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# Analysis of Weights in Central Difference Formulas for Approximation of the First Derivative

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# ANALYSIS OF WEIGHTS IN CENTRAL DIFFERENCE FORMULAS FOR APPROXIMATION OF THE FIRST DERIVATIVE

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Abstract. Manipulations of Taylor series expansions of increasing numbers of terms yield finite difference approximations of derivatives with increasing rates of convergence. In this paper, we consider central difference approximations of arbitrary order of accuracy. We derive explicit formulas for the weights of terms and explore their limits for increasing orders of accuracy.

Keywords: *Finite Difference formulas, first derivative*

MSC (2010): Primary 65L05

## 1. introduction

In this paper, we employ Vandermonde type determinants to derive explicit formulas for the weights in symmetric finite difference approximations to the first order derivative as a function of the number of nodes that are employed in that approximation. It is then a straightforward calculation to compute the limit of the weights as the number of nodes diverges to infinity. This work is motivated by Fornberg [3, 4, 5] and Fornberg and co-authors [2, 6] where the authors were interested in a broad variety of applications of finite difference approximations. Here we focus on the calculation of the weights employing methods related to the calculation of Vandermonde type determinants. There has been considerable interest in Vandermonde type determinants and we refer the interested reader to Lita da Silva [7] and the references therein. The paper is organized as follows. In Section 2, we introduce symmetric difference approximations to the first order derivative along with the Vandermonde determinant which is used throughout the paper. In Section 3, we construct the framework from which we approach the problem and then apply it to derive our results. We conclude in Section 4.

## 2. background material

We begin by deriving a central difference formula for the first order derivative with a quadratic rate of convergence. We employ equally spaced nodes, and we denote the spacing between nodes by $h$. Let $f : \mathbb{R} \to \mathbb{R}$ be three times continuously differentiable, and let $x \in \mathbb{R}$. By Taylor's Theorem with Lagrange remainder, there exist $\xi_1, \xi_2 \in \mathbb{R}$ where

$$x - h < \xi_2 < x < \xi_1 < x + h$$

such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(\xi_1), \tag{2.1}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(\xi_2). \tag{2.2}$$

Subtract (2.2) from (2.1) and apply the Intermediate Value Theorem to $f'''$ to obtain a central difference formula with quadratic rate of convergence:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

Now, if $f$ has sufficiently many continuous derivatives, we can proceed in a similar manner while using more nodes and thus including more terms in the Taylor series expansions of $f$. In that case, we can obtain central difference formulas for $f'$ of higher order. For example, a central difference formula of order four is given by

$$f'(x) = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} + O(h^4).$$

Each of these central difference formulas will have an even order of convergence because the nodes employed are symmetrically and equally placed around $x$. As a result, the first nonzero term in the Taylor series expansion of the finite difference scheme will be at an even power of $h$. In general, a central difference formula of even order $p$ for the first order derivative of a $p$ times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ at $x$ employs $x$ and $p$ additional nodes symmetric around $x$ and can be written as

$$f'(x) = \frac{1}{h} \sum_{j=-p/2}^{p/2} C_j f(x+jh) + O(h^p). \tag{2.3}$$

Fornberg [4, 5] finds recursive algorithms for computing the weights

$$C_j, j = -p/2, \ldots, p/2,$$

in Equation (2.3). Throughout the rest of this paper, we derive explicit formulas which express these weights as a function of $p$, allowing for a straightforward computation of the limit of the weights as $p \to \infty$.

To preserve the completeness of the paper, we state the general known form of a Vandermonde matrix and its determinant, which will be used several times in our calculations. A Vandermonde matrix is a square matrix with rows composed of terms of geometric sequences. Its determinant takes the form

$$\det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i).$$

## 3. Results

Motivated by Eberly [1], we begin by multiplying Equation (2.3) by $h$ to obtain a central difference approximation of $f'$ of the form

$$hf'(x) = \sum_{j=-p/2}^{p/2} C_j f(x+jh) + O(h^{p+1})$$

$$= \sum_{j=-p/2}^{p/2} C_j \left( \sum_{k=0}^{\infty} j^k \frac{h^k}{k!} f^{(k)}(x) \right) + O(h^{p+1})$$

$$= \sum_{k=0}^{\infty} \left( \sum_{j=-p/2}^{p/2} j^k C_j \right) \frac{h^k}{k!} f^{(k)}(x) + O(h^{p+1})$$

$$\approx \sum_{k=0}^{p} \left( \sum_{j=-p/2}^{p/2} j^k C_j \right) \frac{h^k}{k!} f^{(k)}(x). \tag{3.1}$$

