claim. $\{N(x) : x \in X\} \subseteq \max(\text{Idl}(Q))$.

Let $x \in X$. Since $N(x)$ is an ideal, it is contained in a maximal ideal M . As in the previous claim, $M \subseteq N(y)$ for some y . Therefore $N(x) \subseteq M \subseteq N(y)$ and X is T_1 gives $x = y$ and $M = N(x)$.

Having shown that N is a bijection, we next show that it is a homeomorphism. Recall that $\{B(p) : p \in Q\}$ is a base for X and that the collection $\{\uparrow(\downarrow q) \cap \max(\text{Idl}(Q)) : q \in Q\}$ is a base for the subspace topology on the set $\max(\text{Idl}(Q))$. Hence it will suffice to show

Claim. For every $q \in Q$, $\{N(x) : x \in B(q)\} = \uparrow(\downarrow q) \cap \max(\text{Idl}(Q))$.

Fix $q \in Q$. Then, $\{N(x) : x \in B(q)\} = \{N(x) : q \in N(x)\} = \{M \in \max(\text{Idl}(Q)) : q \in M\} = \uparrow(\downarrow q) \cap \max(\text{Idl}(Q))$. \square

The construction above is valid for Scott domains.

Proposition 3.7. *We use the notation of the proof of Theorem 3.6. Let (Q, \ll_Q, B) represent a space X . If Q has the property that every doubleton with an upper bound has a least upper bound, then $\text{Idl}(Q)$ does too. Hence X is Scott domain representable.*

Proof. Let $J, J' \in \text{Idl}(Q)$ have an upper bound J'' . If $q \in J$ and $q' \in J'$, then $\{q, q'\}$ has an upper bound in J'' . Then by hypothesis, $\{q, q'\}$ has a least upper bound, call it $\sqcup(q, q')$. Set $J^* = \bigcup\{\downarrow \sqcup(q, q') : q \in J \text{ and } q' \in J'\}$. J^* is the least upper bound of $\{J, J'\}$. \square

Proposition 3.8. *A space X is generalized subcompact iff there are P, Q , and ϕ satisfying*

- (1) P is a domain with basis Q ,
- (2) the map $q \mapsto \uparrow(q) \cap \max P$ from Q to $\tau^*(\max P)$ is one-to-one, and
- (3) ϕ is a homeomorphism from X to $\max P$.

Proof. If X is generalized subcompact, then X is represented by a triple (Q, \ll, B) with B one-to-one by Lemma 2.5, and then X is domain representable by Lemma 3.6. Moreover, because B is one-to-one, it follows from the definition of $N(x)$ that the map $q \mapsto \uparrow(q) \cap \max P$ from Q to $\tau^*(\max P)$ is one-to-one. The converse follows from Lemma 3.3 and Lemma 2.6. \square

Having shown that the domain representable spaces are exactly those spaces that are represented by triples (Q, \ll, B) satisfying (1) - (5) of Definition 2.4, we define a **countably domain representable space** to be one that is represented by a triple (Q, \ll, B) satisfying (1) - (4) and $(5)_{\omega_1}$.

4. TOPOLOGICAL GAMES

We give a short introduction to topological game and related completeness properties. A more comprehensive introduction to the subject is in the most recent survey of topological games, Telgarsky's 1987 paper [Te87].

The **Banach-Mazur game** on a space X is a two player infinite game denoted $BM(X)$. Player β starts the first round by playing a non-empty open subset U_0 of X and then player α responds with a non-empty open subset $V_0 \subseteq U_0$. In the second round, player β plays a non-empty open set U_1 with $U_1 \subseteq V_0$ and player α with a non-empty open subset $V_1 \subseteq U_1$. They continue in this manner for infinitely many rounds, producing a decreasing nested sequence $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$ of non-empty open subsets of X . If $\bigcap_{i \in \omega} U_i$ is nonempty, then α wins. Otherwise β wins. Player α is sometimes referred to as NONEMPTY and β as EMPTY.

The **strong Choquet game** on a space X , denoted $Ch(X)$, is similar to the Banach Mazur game, but the EMPTY player, β , gets an advantage of selecting a point in addition to an open set. In the first

round, player β starts by selecting a point, x_1 and an open set U_1 containing x_1 and then player α responds with an open set V_1 such that $x_1 \in V_1 \subseteq U_1$. In the second round, player β selects a point x_2 and an open set U_2 such that $x_2 \in U_2 \subseteq V_1$ and α responds with an open set V_2 such that $x_2 \in V_2 \subseteq U_2$. Continuing in this way, the players play infinitely many rounds generating the sequence $(U_1, x_1), V_1, (U_2, x_2), V_2, \dots$. If $\bigcap_{i \in \omega} U_i$ is nonempty, then α wins. Otherwise β wins.

A **strategy** for a player in $Ch(X)$ or $BM(X)$ is a rule for choosing what to play on each round given the full information of moves up until that point. A **winning strategy** for a player is a strategy that produces a win for that player in any game when playing according to that strategy. A **stationary** strategy is a strategy that only depends on the opponents previous move. A **coding strategy** for a player is a strategy that depends only on the last two moves (–to be precise, the opponent’s last move and the player’s previous move). Non-stationary strategies are functions σ with domains that are collections of finite partial plays. It will be convenient to refer to the restriction of σ to the collection of finite partial plays “up to round n ” as σ_n .

If player α has a winning strategy in $Ch(X)$, X is said to be **Choquet complete**. If α has a stationary winning strategy in $Ch(X)$, X is said to be **strongly α -favorable**.

We will use the following well known lemma in the next section.

Lemma 4.1. *Let X be a space. Consider the modified game of $Ch(X)$ where α and β are required to play in a base \mathcal{B} . If α has a stationary winning strategy in the original game, then α has a stationary winning strategy in the modified game.*

Proof. We can consider α ’s winning strategy σ to be a function from $\tau^*(X) \times X$ to $\tau^*(X)$. For each $(B, x) \in \mathcal{B} \times X$, choose $\sigma'(B, x) \in \mathcal{B}$ to satisfy $x \in \sigma'(B, x) \subseteq \sigma(B, x)$. Observe that $\sigma'(B, x)$ is a valid response by α when β plays (B, x) .

If α plays according to σ' , an instance of the game is a sequence

$$(B_0, x_0), \sigma'(B_0, x_0), (B_1, x_1), \sigma'(B_1, x_1), \dots$$

which gives a nested decreasing sequence

$$B_0 \supseteq \sigma(B_0, x_0) \supseteq \sigma'(B_0, x_0) \supseteq B_1 \supseteq \sigma(B_1, x_1) \supseteq \sigma'(B_1, x_1) \supseteq \dots$$

Note that $\bigcap_{i \in \omega} \sigma(B_i, x_i)$ is nonempty because σ is a winning strategy. So $\bigcap_{i \in \omega} \sigma'(B_i, x_i)$ is nonempty because the sequences are entwined. We conclude that α wins when playing according to σ' ; that is, σ' is a stationary winning strategy. \square

Many variations of the previous lemma are true; for example, replacing α with β , replacing stationary winning strategy with winning strategy, the converses of Lemma 4.1 and these variations, etc. The method of proof is the same; entwine the moves of one game with the moves of the other game, then note that one sequence has nonempty intersection iff the other sequence has nonempty intersection. See [DM10, Proposition 2.7], for another example.

Martin related domain representable spaces to topological games.

Theorem 4.2. *Let X be a space.*

- (1) [Ma03, Theorem 5.1] *If X is domain representable, then X is Choquet complete.*
- (2) [Ma03, Theorem 6.1] *If X is Scott domain representable, then X is strongly α -favorable.*

The converse of Theorem 4.2(1) is false. The Σ -product of uncountably many copies of the real line is strongly α -favorable and not domain representable. Corollaries to Lemma 4.2 solve cases of the model problem. A metrizable space is domain representable if and only if it is completely metrizable [Ma03]. Bennett, Lutzer, and Reed [BLR] proved that for the class of Moore spaces, domain representability, strong α -favorability and subcompactness are equivalent.

Now we relate the spaces defined in Section 2 to topological games.

Theorem 4.3. *Let X be a space.*

- (1) *If X is countably subcompact, then X is strongly α -favorable.*
- (2) *If X is countably generalized subcompact, then α has a coding strategy in $Ch(X)$.*
- (3) *If X is countably domain representable, then X is Choquet complete.*

Proof. For the proof of (1), let X be countably subcompact with base \mathcal{B} . For $x \in U \in \tau^*(X)$, let $\sigma(U, x) \in \mathcal{B}$ satisfy $x \in \sigma(U, x) \subseteq \text{cl } \sigma(U, x) \subseteq U$. If α plays according to σ , then a play of $Ch(X)$ gives a sequence

$$U_0 \supseteq \text{cl } \sigma(U_0, x_0) \supseteq U_1 \supseteq \text{cl } \sigma(U_1, x_1) \supseteq \dots$$

Then $\{\sigma(U_n, x_n) : n \in \omega\}$ is countable and \prec_{cl} -downward directed. Hence the intersection is nonempty and α wins.

Towards (2), let X be countably generalized subcompact with base \mathcal{B} and relation \prec . For $x \in U \in \tau^*(X)$, let $\sigma_0(U, x) \in \mathcal{B}$ satisfy $x \in \sigma_0(U, x) \subseteq U$. For $V \in \mathcal{B}$, $U \in \tau^*(X)$, and $x \in U \subseteq V$, let $\sigma(V, U, x)$ satisfy $x \in \sigma(V, U, x) \subseteq U$ and $\sigma(V, U, x) \prec V$. If α plays

according to σ , then a play of $Ch(X)$ gives a sequence

$$U_0 \supseteq \sigma_0(U_0, x_0) = V_0 \supseteq U_1 \supseteq \sigma(V_0, U_1, x_1) = V_1 \supseteq U_2 \supseteq \dots$$

Then $\{\sigma(V_n, U_{n+1}, x_{n+1}) : n \in \omega\}$ is countable and \prec -downward directed. Hence the intersection is nonempty and α wins.

