Locally Compact Groups: Traditions and Trends

Karl Heinrich Hofmann  
Technische Universitat Darmstadt, hofmann@mathematik.tu-darmstadt.de

Wolfgang Herfort

Francesco G. Russo

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“The best mathematics is the most mixed-up mathematics, those disciplines in which analysis, algebra and topology all play a vital role.”
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← Gordon Thomas Whyburn ← Robert Lee Moore
Locally compact groups: Traditions and Trends

Karl Heinrich Hofmann,
TU Darmstadt, Germany
& Tulane University,
New Orleans, USA.

June 2017
Landmarks

1900 David Hilbert
and his Fifth Problem
Landmarks

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1927 Fritz Peter and Hermann Weyl
Representations of Compact Groups
Landmarks

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1934 Lev S. Pontryagin
1937 Egbert van Kampen
Duality of Locally Compact Abelian Groups
Center of Century

1940 André Weil

[L’ intégration dans les groupes topologiques...]
Center of Century

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Center of Century

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[L’ intégration dans les groupes topologiques. . . ]

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1940, 1953, 1965, 1979
1941 Kenkichi Iwasawa

Über die endlichen Gruppen...
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*Über die endlichen Gruppen...*

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Then one knows:

*Every locally compact almost connected group $G$ is approximated by Lie group quotients $G/N$. (Projective limit)*
The Second Half of the 20th Century

1957 R. K. Lashof: Every locally compact group $G$ has a Lie algebra $\mathfrak{g}$ and an exponential function $\exp : \mathfrak{g} \to G$. 
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Solenoid

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\[ \mathbb{R} \times \mathbb{Z}_2 \]
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$$\mathbb{Z}_2$$ is a Cantor set isomorphic to

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$$G$$ is far from a Lie group, yet it has the Lie algebra $$\mathbb{R}$$ and its exponential function $$\exp : \mathbb{R} \to G$$ is a group morphism.
The assignment $G \mapsto \mathfrak{g}$ of a Lie algebra $\mathcal{L}(G) = \mathfrak{g}$ to a locally compact group $G$ is functorial.
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If $f : G \to H$ is a morphism of topological groups, then

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\begin{array}{ccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\
\exp_G & & \exp_H \\
G & \xrightarrow{f} & H
\end{array}
$$

is a commuting diagram.
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is a commuting diagram.

exp is a natural transformation.
The underlying topological vector space of the Lie algebra $\mathfrak{g}$ of a locally compact group is $\cong \mathbb{R}^d$ for a cardinal $d$.

The vector spaces $V = \mathbb{R}^{(d)}$ and $W = \mathbb{R}^d$ are dual to each other, i.e.

$$\operatorname{Hom}(V, \mathbb{R}) \cong \mathbb{R}^d \quad \text{and} \quad \operatorname{Hom}_{\text{cont}}(W, \mathbb{R}) = \mathbb{R}^{(d)}.$$
Dimension

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The cardinal $d$ is assigned canonically to any locally compact group $G$ and is called the dimension $\dim G$ of $G$.

Every cardinal occurs even in the class of compact groups: Let $T = \mathbb{R}/\mathbb{Z}$ be the circle group, then $\dim T^d = d$. 
Topological dimension of locally compact groups

Theorem

On locally compact groups $G$ all concepts of topological dimensions agree and equal $\dim G$. 
Zero-Dimensional Groups

**Corollary**

*A locally compact group is zero-dimensional if and only if it is totally disconnected iff its Lie algebra is singleton.*
Historic roots of zero-dimensional locally compact groups

—Galois groups (of algebraic extensions of fields)
Historic roots of zero-dimensional locally compact groups

—Galois groups (of algebraic extensions of fields)
—nonarchimedian completions of $\mathbb{Q}$ yield the $p$-adic fields $\mathbb{Q}_p$, their linear algebra, eventually the Lie group theory over $\mathbb{Q}_p$.
—also the nonarchimedean ($p$-adic) completion of $\mathbb{Z}$, called $\mathbb{Z}_p$. 
One glance back