The second equality follows since $\sum_{k=0}^{\infty} j^k \frac{h^k}{k!} f^{(k)}(x)$ is a Taylor series expansion of $f(x+jh)$. We treat (3.1) as an equality, and thus for $k = 0, 1, 2, \ldots, p$, we have

$$\sum_{j=-p/2}^{p/2} j^k C_j = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1. \end{cases} \tag{3.2}$$

Let $n = p/2$. Then (3.2) generates the following system of equations:

$$\begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0^0 & 1^0 & \cdots & (n-1)^0 & n^0 \\ (-n)^1 & (-n+1)^1 & \cdots & (-1)^1 & 0^1 & 1^1 & \cdots & (n-1)^1 & n^1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 0^{2n-1} & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0^{2n} & 1^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix} \begin{pmatrix} C_{-n} \\ C_{-n+1} \\ \vdots \\ C_{-1} \\ C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

We solve this system in order to find the weights in Equation (2.3). Let $\mathbf{P}$ denote the coefficient matrix in the above system, and let $\mathbf{P}_j$ denote matrix $\mathbf{P}$ with its $j$th column replaced by the standard unit vector $e_2$ in $\mathbb{R}^{2n+1}$. We will employ Cramer's rule and the known form of a general Vandermonde determinant to solve for each weight. By Cramer's rule,

$$C_j = \frac{\det(\mathbf{P}_j)}{\det(\mathbf{P})}, \quad j = -n, \ldots, n.$$

With respect to the solution of the system given by Cramer's rule, the denominator of each weight is the determinant of $\mathbf{P}$:

$$\det(\mathbf{P}) = \det \begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0^0 & 1^0 & \cdots & (n-1)^0 & n^0 \\ (-n)^1 & (-n+1)^1 & \cdots & (-1)^1 & 0^1 & 1^1 & \cdots & (n-1)^1 & n^1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 0^{2n-1} & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0^{2n} & 1^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

While $\mathbf{P}$ is the transpose of a Vandermonde matrix whose determinant we could readily compute without further manipulation, we proceed to compute $\det(\mathbf{P})$ analogously to how we will compute $\det(\mathbf{P}_{j\neq 0})$ in order to generate forms that ease the simplification of $C_{j\neq 0}$ once $\det(\mathbf{P})$ and $\det(\mathbf{P}_{j\neq 0})$ are each determined in this manner. We begin by noting that column $n+1$ of $\mathbf{P}$ is the standard unit vector $e_1$ in $\mathbb{R}^{2n+1}$. We expand down this column to obtain

$$\det(\mathbf{P}) = (-1)^n \det \begin{pmatrix} (-n)^1 & (-n+1)^1 & \cdots & (-1)^1 & 1^1 & \cdots & (n-1)^1 & n^1 \\ (-n)^2 & (-n+1)^2 & \cdots & (-1)^2 & 1^2 & \cdots & (n-1)^2 & n^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 1^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

We next transpose and factor the first entry from each row of the resulting matrix to obtain

$$\det(\mathbf{P}) = \prod_{i=1}^{n} i^2 \det \begin{pmatrix} 1 & -n & (-n)^2 & \cdots & (-n)^{2n-1} \\ 1 & -n+1 & (-n+1)^2 & \cdots & (-n+1)^{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & (-1)^2 & \cdots & (-1)^{2n-1} \\ 1 & 1 & 1^2 & \cdots & 1^{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n-1 & (n-1)^2 & \cdots & (n-1)^{2n-1} \\ 1 & n & n^2 & \cdots & n^{2n-1} \end{pmatrix}.$$