The proof of (3) mimics Martin's proof of part (1) of the previous theorem. Let the triple (Q, \ll, B) show that X is countably domain representable. Let $<$ well order Q . For $x \in U \in \tau^*(X)$, let a_0 be the $<$ -least element of Q which satisfies $x \in B(a_0) \subseteq U_0$ and set $\sigma_0(U, x) = B(a_0)$. If a game of $Ch(X)$ starts $U_0, x_0, B(a, 0), U_1, x_1, \dots, U_n, x_n$, reconstruct the sequence a_0, a_1, \dots, a_{n-1} . Because $x_n \in U_n \subseteq B(a_{n-1})$, and $\{B(q) : q \in Q\}$ is a base for X , there is $p \in Q$ such that $x_n \in B(p) \subseteq U_n$. Because the set $\{r \in Q : x_n \in B(r)\}$ is \ll -upward directed, the set $\{q \in Q : x_n \in B(q) \subseteq U_n \text{ and } a_{n-1} \ll q\}$ is not empty and has a $<$ -least element, a_n . Set $\sigma_n(U_0, x_0, B(a, 0), U_1, x_1, \dots, U_n, x_n) = B(a_n)$. If α plays according to σ , then a play of $Ch(X)$ gives a sequence

$$U_0 \supseteq B(a_0) \supseteq U_1 \supseteq B(a_1) \supseteq \dots$$

Then $\{a_n : n \in \omega\}$ is countable and \ll -upward directed. Hence the intersection $\bigcap \{B(a_n) : n \in \omega\}$ is nonempty and α wins. \square

5. AN EXTENSION OF DEBS' SPACE

After proving Theorem 4.2, Martin [Ma03] asks "Is there a space with a countably based model in which player α cannot win with a stationary strategy?" Dorais and Mummert [DM10, Corollary 1.4] answer negatively. It is natural to ask the question without the restriction countably based, especially because Theorem 4.2 does not have that restriction. We describe a generalized subcompact space X where α does not have a stationary winning strategy in $BM(X)$. Since subcompact spaces are strongly α -favorable, this example answers Question 5.1 of [BLR], and Questions 3.4 and 5.2 of [BLqa].

As usual, let \mathbb{Q} and \mathbb{R} denote the rationals and the reals. Let \mathcal{I} be the family of open intervals of \mathbb{R} of rational length. Let H contain exactly one element of each coset of the quotient group \mathbb{R}/\mathbb{Q} . We may assume that $0 \in H$. For $h \in H$, let \mathcal{I}_h be the subfamily of \mathcal{I} whose endpoints have the form $h + q$ for some $q \in \mathbb{Q}$.

Let $F = 2^{\mathbb{R}}$ with the countable support topology. In more detail, let $\mathcal{S} = \text{Fn}(\mathbb{R}, 2, \omega_1)$, the family of functions S with $\text{dom } S \subseteq \mathbb{R}$, $\text{ran } S \subseteq \{0, 1\}$, and $|S| < \omega_1$. For $S \in \mathcal{S}$, define $[S] = \{f \in F : S \subset f\}$ and let $\{[S] : S \in \mathcal{S}\}$ is a base for F . Our space is

$$X = \{(f, r) \in F \times \mathbb{R} : f(r) = 1\}.$$

In [De85], Debs constructs a space homeomorphic to the dense subspace $\{(f, r) : 0 < |f^{\leftarrow 0}| \leq \omega \text{ and } r \neq 0\}$ of X .

Elements of X are often denoted $x = (f_x, r_x)$. A base for X is $\mathcal{V} = \{V(S, I) : S \in \mathcal{S} \text{ and } I \in \mathcal{I}\}$, where $V(S, I) = ([S] \times I) \cap X$. For each $h \in H$, set $\mathcal{V}_h = \{V(S, I) : I \in \mathcal{I}_h\}$. For $S \in \mathcal{S}$, let $\{r(S, i) : i \in \omega\}$ list the elements of $S^{\leftarrow 0}$. Abusing notation, when $U = V(S, I)$, we will write $r(U, i)$ for $r(S, i)$. Observe that $|\mathcal{V}| = \mathfrak{c} = |H|$. Fix a one-to-one function $h : \mathcal{V}^2 \rightarrow H \setminus \{0\}$.

We are going to define several things by recursion on $n \in \omega$. After this recursion, we will define a base for X , $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$, and a transitive, antisymmetric relation \prec on \mathcal{B} via $B' \prec B$ iff $B \in \text{pred}(B')$.

Set $\mathcal{B}_0 = \mathcal{V}_0$ and $\text{pred}(B) = \emptyset$ for all $B \in \mathcal{B}_0$.

Suppose that \mathcal{B}_n and $\text{pred}(B)$ for all $B \in \mathcal{B}_n$ have been defined. Let T_n be the set of triples $t = (r, (V(S', I'), V(S'', I''))) \in T_n$ satisfying $V(S', I') \in \mathcal{B}_n$, $V(S'', I'') \in \bigcup_{i \leq n} \mathcal{B}_i$, $r \in I' \cap I''$, $S' \cup S''$ is a function, and $r \notin (S' \cup S'')^{\leftarrow 0}$. (The conjunction of the last three clauses is equivalent to $(r, f) \in V(S', I') \cap V(S'', I'')$ for some $f \in F$). For $t = (r, V(S', I'), V(S'', I'')) \in T_n$, set $\text{pr}(t) = \text{pred}(U') \cup \text{pred}(U'') \cup \{U', U''\}$, and define $R_t = \{r(U, i) : i \leq n \text{ and } U \in \text{pr}(t)\}$. Note that R_t is finite. Define I_t to satisfy

- (1) $r \in I_t$,
- (2) $\text{cl } I_t \subseteq I' \cap I''$,
- (3) $I_t \in \mathcal{I}_{h(U', U'')}$,
- (4) $I_t \cap R_t = \emptyset$, and
- (5) $\text{diam}(I_t) < 2^{-n}$.

Set $B_t = V(S' \cup S'', I_t)$ and $\text{pred}(B_t) = \text{pr}(t)$. Set $\mathcal{B}_{n+1} = \{B_t : t \in T_n\}$. Observe that these concepts are well defined. In particular, item (3) ensures that if $B_t = B_{t'}$, then $\text{pred}(B_t) = \text{pred}(B_{t'})$.

Lemma 5.1. *X , \mathcal{B} , and \prec satisfy the definition of generalized subcompact.*

Proof. Items (1), (2), and (3) are clear from the definitions.

Towards (4), suppose that $(f, r) \in V(S', I') \cap V(S'', I'')$. Note that for some n , $t = (r, (V(S', I'), V(S'', I''))) \in T_n$. Then $(f, r) \in B_t \in \mathcal{B}_{n+1}$, $B_t \prec V(S', I')$, and $B_t \prec V(S'', I'')$.

Towards (5), suppose that $D \subseteq \mathcal{B}$ and (D, \prec) is downward directed. If $B \prec B'$, $B \in \mathcal{B}_n$, and $B' \in \mathcal{B}_{n'}$, then $n' < n$. Hence we can find in D a sequence $\{V(S^m, I^m) : m \in \omega\}$ such that $\text{cl } I^{m+1} \subseteq I^m$ for all m and $\text{diam } I^m \rightarrow 0$. Therefore $\bigcap \{I^m : m \in \omega\}$ is a singleton, $\{r_D\}$. Observe that $r_D \in I$ for all $V(S, I) \in D$. (If $r_D \notin I$ and $V(S, I) \succ V(S', I') \in D$, then $I' \cap I^m = \emptyset$ for some m). Similarly,

$g_D = \bigcup\{S : V(S, I) \in D\}$ is a function and $(r_D, 0) \notin g_D$. (If $S(r_D) = 0$, $r_D = r(S, j)$, and $V(S, I), V(S^j, I^j) \succ V(S', I') \in D$, then $r_D \notin I'$). Let $f \in 2^{\mathbb{R}}$ satisfy $g_D \subseteq f$ and $f(r_D) = 1$. Then $(f, r_D) \in \bigcap D$. \square

Now we show that α does not have a stationary winning strategy. Via Lemma 4.1, we may require both α and β to play from $\mathcal{V}_0 = \{V(S, I) : S \in \mathcal{S} \text{ and } I \in \mathcal{I}_0\}$. Recall that \mathcal{I}_0 is countable. Let μ be a stationary strategy for α . That is, $\mu : \mathcal{V}_0 \rightarrow \mathcal{V}_0$ and $\mu(V(S, I)) \subseteq V(S, I)$ for all $V(S, I) \in \mathcal{V}_0$. We will construct a play of the game in which α plays according to μ and loses. In more detail, given μ , we will define, sequences $\langle I_n : n \in \omega \rangle$ and $\langle J_n : n \in \omega \rangle$ from \mathcal{I}_0 , a point $a \in \mathbb{R}$, and sequences $\langle S_n : n \in \omega \rangle$ and $\langle T_n : n \in \omega \rangle$ from \mathcal{S} which satisfy

- (1) $V(S_n, I_n) \supseteq \mu(V(S_n, I_n)) = V(T_n, J_n) \supseteq V(S_{n+1}, I_{n+1})$ for all $n \in \omega$,
- (2) $\bigcap\{I_n : n \in \omega\} = \{a\}$, and
- (3) $(a, 0) \in S_0$.

For $A \in \mathcal{S}$, set $\mathcal{S}_A = \{S \in \mathcal{S} : A \subseteq S\}$. The prototype of (P, \leq) in the next lemma is (\mathcal{S}, \subseteq) ($-$ not the reversed order used for forcing).