Lie theory $[G \mapsto \mathcal{L}(G)]$ introduces algebra into the topological structure theory of topological groups:
cf. components, arc components, direct decompositions, covering homomorphisms, transformation groups etc.
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Zero-dimensionality nixes the influence of Lie theory on topology: other branches of mathematics must enter: number theory, finite group theory, $p$-adic Lie algebra theory. Abelian locally compact group theory without $\mathbb{R}^m$ and the torus $\mathbb{T}^n$. 
Zero-dimensional locally compact groups: Some historic landmarks

1948 Jean Braconnier: *Sur les groupes topologiques localement compacts*
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Some Background Notes

Some “new” tools

Near abelian groups

Applications

Totally disconnected locally compact groups

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Grand Literature on dim-0 groups

Generic and monographic literature on noncompact zero-dimensional locally compact groups:
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Generic and monographic literature on noncompact zero-dimensional locally compact groups:

NONE as of yet
For a compact space $X$ the set of closed subsets is a compact space called its *hyperspace* $\mathcal{H}(X)$. The function

$$x \mapsto \{x\} : X \to \mathcal{H}(X)$$

is a topological embedding.
Hyperspaces

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is a topological embedding.

[See Leopold Vietoris: Stetige Mengen, Monatshefte 31 (1921)]
[L. Vietoris, 1891—2002.]
The Chabauty space of a locally compact group

**Definition.** For a locally compact group $G$ let $\text{SUB}(G)$ denote the set of all closed subgroups (with the topology of the hyperspace of $G \cup \{\infty\}$ (and the set $\text{SUB}(G)$ embedded into it) is called the *Chabauty space* of $G$. The function

$$g \mapsto \langle g \rangle : G \rightarrow \text{SUB}(G)$$

—is it even continuous? —A natural question!

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A “new” tool: The Chabauty space

“Nostrification”

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The Chabauty space was rediscovered by Bourbaki in the context of Haar measure theory as the space of idempotents in the measure algebra $M(G)$ (uncomplicated only if $G$ is compact).

[see Nicolas Bourbaki, Intégration, Chap. 7 et 8, Hermann, Paris, 1963]
Some results and examples

**Theorem**

*For a locally compact group $G$ the function*

$$g \mapsto \overline{\langle g \rangle} : G \to \text{SUB}(G)$$

*is continuous if and only if*

[Hofmann and Willis, 2015]
Some results and examples

**Theorem**

*For a locally compact group* $G$ *the function*

$$g \mapsto \langle g \rangle : G \to SUB(G)$$

*is continuous if and only if* $G$ *is 0-dimensional.*

[Hofmann and Willis, 2015]
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**Example.** If $G = \mathbb{R}$ then

$$\text{SUB}(\mathbb{R}) = \{ r \cdot \mathbb{Z} : 0 \leq r \} \cup \mathbb{R} \text{ homeom } [0, 1].$$
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**Example.** If $G = \mathbb{R}$ then

$$\text{SUB}(\mathbb{R}) = \{ r \cdot \mathbb{Z} : 0 \leq r \} \cup \mathbb{R}$$ *homeom [0, 1].*

*Inside $\text{SUB}(\mathbb{R})$*
Note: \( G \) is approximated by discrete subgroups \( \cong \mathbb{Z} \) inside \( \text{SUB}(G) \)

Next Example. If \( G = \mathbb{Z}_p \), then

\[
\text{SUB}(G) = \{ p^n \cdot \mathbb{Z}_p : n=0, 1, 2, \ldots \} \cup \{ \{0\} \}
\]

homeomorphic to \( \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N} \} \).
A quite horrible example: The circle group

**Further Example.** If $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, then

$$g \mapsto \langle g \rangle : \mathbb{T} \rightarrow \text{SUB}(\mathbb{T}) = \left\{ \frac{1}{n} \mathbb{Z} : n \in \mathbb{N} \right\} \cup \left\{ \{\mathbb{T}\} \right\}$$

is surjective; the set of all “irrational” points in $\mathbb{T}$ is dense and maps onto $\mathbb{T} \leftrightarrow \infty$. The function $g \mapsto \langle g \rangle$ is continuous at all “irrational” points and discontinuous at all “rational” ones.
Inductively monothetic groups