We are left with a Vandermonde determinant. In view of this, for relevant entries

$$\lambda_1 = -n, \lambda_2 = -n+1, \ldots \lambda_n = -1, \lambda_{n+1} = 1, \ldots, \lambda_{2n-1} = n-1, \lambda_{2n} = n,$$

we first consider the product of terms $(\lambda_j - \lambda_i)$ with $i < j$ and $\lambda_i, \lambda_j$ with the same sign, and we next consider the product of these terms with $i < j$ and $\lambda_i, \lambda_j$ with opposite signs. We compute

$$\begin{aligned} \det(\mathbf{P}) &= \left(\prod_{i=1}^{n} i^2\right) \left[\left(\prod_{i=1}^{n-1} (n-i)^{2i}\right) \left(\frac{\prod_{i=1}^{n} (i+1)^i (2n-i+1)^i}{(n+1)^n}\right)\right] \\ &= n^2 (n+1)^n \prod_{i=1}^{n-i} i^2 (n-i)^{2i} (i+1)^i (2n-i+1)^i. \qquad (3.3) \end{aligned}$$

Thus, (3.3) is the denominator of each weight with respect to Cramer's rule.

We now consider the numerator of each weight, beginning with $C_{j\geq 1}$, then $C_{j\leq -1}$, and then $C_{j=0}$. With respect to Cramer's rule, the numerator of $C_{j\geq 1}$ is the determinant of $\mathbf{P}_{j\geq 1}$:

$$\det(\mathbf{P}_{j\geq 1}) = \det \begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0^0 & 1^0 & \cdots & (j-1)^0 & 0 & (j+1)^0 & \cdots & (n-1)^0 & n^0 \\ (-n)^1 & (-n+1)^1 & \cdots & (-1)^1 & 0^1 & 1^1 & \cdots & (j-1)^1 & 1 & (j+1)^1 & \cdots & (n-1)^1 & n^1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 0^{2n-1} & 1^{2n-1} & \cdots & (j-1)^{2n-1} & 0 & (j+1)^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0^{2n} & 1^{2n} & \cdots & (j-1)^{2n} & 0 & (j+1)^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

Again, we expand down column $n+1$ with sign correction $(-1)^n$ and then expand down column $n+j$ of the resulting matrix with sign correction $(-1)^{n+j-1}$ to find

$$\det(\mathbf{P}_{j\geq 1}) = (-1)^{j+1} \det \begin{pmatrix} (-n)^2 & (-n+1)^2 & \cdots & (-1)^2 & 1^2 & \cdots & (j-1)^2 & (j+1)^2 & \cdots & (n-1)^2 & n^2 \\ (-n)^3 & (-n+1)^3 & \cdots & (-1)^3 & 1^3 & \cdots & (j-1)^3 & (j+1)^3 & \cdots & (n-1)^3 & n^3 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 1^{2n-1} & \cdots & (j-1)^{2n-1} & (j+1)^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 1^{2n} & \cdots & (j-1)^{2n} & (j+1)^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

We next transpose and factor the first entry from each row of the resulting matrix. We obtain

$$\det(\mathbf{P}_{j\geq 1}) = \frac{(-1)^{j+1}}{j^2}\prod_{i=1}^{n} i^4 \det\begin{pmatrix} 1 & -n & (-n)^2 & \cdots & (-n)^{2n-2} \\ 1 & -n+1 & (-n+1)^2 & \cdots & (-n+1)^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & (-1)^2 & \cdots & (-1)^{2n-2} \\ 1 & 1 & 1^2 & \cdots & 1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & j-1 & (j-1)^2 & \cdots & (j-1)^{2n-2} \\ 1 & j+1 & (j+1)^2 & \cdots & (j+1)^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n-1 & (n-1)^2 & \cdots & (n-1)^{2n-2} \\ 1 & n & n^2 & \cdots & n^{2n-2} \end{pmatrix}.$$