Lemma 5.2. *Let (P, \leq) be a poset with the property that every countable chain has an upper bound. Let c be a function from P to a countable set E . For all $p \in P$ there are $p' \geq p$ and $e \in E$ such that for all $q \geq p'$ there is $q' \geq q$ with $c(q') = e$.*

Proof. Assume not. Let p be a counterexample. List E as $\{e_n : n \in \omega\}$. Define $q_n \in P$ by recursion. Set $q_0 = p$. If q_n has been defined, there is $q_{n+1} \geq q_n$ such that $c(q') \neq e_n$ for all $q' \geq q_{n+1}$. Let \bar{q} be an upper bound for $\{q_n : n \in \omega\}$ and set $e_{\bar{n}} = c(\bar{q})$. Then $q_{\bar{n}} \leq \bar{q}$ and $e_{\bar{n}} = c(\bar{q})$. Contradiction. \square

Our first task is to recursively define A_n , I_n and J_n for $n \in \omega$. Set $A_0 = \emptyset$ and $I_0 = (1, 2)$. If A_n and I_n have been defined, define $c_n : \mathcal{S}_{A_n} \rightarrow \mathcal{I}_0$ so that $\mu(V(S, I_n)) = V(S', c_n(S))$ for some $S' \in \mathcal{S}$. Then apply Lemma 5.2 with $(P, \leq) = (\mathcal{S}_{A_n}, \subseteq)$, $c = c_n$, and $E = \mathcal{I}_0$, obtaining $p' = A_{n+1}$ and $e = J_n$ which have the property that for all $T \supseteq A_{n+1}$ there are S and T' satisfying $T \subseteq S \subseteq T'$ and $\mu(V(S, I_n)) = V(T', J_n)$. For $S \in \mathcal{S}$, let $\{\hat{r}(S, i) : i \in \omega\}$ list the elements of $S^{\leftarrow 1}$. After J_n has been defined, choose $I_{n+1} \in \mathcal{I}_0$ to satisfy $\text{cl } I_{n+1} \subset I_n$, $I_{n+1} \cap \{\hat{r}(A_m, k) : m, k \leq n\} = \emptyset$, and $\text{length } I_{n+1} \leq 2^{-n}$.

By construction, $\bigcap\{I_n : n \in \omega\}$ has a unique element, call it a . Set $T_{-1} = \bigcup\{A_n : n \in \omega\} \cup \{(a, 0)\}$. If T_{n-1} has been defined, by the previous paragraph we may choose S_n and T_n to satisfy $T_{n-1} \subseteq S_n \subseteq$

T_n and $\mu(V(S_n, I_n)) = V(T_n, J_n)$. This completes the verification that X does not have a stationary winning strategy.

6. π -COMPLETENESS PROPERTIES

In the previous section, we presented a domain representable space with no stationary winning strategy for α . We used the choice function H to partition the base for X into \mathfrak{c} pieces, each a base for X . This gave us “enough room” to “code” the winning strategy into a coding winning strategy—in fact, we “coded” a triple representing X as a pair showing that X is generalized subcompact. The statement of the next proposition makes “enough room” precise. The proof illustrates what we mean by “code”.

Proposition 6.1. *Suppose that a space X , a base $\mathcal{A} \subseteq \tau^*(X)$ for X , an infinite cardinal κ , and a partition $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ of \mathcal{A} be such that \mathcal{A}_α is a base for X and $|\mathcal{A}_\alpha| = \kappa$ for all $\alpha < \kappa$. If X is domain representable, then X is generalized subcompact.*

Proof. Let (Q, \ll, B) represent X . It is harmless to assume that Q has a least element 0_Q and that $B(0_Q) = X$. Let $h : \mathcal{A}^2 \rightarrow \kappa \setminus \{0\}$. We will define several things by induction on $n \in \omega$. Set $\mathcal{A}_0 = \mathcal{B}_0$. For each $A \in \mathcal{B}_0$, set $p(A) = 0_Q$, and $\text{pred}(A) = \emptyset$.

Suppose that \mathcal{B}_n has been defined, as well as $\text{pred}(A)$ and $p(A)$ for every $A \in \bigcup_{m \leq n} \mathcal{B}_m$. Let T_n be the set of triples $t = (x, A', A'')$ satisfying $A' \in \mathcal{B}_n$, $A'' \in \bigcup_{m \leq n} \mathcal{B}_m$, and $x \in A' \cap A''$. Find $q_t \in Q$ such that $p(A'), p(A'') \ll q_t$ and $x \in B(q_t) \subseteq A' \cap A''$. Find $A_t \in \mathcal{A}_{h(A', A'')}$ satisfying $x \in A_t \subseteq B(q_t)$. Set $p(A_t) = q_t$ and $\text{pred}(A_t) = \{A', A''\} \cup \text{pred}(A') \cup \text{pred}(A'')$. The map $t \mapsto A_t$ is not necessarily one-to-one, but the function h ensures that $\text{pred}(A_t)$ is well defined. Set $\mathcal{B}_n = \{A_t : t \in T_n\}$. Set $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$, and define $B \prec B'$ iff $B' \in \text{pred}(B)$.

Items (1), (2), (3), and (4) of Definition 2.3 are clear from construction. Towards (5), suppose that $\mathcal{F} \subseteq \mathcal{B}$ is downward directed by \prec . If \mathcal{F} has a minimal element A , then $\bigcap \mathcal{F} = A \neq \emptyset$. Otherwise \mathcal{F} and $\{B(p(A)) : A \in \mathcal{F}\}$ are entwined and $\{p(A) : A \in \mathcal{F}\}$ is upward directed, hence $\bigcap \mathcal{F} = \bigcap\{B(p(A)) : A \in \mathcal{F}\} \neq \emptyset$. \square

The unusual hypothesis of Proposition 6.1 is satisfied by many spaces.

Proposition 6.2. *Suppose X is a dense subspace of \mathbb{R}^I . Then X satisfies the hypothesis of Lemma 6.1.*

Proof. If I is countable, then X is a second countable space without isolated points, and a routine induction of length ω suffices.

Suppose $|I| = \kappa > \omega$. Let \mathcal{B} be the family of open intervals of \mathbb{R} with rational endpoints. Let $\Sigma = \text{Fn}(I, \mathcal{B}, \omega)$ be the family of finite nonempty functions σ with domain σ subset of κ and range σ subset of \mathcal{B} . For $\sigma \in \Sigma$ and $\alpha \in \kappa$, set $Y_\sigma(\alpha) = \sigma(\alpha)$ if $\alpha \in \text{dom } \sigma$, and $Y_\sigma = \mathbb{R}$ otherwise. Set $B_\sigma^0 = \prod_{\alpha \in \kappa} Y_\sigma(a)$ and $B_\sigma = B_\sigma^0 \cap X$. Then $\{B_\sigma : \sigma \in \Sigma\}$ is a base for X . Because X is dense and $B \neq B' \in \mathcal{B}$ implies $\text{int}(B \setminus B' \cup B' \setminus B) \neq \emptyset$, we obtain that $\sigma \mapsto B_\sigma$ is one-to-one.

Let $\{J_\alpha : \alpha < \kappa\}$ partition κ so that each J_α has cardinality κ and each J_α is cofinal in κ . For $\alpha \in \kappa$, define $\mathcal{A}_\alpha = \{B_\sigma : \max \text{dom } \sigma \in J_\alpha\}$, and $\mathcal{A} = \bigcup_{\alpha \in \kappa} \mathcal{A}_\alpha$. Towards showing that \mathcal{A}_α is a base, suppose that $x \in U$, open in X . There is B_σ satisfying $x \in B_\sigma \subseteq U$. Let $\beta \in J_\alpha$ satisfy $\max \text{dom } \sigma < \beta$, and let $B \in \mathcal{B}$ satisfy $x(\beta) \in B$. Set $\sigma' = \sigma \cup \{(\beta, B)\}$. Then $B_{\sigma'} \in \mathcal{A}_\alpha$ and $x \in B_{\sigma'} \subseteq U$. \square

The proof above is valid for X a dense subspace of M^I when M is a space with a base \mathcal{B} satisfying $B \neq B' \in \mathcal{B}$ implies $\text{int}(B \setminus B' \cup B' \setminus B) \neq \emptyset$ (every regular space M has such a base) and $|\mathcal{B}| \leq |I|$.

Lemma 6.1 has an unusual hypothesis because the lemma borrows a technique from π -completeness properties. Several important concepts of the study of general completeness properties are “point-free” – for example, the conclusion of the Baire category theorem, the Banach-Mazur game, and Oxtoby’s notion of pseudocompleteness[Ox61]. We suggest the term π -completeness because the point-free analog of base is usually called a π -base.

Definition 6.3. Let X be a space. A family $\mathcal{P} \subseteq \tau^*(X)$ is called a π -base for X if for every $U \in \tau^*(X)$ there is $P \in \mathcal{P}$ satisfying $P \subseteq U$. The π -weight of X , denoted $\pi w(X)$ is the least cardinality of a π -base for X . We define the notions π -subcompact, π -generalized subcompact, and π -domain representable by modifying the definitions in Section 2 appropriately. We define π -generalized subcompact by replacing items (1) and (4) of Definition 2.3 with

- 1_π $\mathcal{B} \subseteq \tau^*(X)$ is a π -base for $\tau(X)$
- 4_π if $B, B' \in \mathcal{B}$ satisfy $B \cap B' \neq \emptyset$, then there is $B'' \in \mathcal{B}$ satisfying $B'' \prec B, B'$.

We define π -domain representable by replacing items (1) and (4) of Definition 2.4 with

- 1_π $\{B(q) : q \in Q\} \subseteq \tau^*(X)$ is a π -base for $\tau(X)$
- 4_π if $q, q' \in Q$ satisfy $B(q) \cap B(q') \neq \emptyset$, there is $q'' \in Q$ satisfying $q, q' \ll q''$.