**Definition.** A subgroup $H$ of a topological group is called *monothetic* if it is closed and has a dense cyclic subgroup. If is called *inductively monothetic* if it is closed and any closed subgroup having a dense finitely generated subgroup is monothetic.
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- $\mathbb{Q}$ (discrete) is inductively monothetic but not monothetic.
- $\mathbb{T}^2$ (compact 2-torus) is monothetic but not inductively monothetic. (It contains the 4 element subgroup $\left\{\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right\}^2$ which is not monothetic.)
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- However: Fact. All 0-*dimensional* monothetic groups are inductively monothetic.
A “new” tool: Inductively monothetic groups

Classification of Inductively Monothetic Groups

Theorem

[Herfort, Hofmann, Russo] A locally compact group $G$ is inductively monothetic if and only if one of the following conditions is satisfied:
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Theorem

[Herfort, Hofmann, Russo] A locally compact group $G$ is inductively monothetic if and only if one of the following conditions is satisfied:

1. $G$ is a one-dimensional monothetic group.
2. $G$ is discrete and isomorphic to a subgroup of $\mathbb{Q}$.
3. $G$ is isomorphic to a local product

\[
\prod_{p \text{ prime}} (G_p, C_p),
\]

where $G_p$ is either $\cong \mathbb{Z}(p^n)$, $n = 0, 1, \ldots, \infty$, or $\mathbb{Z}_p$ or $\mathbb{Q}_p$. 
A “new” tool: Inductively monothetic groups

Here: $\prod_{j \in J}^{\text{loc}} (G_j, C_j)$ is a subgroup of $\prod_{j \in J} G_j$ containing all $(g_j)_{j \in J}$ such that $\{j \in J : g_j \notin C_j\}$ is finite.
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Near abelian groups

Applications

A “new” tool: Inductively monothetic groups

Here: \( \prod_{j \in J}^{\text{loc}} (G_j, C_j) \) is a subgroup of \( \prod_{j \in J} G_j \) containing all \( (g_j)_{j \in J} \) such that \( \{ j \in J : g_j \notin C_j \} \) is finite.

The topology on the local product is obtained by declaring \( \prod_{j \in J} C_j \) with its product topology open. For infinite \( J \) it is finer than the topology induced from \( \prod_{j \in J} G_j \).
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The class of inductive monothetic groups is called $\mathcal{IM}$. 
Selfduality of the class $\mathcal{IM}$

Corollary

*The class $\mathcal{IM}$ is closed under passage to the character group.*
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*The class $\mathcal{IM}$ is closed under passage to the character group.*

Note: This property makes the class $\mathcal{IM}$ formally quite different from the class of monothetic groups.
Groups $G$ approximable by groups $\cong \mathbb{Z}$ in $\text{SUB}(G)$

For a topological group $G$ let $G_0$ denote the connected component of 1,
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For a topological group $G$ let $G_0$ denote the connected component of 1, and let $\text{comp}(G)$ denote the union of all compact subgroups in $G$.

**Theorem**

Let $G$ be a locally compact group. Then the following statements are equivalent:

1. In $\text{SUB}(G)$ we have $G \in \{ H \in \text{SUB}(G) : H \cong \mathbb{Z} \}$. 

[Hamrouni, Hofmann]
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For a topological group $G$ let $G_0$ denote the connected component of 1, and let $\text{comp}(G)$ denote the union of all compact subgroups in $G$.

**Theorem**

*Let $G$ be a locally compact group. Then the following statements are equivalent:*

1. In $\text{SUB}(G)$ we have $G \in \{ H \in \text{SUB}(G) : H \cong \mathbb{Z} \}$.
2. Either $G \cong \mathbb{R} \times \text{comp}(G)$ and $G/G_0$ is inductively monothetic, or else $G$ is $\cong$ to a subgroup of $\mathbb{Q}$.

[Hamrouni, Hofmann]
The Definition of Near Abelian Groups

**Definition**

A topological group will be called *near abelian* if it is locally compact and and there is a closed normal subgroup $A$ such that

1. $G/A$ is inductively monothetic.
2. Every closed subgroup of $A$ is normal in $G$. 

There is a representation $\psi: G \to \text{Aut}(A)$, $\psi(g)(a) = gag^{-1}$.

$\ker \psi = C_G(A) = \{g \in G : (\forall a \in A : ag = ga)\}$, is the centralizer of $A$ in $G$. 


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The centralizer $C_G(A)$

Clearly, $A \subseteq C_G(A)$; $C_G(A)$ is the unique largest normal abelian subgroup of $G$ containing $A$. 
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containing $A$.

If $\psi(A) \subseteq \{\text{id}, -\text{id}\}$, then $G$ is said to be $A$-trivial.
First Results

Theorem

Let $G$ be an $A$-nontrivial near abelian group. Then

1. $A$ is periodic (i.e. $A$ totally disconnected and $A \subseteq \text{comp}(A)$.)
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3. $G$ is periodic iff $G/A$ is periodic iff $G/A \not\cong$ a subgroup of $\mathbb{Q}$.

This explains why in the context of near abelian groups the theory of periodic groups is significant.
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This explains why in the context of near abelian groups the theory of *periodic groups* is significant.
Factorisation and Scaling

Definition

Let $G$ be a near abelian group with base $A$. A closed subgroup $H$ of $G$ is called a scaling subgroup if

(i) $H$ is inductively monothetic, and

(ii) $G = AH$
Factorisation and Scaling

Definition

Let $G$ be a near abelian group with base $A$. A closed subgroup $H$ of $G$ is called a scaling subgroup if

(i) $H$ is inductively monothetic, and

(ii) $G = AH$

If $H$ satisfies (i) and

(ii′) $G = C_G(a)H$,

then $H$ is called a small scaling subgroup.
A Note on the Significance of Scaling

If $G = AH$, since $H$ is $\sigma$-compact and has an open compact subgroup, there is a quotient homomorphism

$$A \rtimes H \to AH = G \text{ with kernel } \{(-h, h) : h \in A \cap H\} \cong A \cap H.$$
A Note on the Significance of Scaling

If $G = AH$, since $H$ is $\sigma$-compact and has an open compact subgroup, there is a quotient homomorphism

$$A \rtimes H \to AH = G$$

with kernel $\{(h, -h) : h \in A \cap H\} \cong A \cap H$.

The product $AH$ is not too far from the semidirect product $A \rtimes H$. 

A Fundamental Theorem

Theorem

Let $G$ be a periodic near abelian group with base $A$ such that $G$ is $A$-nontrivial. Then there is a small scaling subgroup, i.e. $G = C_G(A)H$. 

[Herfort, Hofmann, Russo]

We call such groups $H\prod$-procyclic. These groups are not necessarily compact.
A Fundamental Theorem

Theorem

Let $G$ be a periodic near abelian group with base $A$ such that $G$ is $A$-nontrivial. Then there is a small scaling subgroup, i.e. $G = C_G(A)H$.

Moreover, $H = \prod_{p \text{ prime}}^{loc} (G_p, C_p)$ and $G_p$ is either a finite cyclic $p$-group or a $p$-adic group $\cong \mathbb{Z}_p$.

[Herfort, Hofmann, Russo]
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Moreover, $H = \prod_{p \text{ prime}}^{loc}(G_p, C_p)$ and $G_p$ is either a finite cyclic $p$-group or a $p$-adic group $\cong \mathbb{Z}_p$.

[Herfort, Hofmann, Russo]

We call such groups $H \text{ } \Pi$-procyclic. These groups are not necessarily compact.
Another way of saying that a group $H$ is $\Pi$-procyclic is that $H$ is periodic locally compact abelian such that each $p$-Sylow subgroup is procyclic.
An Alternative Fact (Theorem)

Another way of saying that a group $H$ is $\Pi$-procyclic is that $H$ is periodic locally compact abelian such that each $p$-Sylow subgroup is procyclic.

**Theorem**

Let $G$ be a locally compact group and $A$ any compact normal subgroup such that $G/A$ is $\Pi$-procyclic. Then there is a $\Pi$-procyclic subgroup $H$ such that $G = AH$. 
Another Alternative Fact (Theorem)

**Theorem**

*Let $G$ be a locally compact group and $A$ any compact normal subgroup such that $G/A$ is isomorphic to a discrete subgroup of $\mathbb{Q}$. Then there is a closed subgroup $H$ such that $G = A \rtimes H$.***
Another Alternative Fact (Theorem)

**Theorem**

Let $G$ be a locally compact group and $A$ any compact normal subgroup such that $G/A$ is isomorphic to a discrete subgroup of $\mathbb{Q}$. Then there is a closed subgroup $H$ such that $G = A \rtimes H$.