The following result is analogous to Theorem 7 of [GT86].

Lemma 6.4. *A π -domain representable space X is π -generalized subcompact.*

Proof. Say that $U \in \tau^*(X)$ is π -weight homogeneous if $\pi w(V) = \pi w(U)$ for every nonempty open subset V of U . Let \mathcal{U} be a maximal pairwise disjoint family of π -weight homogeneous open subsets of X . For each $U \in \mathcal{U}$, let \mathcal{P}_U be a π -base for U of cardinality $\pi w(U)$. It is straightforward to partition \mathcal{P}_U into $\pi w(U)$ many subfamilies, each of which is a π -base for U . Apply the construction of Lemma 6.1 to each $U \in \mathcal{U}$. \square

7. BASIC CONSTRUCTIONS

In this section we investigate when certain constructions (for example, products, open subspaces, closed subspaces) preserve subcompactness, generalized subcompactness, and domain representability.

DeGroot [dG63] observes that subcompactness is preserved by arbitrary products and topological unions. Corresponding results are immediate for generalized subcompactness and domain representability. It is easy to show that an open subspace of a subcompact space is subcompact. Let \mathcal{B} be a subcompact base for X and let U be an open subspace of X . Set $\mathcal{B}_U = \{B \in \mathcal{B} : B \subseteq U\}$. The same method works for generalized subcompact spaces and domain representable spaces.

By definition, every completely regular realcompact space – in particular, \mathbb{Q} – is homeomorphic to a closed subspace of a product of real lines. Because a product of real lines is subcompact, and \mathbb{Q} is not domain representable, these classes of spaces are not closed hereditary.

We now discuss when assuming that certain subspaces are subcompact (or domain representable) imply that the entire space is subcompact (or domain representable). The first part of Question 3.6 of [BLqa] asks whether the union of two open subcompact subspaces is subcompact. With Tkachuk [FTY] we show more – the union of two not necessarily open subcompact subspaces is subcompact. The analogous result holds for domain representable.

Proposition 7.1. *The union of two domain representable subspaces is domain representable.*

Proof. Let $Y = X \cup X'$. Let (Q, \ll, B) represent X and let (Q', \ll', B') represent X' . We may assume that Q and Q' are disjoint. Let us say that \bar{c} is **nice chain to** r, V when \bar{c} is a finite sequence $\langle q_0, U_0, q_1, U_1, \dots, q_n, U_n \rangle$ satisfying

- (1) $q_m \in Q \cup Q'$ and $U_m \in \tau^*(Y)$ for all $m \leq n$,
- (2) $B(q_m) = U_m \cap X$ if $q_m \in Q$,

- (3) $B'(q_m) = U_m \cap X'$ if $q_m \in Q'$,
- (4) $U_{m+1} \subseteq U_m$ for all $m < n$,
- (5) $q_m \ll q_\ell$ if $q_m \in Q$, $q_\ell \in Q$, and $m < \ell$,
- (6) $q_m \ll' q_\ell$ if $q_m \in Q'$, $q_\ell \in Q'$, and $m < \ell$,
- (7) $q_n = r$ and $U_n = V$.

Let P be the collection of p which, for some fixed r_p and V_p , are a finite set of nice chains to r_p, V_p . For $p \in P$, set $A(p) = V_p$. Set $p_0 \ll^* p_1$ if every $\bar{c} \in p_0$ is a initial segment of some $\bar{c}' \in p_1$.

We claim that (P, \ll^*, A) represents Y . It is routine to verify items (1), (2), and (3) of Definition 2.4.

Towards item (4), suppose $x \in Y$ and $p_0, p_1 \in P$ are such that $x \in V_{p_0} \cap V_{p_1}$. We may assume that $x \in X$. Because p_0 and p_1 are finite sets of finite chains, there is $\hat{q} \in Q$ such that $x \in B(\hat{q})$ and $q \ll \hat{q}$ for every $q \in Q$ which is a q_m for some $\bar{c} \in p_0 \cup p_1$. If necessary, we can extend again so that $B(\hat{q}) \subseteq V_{p_0} \cap V_{p_1}$. Let \hat{U} satisfy $B(\hat{q}) = \hat{U} \cap X$ and $\hat{U} \subseteq V_{p_0} \cap V_{p_1}$. For $\bar{c} \in p_0 \cup p_1$, let \bar{c}^+ be \bar{c} extended by \hat{q}, \hat{U} . Set $p_2 = \{\bar{c}^+ : \bar{c} \in p_0 \cup p_1\}$. Then $p_0, p_1 \ll^* p_2$ and $x \in A(p_2)$.

Towards item (5), let $D \subseteq P$ be \ll^* -upward directed. Set $E = \{p \in D : r_p \in Q\}$. If E is cofinal in D , then $\{r_p : p \in E\}$ is \ll -upward directed and $\bigcap_{p \in D} A(p) = \bigcap_{p \in E} A(p) \neq \emptyset$. If E is not cofinal in D , there is $d \in D$ with no extension in E . Set $E' = \{p \in D : r_p \in Q'\}$. If E' is cofinal in D , then $\{r_p : p \in E'\}$ is \ll' -upward directed and $\bigcap_{p \in D} A(p) = \bigcap_{p \in E'} A(p) \neq \emptyset$. If E' is not cofinal in D , there is $d' \in D$ with no extension in E . However, then d and d' have no extension in D , contradicting the assumption that D is directed upward. \square

The next result answers the second part of Question 3.6 of [BLqa].

Theorem 7.2. *If a space X is the union of open domain representable subspaces, then X is domain representable.*

Proof. Let $\{Y_i : i \in I\}$ be a family of open domain representable subspaces of X such that $X = \bigcup\{Y_i : i \in I\}$. For each $i \in I$, let (Q_i, \ll_i, B_i) represent Y_i . Let Q be the set of triples $q = (a, f, W)$ satisfying

- (1) $a \in [I]^{<\omega}$,
- (2) f is a function with $\text{dom } f = a$ and $f(i) \in Q_i$ for all $i \in a$,
and
- (3) $W = \bigcap\{B_i(f(i)) : i \in a\} \in \tau^*(X)$.

For $q \in Q$, set $B(q) = W$. For $q = (a, f, W)$ and $q' = (a', f', W')$ in Q , set $q \ll q'$ if $a \subseteq a'$, $f(i) \ll_i f'(i)$ for all $i \in a$, and $B_i(f'(i)) \subseteq W'$ for all $i \in a$.

We verify that (Q, \ll, B) satisfies Definition 2.4. As usual, items (1), (2), and (3) are routine.

Towards item (4), suppose $x \in B(q) \cap B(r)$ for $q = (a_q, f_q, W_q)$ and $r = (a_r, f_r, W_r)$. Set $a = a_q \cup a_r$. For $i \in a$, choose $p_0(i) \in Q_i$ satisfying $x \in B_i(p_0(q, r)) \subseteq B(q) \cap B(r)$. Then choose $f(i) \in Q_i$ satisfying $p_0(i) \ll_i f(i)$, $f_q(i) \ll_i f(i)$ if $i \in a_q$, and $f_r(i) \ll_i f(i)$ if $i \in a_r$. Set $W = \bigcap \{B_i(f(i)) : i \in a\}$. Then $p = (a, f, W)$ satisfies $q, r \ll p$ and $x \in B(p)$.

Towards item (5), suppose $D \subseteq Q$ is upward directed by \ll . Fix $p \in D$ and $j \in a_p$. Note that $E = \{q \in D : p \ll q\}$ is cofinal in D . Now $\{f(j) : (a, f, W) \in E\}$ is directed in Q_j , hence there is $x \in \bigcap B_j(f(j) : (a, f, W) \in E) = \bigcap \{B(q) : q \in D\} \neq \emptyset$. \square

Our next goal is to show that every completely regular (not necessarily realcompact) space is homeomorphic to a closed subset of a generalized subcompact space. First we introduce some special notation. Suppose τ is a topology on a space X and $S \subseteq X$. Let τ^S be the topology on X generated by $\tau \cup \{\{s\} : s \in S\}$. This new topology isolates all points of S and agrees with τ on $X \setminus S$. More generally, suppose σ is any topology finer than the subspace topology $\tau|_S = \{U \cap S : U \in \tau\}$. Then let $\tau \vee \sigma$ be the topology on X generated by $\tau \cup \sigma$.

Proposition 7.3. *Suppose (X, τ) is generalized subcompact and (S, σ) is generalized subcompact, where S is a subset of X with $\tau|_S \subseteq \sigma$. Then $(X, \tau \vee \sigma)$ is generalized subcompact. Hence, for example, if X is generalized subcompact, and $S \subseteq X$, then (X, τ^S) is generalized subcompact.*

Proof. Let (X, τ) be a generalized subcompact space and let $S \subseteq X$ be equipped with a GCS topology σ finer than $\tau|_S$. Let $\mathcal{B}_X, \mathcal{B}_S, \prec_X$ and \prec_S satisfy the definition of generalized subcompact for (X, τ) and (S, σ) respectively. We define a base $\mathcal{B} = \mathcal{B}_X \cup \mathcal{B}_S$ for $(X, \tau \vee \sigma)$ and an order \prec on \mathcal{B} . We define $U \prec W$ in cases.

- (1) If $U, W \in \mathcal{B}_X \setminus \mathcal{B}_S$, then $U \prec W$ if $U \prec_X W$.
- (2) If $U, W \in \mathcal{B}_S$, then $U \prec W$ if $U \prec_S W$.
- (3) If $U \in \mathcal{B}_S, W \in \mathcal{B}_X \setminus \mathcal{B}_S$, then $U \prec W$ if $U \prec_S W'$, for some $W' \in \mathcal{B}_S$ with $W' \subseteq W \cap S$.