It is unknown to which extent these facts remain valid if the compactness of $A$ is relaxed to closedness.
Lemma

Let $G$ be a locally compact abelian periodic group. and $\alpha$ a continuous automorphism. Then the following statements are equivalent:

(1) $\alpha(H) \subseteq H$ for every closed subgroup $H \leq G$. 
Lemma

Let $G$ be a locally compact abelian periodic group. and $\alpha$ a continuous automorphism. Then the following statements are equivalent:

1. $\alpha(H) \subseteq H$ for every closed subgroup $H \leq G$.
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1. $\alpha(H) \subseteq H$ for every closed subgroup $H \leq G$.
2. $\alpha(\langle g \rangle) \subseteq \langle g \rangle$ for every $g \in G$.
3. $\alpha(g) \in \langle g \rangle$ for every $g \in G$.
4. There is an $r \in \widehat{\mathbb{Z}}$ such that $\alpha(g) = r \cdot g$ for all $g \in G$. 
Here
\( \tilde{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p \) is the universal profinite compactification of \( \mathbb{Z} \).
Its (discrete) character group is \( \cong \mathbb{Q}/\mathbb{Z} \).

The subgroup of \( \text{Aut} \ G \) containing all scalar automorphisms is written \( \text{SAut} \ G \).
The Lemma shows that the representation

\[ r \mapsto \{ x \mapsto r \cdot x \} : (\tilde{\mathbb{Z}})^\times \to \text{SAut } G \]

is surjective for each locally compact abelian periodic group \( G \).

(For any ring \( R \) let \( R^\times \) denote the group of units, i.e. invertible elements.)
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For each near abelian group \( G \) with base \( A \) and periodic \( G/A \) we have a representation \( G/A \to \text{SAut } A \) therefore \( \text{SAut } A \) and so \((\tilde{\mathbb{Z}})^\times \) must be understood.
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Each profinite (=compact 0-dimensional) Abelian group \( G \) is the product \( \prod_{p \text{ prime}} G_p \) of its \( p \)-Sylow subgroups \( G_p \) (also called \( p \)-primary components).
The Structure of $\mathbb{Z}_p^\times$

Let $p$ be an odd prime. Then the $p$-Sylow subgroup of $\mathbb{Z}_p^\times$ contains $1 + p\mathbb{Z}_p$ ($=$the image of $p\mathbb{Z}_p$ under the exponential function).
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$$x \mapsto \exp p \cdot x : \mathbb{Z}_p \to (1 + p \cdot \mathbb{Z}_p, \times) \subseteq \mathbb{Z}_p^\times$$

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Also, $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ is the field of $p$ elements, and its multiplicative group of units is cyclic of order $p - 1$. 
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Also, \( \mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{Z} / p\mathbb{Z} \) is the field of \( p \) elements, and its multiplicative group of units is cyclic of order \( p - 1 \).

This group lifts to \( \mathbb{Z}_p^\times \), and in the end we have

\[
(\forall p > 2) \quad \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}(p - 1) \text{ (additively written)}.
\]
The exceptional case is $p = 2$:

$$\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times \mathbb{Z}(2) \text{ (additively written).}$$
Scalar Automorphisms

Decomposing $\tilde{\mathbb{Z}}^\times$

We know $\tilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, therefore,

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Thus we have

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Thus we have

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But we want

$$\widetilde{\mathbb{Z}}^\times = \prod_p (\widetilde{\mathbb{Z}}^\times)_p.$$
and for this purpose we need the Sylow decomposition of

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where

\[ q - 1 = p_1^{k_1} \cdots p_n^{k_n}, \text{ with } n = n(q) \text{ and } p_j | (q - 1). \]

At this point one loses track without some simple graph theory.
Figure 1. The initial part of the master graph.
The Mastergraph

Graph theoretical interpretation for $\tilde{\mathbb{Z}}^\times$

We have a sloping edge $e = ((m, 1), (n, 0))$ with $m < n$ (that is, prime $p_m \leftrightarrow (m, 1)$ in the top row and prime $q_n \leftrightarrow (n, 0)$ in the bottom row) iff $p_m | (q_n - 1)$. 
Graph theoretical interpretation for $\hat{\mathbb{Z}}^\times$