We first show that $\{U \in \mathcal{B} : x \in U\}$ is downward directed by \prec for all $x \in X$. Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. If $x \in X \setminus S$ and hence $B_1, B_2 \in \mathcal{B}_X$, then there is $B_3 \in \mathcal{B}_X \setminus \mathcal{B}_S$ with $x \in B_3$ and $B_3 \prec_X B_1$ and $B_3 \prec_X B_2$. So, $B_3 \prec B_1$ and $B_3 \prec B_2$. If $x \in S$, let $B'_1, B'_2 \in \mathcal{B}_S$ be such that $B'_1 \subseteq B_1 \cap S, B'_2 \subseteq B_2 \cap S$ and $x \in B'_1 \cap B'_2$. (In the case

that $B_i \in \mathcal{B}_S$, we assume that $B'_i = B_i$.) Then, let $B_3 \in \mathcal{B}_S$ be such that $x \in B_3$, $B_3 \prec_S B'_2$ and $B_3 \prec_S B'_1$. Then, $x \in B_3$, $B_3 \prec B_1$ and $B_3 \prec B_2$ and we are done.

Now we show that if $\mathcal{F} \subseteq \mathcal{B}$ is downward directed by \prec , then $\bigcap \mathcal{F} \neq \emptyset$. Case 1: For all $B \in \mathcal{F}$ there is $A \in \mathcal{F} \cap \mathcal{B}_S$ such that $A \prec B$. Then $\mathcal{F} \cap \mathcal{B}_S$ is downward directed by \prec_S . Hence $\bigcap \mathcal{F} = \bigcap (\mathcal{F} \cap \mathcal{B}_S) \neq \emptyset$. Case 2: For all $B \in \mathcal{F}$ there is $A \in \mathcal{F} \setminus \mathcal{B}_S$ such that $A \prec B$. Then $\mathcal{F} \setminus \mathcal{B}_S$ is downward directed by \prec_X . Hence $\bigcap \mathcal{F} = \bigcap (\mathcal{F} \setminus \mathcal{B}_S) \neq \emptyset$. Towards a contradiction, assume that $B_0 \in \mathcal{F}$ witnesses that Case 1 fails and $B_1 \in \mathcal{F}$ witnesses that Case 2 fails. Because \mathcal{F} is downward directed, there is $A \in \mathcal{F}$ satisfying $A \prec B_0, B_1$. Then either $A \in \mathcal{B}_S$ contradiction, or $A \notin \mathcal{B}_S$ contradiction. \square

Bennett and Lutzer [BL06, Example 3.1] show the next two results with generalized subcompact replaced by domain representable.

Corollary 7.4. *If X is a generalized subcompact space and $S \subseteq X$, then (X, τ^S) is generalized subcompact.*

Corollary 7.5. *Every completely regular space is homeomorphic to a closed subspace of a generalized subcompact space.*

Proof. Let Y be a completely regular space. Let (X, τ) be a compactification of Y (for example $X = \beta Y$), and set $S = X \setminus Y$. Then Y is homeomorphic to a closed subspace of the generalized subcompact space (X, τ^S) . \square

8. G_δ -DIAGONALS

We say that a space X has a G_δ -diagonal if there is a family $\{G_n : n \in \omega\}$ of open subsets of X^2 such that $\bigcap \{G_n : n \in \omega\} = \{(x, x) \in X^2 : x \in X\}$. In the class of spaces with G_δ -diagonals, we have partial converses to Lemmas 4.3 and 4.2. We improve in two ways Proposition 4.3 [BLHo]. The next lemma is proved by the method of Lemma 4.1

Lemma 8.1. [BLHo, Lemma 4.2] *If α has a stationary winning strategy in $Ch(X)$ and X has a G_δ -diagonal, then α has a stationary winning strategy σ such that $|\bigcap \{V_i : i \in \omega\}| = 1$ whenever*

$$V_0, x_0, \sigma(V_0, x_0), V_1, x_1, \sigma(V_1, x_1), \dots$$

is a play in $Ch(X)$.

Proposition 8.2. *If X is strongly α -favorable and X has a G_δ -diagonal then X is generalized subcompact.*

Proof. Let σ be a stationary winning strategy as in Lemma 8.1. Let $\mathcal{B} = \tau^*(X)$ and for $U, V \in \tau^*(X)$, define $U \prec V$ if and only if $x \in U \subseteq \sigma(V, x)$ for some $x \in X$. Items (1)-(3) of generalized subcompact are clear. Let $x \in X$ and $U, V \in \tau^*(X)$ with $x \in U \cap V$. Then $W = \sigma(U, x) \cap \sigma(V, x)$ satisfies $x \in W \prec U, V$. So, $\{W \in \mathcal{B} : x \in W\}$ is directed by \prec . Towards (5), let \mathcal{F} be a \prec -directed subcollection of \mathcal{B} . First, let U_0, U_1, \dots be a \prec -decreasing sequence from \mathcal{F} . So, there are x_0, x_1, \dots with

$$U_0 \supseteq \sigma(U_0, x_0) \supseteq U_1 \supseteq \sigma(U_1, x_1) \supseteq U_2 \supseteq \dots$$

and $x_i \in U_{i+1}$ for all $i \geq 0$. Therefore, by Lemma 8.1, $\bigcap\{U_i : i \in \omega\} = \{x\}$ for some $x \in X$. Now, let $W \in \mathcal{F}$. Define V_1, V_2, \dots in \mathcal{F} such that $V_1 \prec W$, $V_{i+1} \prec V_i$ and $V_i \prec U_{i-1}$ for all $i \geq 1$. So, there are $y_0, y_1, y_2, \dots \in X$ such that

$$W \supseteq \sigma(W, y_0) \supseteq V_1 \supseteq \sigma(V_1, y_1) \supseteq V_2 \supseteq \dots$$

and $y_i \in V_{i+1}$ for all $i \geq 0$. Then, by Lemma 8.1, $|\bigcap\{V_i : i \in \omega\}| = 1$. However, since $V_i \subset U_{i-1}$ for each $i \geq 1$, we have $\bigcap\{V_i : i \in \omega\} = \{x\}$. Hence, $x \in \bigcap \mathcal{F}$. \square

Proposition 8.3 answers question 7.7(c) and the first question in question 5.4(a) of [BLqa].

Proposition 8.3. *If X is Choquet complete and has a G_δ -diagonal, then X is domain representable.*

Proof. Let the diagonal of X^2 be $\bigcap\{G_n : n \in \omega\}$. Recall that we say that X is Choquet complete when player α has a winning strategy $\langle \sigma_n \rangle$ in the game $Ch(X)$. We may assume that if V is in the range of σ_n , then $V^2 \subset G_n$. Hence the analog of Lemma 8.1 holds.

Suppose that q is a finite set of partial plays according to $\langle \sigma_n \rangle$ where the last move is the open set V of a β move (without the point). It means that if s is an element of q , then for some $m(s) \in \mathbb{N}$, s has the form

$$U_1, x_1, \sigma_1(U_1, x_1), \dots, \sigma_{m(s)}(U_1, x_1, \dots, U_{m(s)}, x_{m(s)}), V.$$

In this case put q in Q and set $B(q) = V$. Define $q \ll q'$ when every member of q is a proper initial segment of some member of q' . Items (1)-(3) of Definition 2.4 are clear. Towards item (4) of Definition 2.4, suppose that $x \in B(q) \cap B(q')$. Extend every $s \in q \cup q'$ by one more inning. The extension, \hat{s} , of s is

$$U_1, x_1, \sigma_1(U_1, x_1), \dots, \sigma_{m(s)}(U_1, x_1, \dots, x_{m(s)}), V, x, \sigma_{m(s)+1}(U_1, \dots, V, x).$$

Let $W(\hat{s})$ denote the last term in the \hat{s} sequence. The elements of q'' are $\hat{s} \cap \bigcap_{r \in q \cup q'} W(\hat{r})$ for $s \in q \cup q'$.

Towards item (5) of Definition 2.4, suppose D is a \ll -upward directed subset of Q . Let $\{q_n : n \in \omega\}$ be \ll -increasing in D . Then, define a sequence $\{s_n : n \in \omega\}$ of partial plays such that $s_n \in q_n$ and s_{n+1} properly extends s_n for each n . Hence $\bigcup_{n \in \omega} s_n$ is a full play of the game in which α is playing according to σ . Furthermore, for each $n \in \omega$, $B(q_n)$ appears as the open set in one of β 's moves. So, by the analog of Lemma 8.1, $\bigcap B(q_n) = \{x\}$ for some $x \in X$. To see that $\{x\} \subseteq \bigcap_{q \in D} B(q)$, let $q \in D$ and create another \ll -increasing sequence $\{r_n : n \in \omega\}$ with $q_0, q \ll r_0$ and $q_n \ll r_n$ for each $n \in \omega$. Then $\emptyset \neq \bigcap_{n \in \omega} B(r_n) \subseteq \bigcap_{n \in \omega} B(q_n) = \{x\}$. So, $\{x\} \subseteq \bigcap_{n \in \omega} B(r_n) \subseteq B(q)$. \square

9. G_δ -SUBSPACES

The situation with G_δ subspaces is more complex than the situation with open subspaces or closed subspaces. Theorem 3.2 of [BL06] asserts that any G_δ of a domain representable space is a domain representable space. Because the representation in Definition 2.4 is simpler than usual definition of domain representability, we can present an easier proof.

Proposition 9.1. *If Y is a G_δ subspace of a domain representable space X , then Y is domain representable.*

Proof. Suppose (Q, \ll, B) represents X and let $\{U_n : n \in \omega\}$ be a nested decreasing sequence of nonempty open subsets of X such that $Y = \bigcap \{U_n : n \in \omega\}$. We write $U_\omega = Y$. Let $Q^* = \{q \in Q : B(q) \cap Y \neq \emptyset\}$. For $q \in Q^*$, define $n(q) = \sup\{n \in \omega : B(q) \subseteq U_n\}$ and $B^*(q) = B(q) \cap Y$. For $q, r \in Q^*$, define $q \ll^* r$ if and only if $q \ll r$ and $(n(r) > n(q) \text{ or } n(r) = n(q) = \omega)$. We now verify that (Q^*, \ll^*, B^*) represents Y . We show (4) and (5) of Definition 2.4, the rest is easy to verify.

Suppose $x \in B^*(q) \cap B^*(r)$. Let $n = 1 + \max\{n(q), n(r)\}$ and let $p' \in Q$ be such that $x \in B(p') \subseteq B(q) \cap B(r) \cap U_n$. Then let $p \in Q$ be such that $p', q'r \ll p$ and $x \in B(p)$. Then, $p \in Q^*$, $x \in B^*(p)$ and $q, r \ll^* p$. So $\{q \in Q^* : x \in B^*(q)\}$ is upward directed by \ll^* .

Now suppose that $D \subset Q^*$ is upward directed by \ll^* . Then, D is also upward directed by \ll . If there is $p \in D$ with $n(p) = \omega$, then $\bigcap \{B(q) : q \in D\} \subseteq B(p) \subseteq Y$. Hence $\bigcap \{B^*(q) : q \in D\} = \bigcap \{B(q) : q \in D\} \neq \emptyset$. If there is no such p , then there is a sequence $\{p_i : i \in \omega\} \subseteq D$ with $\{n(p_i) : i \in \omega\}$ unbounded in ω . In this case $\bigcap \{B(q) : q \in D\} \subseteq \{B(p_i) : i \in \omega\} \subseteq Y$. Again, this means $\bigcap \{B^*(q) : q \in D\} = \bigcap \{B(q) : q \in D\} \neq \emptyset$. \square

As discussed in Section 2, the above proof does not work for subcompact spaces because B is not necessarily one-to-one. However, we can overcome this difficulty when Y is a dense G_δ of X .

Proposition 9.2. *If Y is a dense G_δ subspace of a subcompact space X , then Y is generalized subcompact.*

Proof. Let X be subcompact, let $\{U_n : n \in \omega\}$ be a nested decreasing sequence of nonempty open subsets of X , and let $Y = \bigcap \{U_n : n \in \omega\}$. It is harmless to assume that $U_0 = X$ and we write $U_\omega = Y$. Let \mathcal{B} be a subcompact base for X and define $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$. For each $U \in \mathcal{B}_Y$, choose $B(U)$ to be some member of \mathcal{B} with $B(U) \cap Y = U$ and let $n(U) = \sup\{n \in \omega : \text{cl}_X(B(U)) \subseteq U_n\}$. Since Y is dense in X , if $U \in \mathcal{B}_Y$, then any $B' \in \mathcal{B}$ with $B' \cap Y = U$ has $\text{cl}_X(B') = \text{cl}_X(B(U))$. Now we define a generalized subcompact relation on \mathcal{B}_Y . Let $U \prec_Y V$ iff $B(U) \prec_{cl} B(V)$ and $(n(U) > n(V) \text{ or } n(U) = n(V) = \omega)$.

We verify that \mathcal{B}_Y and \prec_Y satisfy the definition of generalized subcompact for Y . First, \mathcal{B}_Y is a base for Y consisting of nonempty open sets. Second, $W \prec_Y V$ implies $W \subseteq \text{cl}_X(W) = \text{cl}_X(B(W) \cap Y) = \text{cl}_X(B(W)) \subseteq B(V)$. Hence $W = W \cap Y \subseteq B(V) \cap Y = V$. So, we have shown that $W \prec_Y V$ implies $W \subseteq V$.

Suppose that $x \in Y$ and that $W, V \in \mathcal{B}_Y$ are such that $x \in W \cap V$. Set $m = 1 + \max\{n(W), n(V)\}$. Choose $B_0 \in \mathcal{B}$ satisfying $x \in B_0 \subseteq \text{cl}_X(B_0) \subseteq B(W) \cap B(V) \cap U_m$. Then, let $U = B_0 \cap Y$. As we noticed before, $\text{cl}_X(B_0) = \text{cl}_X(B(U))$. So, $\text{cl}_X(B(U)) \subseteq B(W) \cap B(V)$. Moreover, since $\text{cl}_X(B(U)) = \text{cl}_X(B_0) \subseteq U_m$, we have $n(U) > n(W), n(V)$ (or $n(U) = n(W) = n(V) = \omega$).

Now, let $\mathcal{F} \subseteq \mathcal{B}_Y$ be downward directed by \prec_Y . Then, $\mathcal{F}' = \{B(W) : W \in \mathcal{F}\}$ is downward directed by \prec_{cl} in \mathcal{B} . Since X is subcompact, $\bigcap \mathcal{F}' \neq \emptyset$. We argue that $\bigcap \mathcal{F}' \subseteq Y$ and hence $\bigcap \mathcal{F} = \bigcap \mathcal{F}' \cap Y$ is nonempty. Since \mathcal{F} is directed by \prec_Y , we can select an infinite decreasing chain $V_0 \succ_Y V_1 \succ_Y V_2 \succ_Y \dots$ of (not necessarily distinct) elements of \mathcal{F} . Then, $\{n(V_i) : i \in \omega\}$ is increasing and unbounded in ω , or there is $j \in \omega$ with $n(V_i) = \omega$ for all $i \geq j$. In either case, we conclude that $\bigcap \{V_i : i \in \omega\} \subseteq Y$. Hence $\bigcap \mathcal{F}' \subseteq \bigcap \{V_i : i \in \omega\} \subseteq Y$. \square

Corollary 9.3. *If Y is Čech complete, then Y is generalized subcompact.*

The G_δ question for subcompact spaces had few positive results. With Tkachuk we give partial answers in [FTY]. Recently van Mill and Tkachuk [vMTk] proved that if a Čech-complete space X is the

union of a countable family of closed subcompact subspaces (in particular, if X is separable or k -separable), then X is subcompact. The next result, that Čech-complete generalized ordered spaces are subcompact, is even more recent. Discussions with David Lutzer were helpful in preparing this theorem. He suggested that proving that the space $E(Y, X)$ from [BL98] is subcompact probably would lead to a general theorem. He also read preliminary versions of the theorem.

Recall that a space $(X, <_X, \tau(X))$ is called a generalized ordered space if $<_X$ is a linear order on X such that there is a base for $\tau(X)$ consisting of open convex sets. We will use the characterization that a space is a generalized ordered space iff it is a subspace of a linearly ordered space.

Theorem 9.4. *If $(X, <_X, \tau(X))$ is a Čech-complete generalized ordered space, then X is subcompact.*

We begin by specifying how X is G_δ in a compact space.

Lemma 9.5. *There are $(K, <_K)$, $\mathcal{G} = \{G_n : n \in \omega\}$, and $\{\mathcal{U}_n : n \in \omega\}$, which satisfy*

- (1) X is a dense subspace of K , a compact linearly ordered topological space,
- (2) $<_K$ extends $<_X$,
- (3) \mathcal{G} is a nested decreasing sequence of open subsets of K ,
- (4) $\bigcap \mathcal{G} = X$,
- (5) for each n , \mathcal{U}_n is the collection of maximal open convex subsets of G_n
- (6) If N is an infinite subset of ω , $\{(a_n, b_n) : n \in N\}$ is nested, and $(a_n, b_n) \in \mathcal{U}_n$ for all $n \in N$, then $\bigcap \{(a_n, b_n) : n \in N\} \cap X \neq \emptyset$.

Proof. Because X is generalized ordered, X is a subspace of a linearly ordered space, X' , which has a compact order completion K' . Set $K = \text{cl}_{K'} X$. Set $K^+ = K \cup \{-\infty, \infty\}$. Every open convex subset U of X has the form $(a(U), b(U)) \cap X$ where $a(U)$ and $B(U)$ are points in K^+ .

Because X is Čech-complete, there is a family $\{G'_n : n \in \omega\}$ of open subsets of K such that $\bigcap \{G'_n : n \in \omega\} = X$. Now we mimic the classical representation of the irrationals as continued fractions. By induction on $n \in \omega$, we define G_n so that item (6) holds. Set $G_0 = G'_0$. Suppose that G_n has been defined. Define \mathcal{U}_n to be the collection of maximal open convex subsets of G_n . For each $(a, b) \in \mathcal{U}_n$, we define a subset $F(n, a, b)$. Let n be even. If there is an ω -sequence from $(a, b) \setminus X$ converging to a , let $F(n, a, b)$ be such. If not, let $F(n, a, b) = \emptyset$. Let n be odd. If there is an ω -sequence from

$(a, b) \setminus X$ converging to b , let $F(n, a, b)$ be such. If not, let $F(n, a, b) = \emptyset$. Let $F(n) = \text{cl}_K \bigcup \{F(n, a, b) : (a, b) \in \mathcal{U}_n\}$. Because $X \subseteq G_n$, each $x \in X$ has a neighborhood of the form $(a, b) \setminus F(n, a, b)$, showing that $X \cap F(n) = \emptyset$. Set $G_{n+1} = G_n \cap G'_{n+1} \setminus F(n)$ and continue the induction. Because $X = \bigcap \{G'_n : n \in \omega\}$ and $X \subseteq G_n \subseteq G'_n$, we conclude that $X = \bigcap \mathcal{G}$.

Towards item (6), let $\{(a_n, b_n) : n \in N\}$ be decreasing with $(a_n, b_n) \in \mathcal{U}_n$ for all n .