We have a sloping edge $e = ((m, 1), (n, 0))$ with $m < n$ (that is, prime $p_m \leftrightarrow (m, 1)$ in the top row and prime $q_n \leftrightarrow (n, 0)$ in the bottom row) iff $p_m | (q_n - 1)$. Keep in mind $\mathbb{Z}_{q_n}^\times \cong \mathbb{Z}_{q_n} \times \mathbb{Z}(q_n - 1)$, and so we have a cyclic $p_m$- group $S_e \cong \mathbb{Z}(p_m^{k_e})$ for a suitable natural number $k_e$. For the vertical edge $e$ coming down to $q_n$ write $S_e = \mathbb{Z}_{q_n}$. 
Let $\mathcal{F}_q$ denote the finite set of sloping edges ending up in the lower vertex $q = q_n \leftrightarrow (n, 0)$ and $\mathcal{E}_p$ the set of all edges coming down from $p = p_m \leftrightarrow (m, 1)$. 
Structure of $\mathbb{Z}_q^\times$

Let $\mathcal{F}_q$ denote the finite set of sloping edges ending up in the lower vertex $q = q_n \leftrightarrow (n, 0)$ and $\mathcal{E}_p$ the set of all edges coming down from $p = p_m \leftrightarrow (m, 1)$. Then

$$\mathbb{Z}_q^\times = \prod_{e \in \mathcal{F}_q} S_e = \mathbb{Z}_q \times \prod_{e \in \mathcal{F}_q, \text{sloping}} \mathbb{Z}(p_e^{k(e)}).$$
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Remember $\tilde{\mathbb{Z}}^\times = \prod_{p \text{ prime}} \tilde{\mathbb{Z}}_p^\times$.
The Mastergraph

\[ \widetilde{\mathbb{Z}}_p^\times = \prod_{e \in \mathcal{E}_p} S_e \]

is the \( p \)-Sylow subgroup of \( \widetilde{\mathbb{Z}}^\times \) with \( e \) ranging over the set \( \mathcal{E}_p \) descending from \( p \) in the mastergraph.
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Thus \( \tilde{\mathbb{Z}}^\times \) is the product of the procyclic \( p \)-groups \( \mathbb{Z}_e \):
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This set is infinite according to Dirichlet’s Theorem on the infinity of the set of primes in arithmetic progressions.
Thus $\tilde{\mathbb{Z}}^\times$ is the product of the procyclic $p$-groups $\mathbb{Z}_e$:

$$\tilde{\mathbb{Z}}^\times = \prod_{e} S_e$$

where $e$ ranges through ALL edges of the mastergraph.
The Master-Graph of a Near Abelian Group

If $G$ is near abelian with base $A$, then we define the *graph $G$ of $G$* as a subgraph of the master-graph of $\tilde{\mathbb{Z}}$

—an upper vertex $p$ in the Mastergraph of $\tilde{\mathbb{Z}}$ is an upper vertex of the *Graph $G$ of $G$* if $(G/A)_p \neq \{1\}$,
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—an edge $e$ from $p$ to $q$ is an edge in $\mathcal{G}$ if $[G_p, A_q] \neq \{1\}$

Then $\mathcal{G}$ gives considerable insight into the structure of $G$. 
The Benefit from the use of Graphs

The examples show that the use of even technically simple graphs can help in the organization of an almost impenetrable Sylow subgroup structure of locally compact periodic groups such as $\tilde{\mathbb{Z}}$ and, more generally of all near abelian locally compact groups.
Summarizing Applications

This presentation is a preview of a monograph by W. Herfort, K. H. Hofmann, and R. G. Russo entitled

*Periodic Locally Compact Groups*,

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(B) Topologically modular groups $G$

\[ (\forall A, B, C \leq G \text{ closed}) \quad A \leq C \Rightarrow A \lor (B \cap C) = (A \lor B) \cap C. \]
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\[ (\forall A, B, C \leq G \text{ closed}) \quad A \leq C \Rightarrow A \lor (B \cap C) = (A \lor B) \cap C. \]

(C) Strongly topologically quasihamiltonian groups $G$

\[ (\forall A, B \leq G \text{ closed}) \quad AB = \overline{AB} \leq G. \]