Case 1: There is m such that for all $n \geq m$ we have $a_n = a_m$ and $b_n = b_m$. Then the intersection is $(a_m, b_m) \neq \emptyset$.

Case 2: Neither $\{a_n : n \in N\}$ nor $\{b_n : n \in N\}$ is eventually constant. Then $\bigcap \{(a_n, b_n) : n \in N\} = \bigcap \{\text{cl}_K(a_n, b_n) : n \in N\}$ which is not empty because K is compact, and a subset of X because $(a_n, b_n) \subseteq G_n$.

Case 3: $\{b_n : n \in N\}$ is eventually constant and $\{a_n : n \in N\}$ is not eventually constant. To be specific, let $m \in \omega$ and $a \in K^+$ be such that m is even and $a_n = a$ for all $n \in \{n \in N : m \leq n\}$. Case 3a: $\inf\{b_n : n \in N\} = a$. Then $F(n, a_m, b_m) \neq \emptyset$, hence a is not a left endpoint of an interval in \mathcal{U}_n for any $n > m$. Contradiction. Case 3b: $a < \inf\{b_n : n \in N\}$. Then $\inf\{b_n : n \in N\} \in \bigcap \{(a_n, b_n) : n \in N\} \cap X \neq \emptyset$.

Case 4: $\{a_n : n \in N\}$ is eventually constant and $\{b_n : n \in N\}$ is not eventually constant. Analogous to Case 3. \square

For each n , let $\mathcal{V}_n = \{V \cap X : V \in \mathcal{U}_n\}$. For each $x \in X$, let $V(n, x)$ be the unique member of \mathcal{V}_n containing x . Set $C(x) = \bigcap \{V(n, x) : n \in \omega\}$, $a(x) = \sup\{y \in K \setminus X : y < x\}$, and $b(x) = \inf\{y \in K \setminus X : x < y\}$. For each n , set $a(n, x) = \sup\{y \in K \setminus G_n : y < x\}$ and set $b(n, x) = \inf\{y \in K \setminus G_n : x < y\}$. Note that for each x , the sequence $\{a(n, x) : n \in \omega\}$ is nondecreasing and $\sup a(n, x) = a(x)$. Moreover, $a(n, x) = a(V(n, x))$. Similarly, for each x , $b(n, x)$, $n \in \omega$, is nonincreasing, $\inf b(n, x) = b(x)$, and $b(n, x) = b(V(n, x))$.

The base \mathcal{B} showing that X is subcompact is the union of four families. Let \mathcal{B}_1 be the family of open convex subsets B of X such that $\text{cl}_K B \subseteq X$. Let $\mathcal{B}_2 = \bigcup \{\mathcal{V}_n : n \in \omega\}$.

Towards defining \mathcal{B}_3 , fix $n \in \omega$ and $V \in \mathcal{V}_n$. The order $<$ on K induces an order $< = <_{n, V}$ on $\mathcal{W} = \mathcal{W}_{n, V} = \{W \in \mathcal{V}_{n+1} : W \subseteq V\}$. Let \mathcal{Z} be well ordered and cofinal in \mathcal{W} . (If \mathcal{W} has a maximal element, Z , set $\mathcal{Z} = \{Z\}$). Let $T \subseteq V$ satisfy $|Z \cap T| = 1$ for all $Z \in \mathcal{Z}$ plus if $t \in T$ and $C(t)$ has a maximal element, then $t = \max C(t)$. Let $\mathcal{B}(3, n, V)$ be the collection of convex open subsets B of V such that there are $c(B)$, $x(B)$, and $t(B)$ satisfying

- (1) $c(B) < x(B) \leq t(B)$,
- (2) $B = (c(B), b(V(n+1, t(B))))$,
- (3) $B \cap T = \{t(B)\}$,
- (4) $(c(B), x(B)) \subseteq C(x(B))$,
- (5) $x(B) = \max C(x(B))$.

Observe that if B , B' , and B'' are members of $\mathcal{B}(3, n, V)$ which satisfy $B \subseteq B' \cap B''$, then $t(B) = t(B') = t(B'')$. Further, note that if $x' \in B \in \mathcal{B}(3, n, V)$, then $V(n+1, x') \subseteq B$ or $C(x')$ has a maximal element, namely $x(B)$ (or both). Unfix n and V . Set $\mathcal{B}_3 = \bigcup \{\mathcal{B}(3, n, V) : n \in \omega \wedge V \in \mathcal{V}_n\}$. Define \mathcal{B}_4 analogously to \mathcal{B}_3 with the order reversed.

Lemma 9.6. *Suppose that $b(x) = x$. For every n , there is $x' \in V(n, x) \cap X$ satisfying $x < x'$. Moreover, there is $m \geq n$ such that $V(m+1, x)$ is not the maximal member of $\mathcal{W}_{m, V(m, x)}$. Similar results hold when $a(x) = x$.*

Proof. Because $x = b(x) < b(n, x)$, there is y from $K \setminus X$ satisfying $x < y < b(n, x)$. Then $(x, b(n, x))$ is a nonempty open subset of K . Because X is dense in K , there is $x' \in X$ satisfying $x < x' < b(n, x)$. Then $x' \in (x, b(n, x)) \subseteq V(n, x)$, and x is not the largest member of $V(n, x)$. Choose m least such that $b(m+1, x) < x'$. Then $V(m+1, x) < V(m+1, x')$, and both are in $\mathcal{W} = \mathcal{W}_{m, V(m, x)}$. \square

Lemma 9.7. $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ is a base for X .

Proof. Let $c < x < d$, where $x \in X$, and $c, d \in K^+$. We must find $B \in \mathcal{B}$ such that $x \in B \subseteq (c, d)$. We proceed by cases.

Case 1. $a(x) < x < b(x)$. Let c' satisfy $\max\{a(x), c\} < c' < x$ if there is such a point; let $c' = \max\{a(x), c\}$ if not. Choose d' analogously. Then $(c', d') \in \mathcal{B}_1$.

Case 2. $a(x) = x = b(x)$. Choose n large enough that $c < a(n, x) < x < b(n, x) < d$. Then $V(n, x) \in \mathcal{B}_2$.

Case 3. $a(x) < x = b(x)$. We may assume that $a(x) \leq c$. (Replace c with $\max\{c, a(x)\}$ if necessary). Choose n large enough that $b(n, x) < d$. By Lemma 9.6, there is $m \geq n$ such that $V(m+1, x)$ is not the maximal member of $\mathcal{W}_{m, V(m, x)}$. Because \mathcal{Z} is well ordered and cofinal in \mathcal{W} , $\{t \in T : c < t\} \neq \emptyset$ has a least element t' . Then $B = (c, b(V(m+1, t')))$ is a convex open subset of $V(m, x)$ and $B \cap T = \{t'\}$. By definition, $B \in \mathcal{B}_3$.

Case 4. $a(x) = x < b(x)$. Analogous to Case 3. \square

Lemma 9.8. *If $\mathcal{F} \subseteq \mathcal{B}$ be a regular open filter base, then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. There are four cases.

Case 1: \mathcal{B}_1 is cointial in \mathcal{F} . Then $\bigcap \mathcal{F} \neq \emptyset$ because \mathcal{F} contains a compact set.

Case 2: \mathcal{B}_2 is cointial in \mathcal{F} . Let $N(\mathcal{F})$ be the set of n such that $\mathcal{F} \cap \mathcal{V}_n \neq \emptyset$. Because \mathcal{V}_n is pairwise disjoint, there is only such V , let's call it $V(n, \mathcal{F})$. If $n^* = \max N(\mathcal{F})$, then $\bigcap \mathcal{F} = V(n^*, \mathcal{F}) \neq \emptyset$. If $N(\mathcal{F})$ is infinite, then Lemma 9.5(6) implies that $\bigcap \mathcal{F} = \bigcap \{V(n, \mathcal{F}) : n \in N(\mathcal{F})\} \neq \emptyset$, and hence the intersection is $C(x)$ for some $x \in X$.

Case 3: \mathcal{B}_3 is cointial in \mathcal{F} . Let $N(\mathcal{F})$ be the set of n such that $\mathcal{F} \cap \mathcal{B}(3, n, V) \neq \emptyset$ for some $V \in \mathcal{V}_n$. Because \mathcal{V}_n is pairwise disjoint, there is only such V , let's call it $V(n, \mathcal{F})$. If $n^* = \max N(\mathcal{F})$, then $t^* \in \bigcap \mathcal{F} \neq \emptyset$, where $t^* = t(B)$ for all $B \in \mathcal{F} \cap \mathcal{B}(3, n, V(n, \mathcal{F}))$.

Suppose that $N(\mathcal{F})$ is infinite. Then $\bigcap \{V(n, \mathcal{F}) : n \in N(\mathcal{F})\} = C(x)$, for some $x \in X$. If $c(B) \notin C(x)$ for all $B \in \mathcal{F}$, then $\bigcap \mathcal{F} = C(x) \neq \emptyset$. If $c(B^*) \in C(x)$ for some $B^* \in \mathcal{F}$, then $C(x)$ has a maximal element $x(B^*)$, and $x(B^*) \in \bigcap \mathcal{F} \neq \emptyset$.

Case 4: \mathcal{B}_4 is cointial in \mathcal{F} . Analogous to Case 3. \square

10. SUBCOMPACT BASES ARE FRAGILE

One reason that it is difficult to prove results about subcompact spaces is that subcompact bases are fragile. Here “fragile” is an imprecise term. We mean that a number of plausible statements of the form “if X has a subcompact base \mathcal{B} then \mathcal{B} can be modified to become a nicer subcompact base” are false.

Example 10.1. A completely metrizable space X with a subcompact base \mathcal{B} such that $\{\text{int cl } B : B \in \mathcal{B}\}$ is not a subcompact base.

Let $X = \mathbb{R}$ and let $\mathcal{B}_0 = \{(x, y) : x < y\}$ be the usual base. Let $\mathcal{B}_1 = \{(n, \infty) \setminus \mathbb{Z} : 0 < n \in \mathbb{Z}\}$ and $\mathcal{B}_2 = \{(n, \infty) : 0 < n \in \mathbb{Z}\}$. Then $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ is a subcompact base for X , but $\{\text{int cl } B : B \in \mathcal{B}\} = \mathcal{B}_0 \cup \mathcal{B}_2$ is not.

Example 10.2. A completely regular subcompact space with no subcompact base closed under finite intersection.

A rough description of this space is a Cantor tree with the isolated points split. Now a precise description. Let Y be the set of functions from ω to $\{0, 1\}$. Let Z_0 be the set of functions from a natural number to $\{0, 1\}$ – in other words, the set of proper initial segments of elements of Y . Let $Z = Z_0 \times [Y]^{<\omega}$. Here $[Y]^{<\omega}$ is the collection of finite subsets of Y . For $y \in Y$ and $n \in \omega$, set

$$B(y, n) = \{y\} \cup \{(y|_m, s) \in Z : n \leq m < \omega \text{ and } y \in s \in [Y]^{<\omega}\}.$$

Set $\mathcal{B}_0 = \{\{z\} : z \in Z\}$. Set $\mathcal{B}_1 = \{B(y, n) : y \in Y \text{ and } n \in \omega\}$, $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ and $X = Y \cup Z$. It is straightforward to verify that \mathcal{B} is a subcompact base for a T_1 , regular topology on X .

Let \mathcal{A} be a base for X . We will find subfamily \mathcal{C} of \mathcal{A} such that $\bigcap \mathcal{C}' \neq \emptyset$ for every finite subset \mathcal{C}' of \mathcal{C} , and yet $\bigcap \mathcal{C} = \emptyset$. For each $y \in Y$, first choose $A_y \in \mathcal{A}$ and then $k_y \in \omega$ to satisfy

$$y \in B(y, k_y) \subseteq A_y \subseteq B(y, 1).$$

Briefly consider Y to be the Cantor set via the usual homeomorphism. Apply the Baire category theorem to obtain $k \in \omega$ and a clopen subset W of Y such that $D = \{y \in W : k_y = k\}$ is dense in W . Restrict, if necessary, so that W has the form $\{y \in Y : \zeta \subset y\}$ for some finite function ζ from $n \geq k$ to $\{0, 1\}$. Set $\mathcal{C} = \{A_y : y \in D\}$. If s is a finite subset of D , then for some $n' \geq k$ and any $y \in s$, we have

$$\bigcap \{A_{y'} : y' \in s\} = \{(y|_m, t) : n' \leq m \leq k \text{ and } s \subseteq t\},$$

a closed set of isolated points. We conclude that if \mathcal{A} is closed under finite intersection, then \mathcal{A} is not a subcompact base for X .

Example 10.2 has many interesting properties. It is a Čech complete, Moore complete metacompact Moore space which is not cocompact nor Scott domain representable. These properties and more are defined and discussed in [Ta73] (where Tall introduced this space), [Mi83], and [BL07]. In particular, Bennett and Lutzer show that X is a (not dense) G_δ subspace of a Scott domain representable space X^+ . We present a brief definition of X^+ . Following the definition of X above, let $Z^+ = Z_0 \times \mathcal{P}(Y)$. Here $\mathcal{P}(Y)$ is the family of all subsets of Y . Define $B^+(y, n)$ with $\mathcal{P}(Y)$ replacing $[Y]^{<\omega}$ in the definition of $B(y, n)$. Continue with $\mathcal{B}_1^+ = \{B^+(y, n) : y \in Y \text{ and } n \in \omega\}$, $\mathcal{B}^+ = \mathcal{B}_0 \cup \mathcal{B}_1^+$, and $X^+ = Y \cup Z^+$.

Example 10.3. There is a subcompact space X with a base \mathcal{B} such that every subfamily \mathcal{A} of \mathcal{B} is not a subcompact base.

Let X be the Sorgenfrey line. In more detail, the point set of X is \mathbb{R} , and the usual base is the family of left closed, right open intervals, $\mathcal{J} = \{[x, y) : x < y\}$. By a well-ordering argument, Aarts and Lutzer find a subfamily \mathcal{J}' of \mathcal{J} , such that \mathcal{J}' is a base for X and if $[x, y), [x', y') \in \mathcal{J}'$, and $y = y'$, then $x = x'$. \mathcal{J}' is a subcompact base for X .

We will work with the subfamily $\mathcal{I} = \{[x, q) : x < q \in \mathbb{Q}\}$. Note that these intervals are clopen, hence every open filter base on \mathcal{I} is a regular open filterbase. If subfamily \mathcal{U}' of \mathcal{I} satisfies $\inf\{y : [x, y) \in \mathcal{U}'\} = x$ then \mathcal{U}' contains a neighborhood base at x .

Enumerate \mathbb{Q} as $\{q_k : k \in \omega\}$, and let \mathcal{F} be the collection of functions from $\omega \rightarrow \mathcal{I}$ such that the right endpoint of $f(k)$ is q_k and the length of $f(k)$ approaches 0 as k approaches 0.

Claim. *If $\mathcal{A} \subseteq \mathcal{U}$ is a subcompact base for X , then there is $f \in \mathcal{F}$ such that for all $k \in \omega$, if $x \in f(k)$, then $[x, q_k] \notin \mathcal{A}$.*

Proof. If not, then for some k , $\bigcap \{[x, q_k] : [x, q_k] \in \mathcal{A}\} = \emptyset$. \square

For each $f \in \mathcal{F}$ and $k \in \omega$, $V(f, k) := \bigcup_{k < n < \omega} f(n)$ is a dense open subset of \mathbb{R} , and for every $f \in \mathcal{F}$, $G_f := \bigcap_{k \in \omega} V(f, k)$ is a dense G_δ of \mathbb{R} and hence G_f has cardinality \mathfrak{c} . There is a one-to-one map $f \mapsto x_f \in G_f$ because the cardinality of \mathcal{F} is also \mathfrak{c} . Observe that if $x \in G_f$, then $\mathcal{B}(f, x) := \{[x, q_k] : x \in V(f, k)\}$ is infinite; in fact $\mathcal{B}(f, x)$ is a base at x because length $f(n)$ goes to 0.

For each $x \in X$, we define a neighborhood base, $\mathcal{B}(x)$, at x . If $x = x_f$ for some f , let $\mathcal{B}(x_f)$ be the collection $\{[x_f, q_k] : x_f \in V(f, k)\}$. Otherwise, let $\mathcal{B}(x) = \{[x, q] : q \in \mathbb{Q}\}$. Let $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}(x)$.

Suppose by way of contradiction that $\mathcal{A} \subseteq \mathcal{B}$ is a subcompact base for X . Let $g \in \mathcal{F}$ be as in the claim. Because \mathcal{A} is a base and $\mathcal{A} \subseteq \mathcal{B}$, there is $[x_g, q_k] \in \mathcal{A}$ satisfying $x_g \in [x_g, q_k] \subset [x_g, \infty)$, contradicting the claim.

11. QUESTIONS

We begin by repeating Question 3.1 of [BLqa].

Question 11.1. *If Y is a G_δ subset of a subcompact space X , must Y be subcompact? What if Y is a dense G_δ in X ? Is every Čech complete space subcompact?*

The G_δ question is open for other completeness properties.

Question 11.2. a. *If Y is a dense G_δ subspace of a quasiregular pseudocomplete space X , must Y be pseudocomplete? (See [AL74]).*

b. *If Y is a dense G_δ of a Scott representable space X , must Y be Scott domain representable? (See Example 10.2 and [BL07]).*

c. *If Y is a dense G_δ of a generalized subcompact space X , must Y be generalized subcompact?*

The results of Section 3 suggest the next question.

Question 11.3. *Is every domain representable space generalized subcompact? Equivalently, if X is domain representable must there be a domain P , a basis Q for P such that $q \mapsto \uparrow \cap \max P$ from Q to $\tau^*(\max P)$ is one-to-one, and a homeomorphism $\phi : X \rightarrow \max P$. (See Proposition 3.8).*

The next question is Question 5.2a of [BLqa].

Question 11.4. *If X is domain representable space, must α have a coding winning strategy in $Ch(X)$?*

Question 11.5. *Do the analogs of Proposition 7.3, Corollary 7.4, and Corollary 7.5 hold for subcompact spaces? (See [BLqa, Question 3.2]).*

The answer to the previous question is yes for spaces X with a subcompact base of clopen sets. However, this does not automatically extend to all zero-dimensional subcompact spaces unless we have a yes answer to the following.

Question 11.6. *Does a zero-dimensional subcompact space have a subcompact base of clopen sets?*

The naive, first attempt (hope that $\{\text{int cl } B : B \in \mathcal{B}\}$ is a subcompact base – see Example 10.1) does not answer the following question.

Question 11.7. *Does every completely regular subcompact space have a subcompact base of regular open sets?*

The next question rephrases Question 5.4 of [BLqa]

Question 11.8. *Does the full converse of Theorem 4.3 hold in the class of spaces with G_δ -diagonals?*

Question 11.9. *Debs [De85] and Galwin-Telgarski [GT86] showed that if α has a winning strategy in $BM(X)$, then α has a coding winning strategy in $BM(X)$. Does the analogous result hold for $Ch(X)$?*

The following are projects, rather than specific questions.

Question 11.10. *Given that a space X is domain representable, find the domain P (and/or a basis Q of P) which best represents X . What does “best” mean in this context?*

Question 11.11. *Extend the Banach-Mazur and strong Choquet games to transfinite games in a useful way.*

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