Again, a Vandermonde determinant remains. As before, we first consider the product of differences in relevant entries with similar signs and then opposite signs. However, we must here adjust the calculation employed to obtain (3.3) in order to account for the skipped $(n+j)$th term. We thus compute

$$\begin{aligned} \det(\mathbf{P}_{j\geq 1}) &= \frac{(-1)^{j+1}}{j^2}\left(\prod_{i=1}^{n} i^4\right)\left[\left(\frac{\prod_{i=1}^{n-1}(n-i)^{2i}}{(j-1)!(n-j)!}\right)\left(\frac{\prod_{i=1}^{n}(i+1)^i(2n-i+1)^i}{\frac{(n+j)!}{j!}(n+1)^n}\right)\right] \\ &= (-1)^{j+1}\frac{n^4(n+1)^n}{j(n-j)!(n+j)!}\prod_{i=1}^{n-1} i^4(n-i)^{2i}(i+1)^i(2n-i+1)^i. \end{aligned} \tag{3.4}$$

We can repeat the above procedure to find $\det(\mathbf{P}_{j\leq -1})$. Note that in the first step, we expand down column $n+1$ with sign correction $(-1)^n$ and then expand down column $n+j$ with sign correction $(-1)^{n+j}$ instead of $(-1)^{n+j-1}$ as before. The rest of the computation is analogous, and it is clear that $\det(\mathbf{P}_{j\leq -1})$ is the additive inverse of $\det(\mathbf{P}_{j\geq 1})$. That is,

$$\det(\mathbf{P}_{j\leq -1}) = (-1)^{j}\frac{n^4(n+1)^n}{j(n-j)!(n+j)!}\prod_{i=1}^{n-1} i^4(n-i)^{2i}(i+1)^i(2n-i+1)^i.$$

It now remains to find the numerator of $C_0$:

$$\det(\mathbf{P}_0) = \det\begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0 & 1^0 & \cdots & (n-1)^0 & n^0 \\ (-n)^1 & (-n+1)^1 & \cdots & (-1)^1 & 1 & 1^1 & \cdots & (n-1)^1 & n^1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 0 & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0 & 1^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

Expanding down column $n+1$, we obtain

$$\det(\mathbf{P}_0) = (-1)^{n+1}\det\begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 1^0 & \cdots & (n-1)^0 & n^0 \\ (-n)^2 & (-n+1)^2 & \cdots & (-1)^2 & 1^2 & \cdots & (n-1)^2 & n^2 \\ (-n)^3 & (-n+1)^3 & \cdots & (-1)^3 & 1^3 & \cdots & (n-1)^3 & n^3 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 1^{2n} & \cdots & (n-1)^{2n} & n^{2n} \end{pmatrix}.$$

As the remaining matrix is not directly mutable to a Vandermonde matrix, we perform elementary matrix operations and show that its determinant is zero. To begin, we subtract column $i$ from column $2n-i+1$ for each $i=1,\dots,n$ to obtain

$$\begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0 & \cdots & 0 & 0 \\ (-n)^2 & (-n+1)^2 & \cdots & (-1)^2 & 0 & \cdots & 0 & 0 \\ (-n)^3 & (-n+1)^3 & \cdots & (-1)^3 & 2(1^3) & \cdots & 2(n-1)^3 & 2n^3 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (-n)^{2n-1} & (-n+1)^{2n-1} & \cdots & (-1)^{2n-1} & 2(1^{2n-1}) & \cdots & 2(n-1)^{2n-1} & 2n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We now add $1/2$ of column $2n-i+1$ to column $i$ for each $i=1,\dots,n$ and then multiply odd rows by $1/2$. We have

$$\begin{pmatrix} (-n)^0 & (-n+1)^0 & \cdots & (-1)^0 & 0 & \cdots & 0 & 0 \\ (-n)^2 & (-n+1)^2 & \cdots & (-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1^3 & \cdots & (n-1)^3 & n^3 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \\ (-n)^{2n} & (-n+1)^{2n} & \cdots & (-1)^{2n} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Finally, we interchange rows to obtain

$$\begin{pmatrix} n^0 & (n-1)^0 & \cdots & (-1)^0 & 0 & \cdots & 0 & 0 \\ n^2 & (n-1)^2 & \cdots & (-1)^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ n^{2n} & (n-1)^{2n} & \cdots & (-1)^{2n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1^3 & \cdots & (n-1)^3 & n^3 \\ 0 & 0 & \cdots & 0 & 1^5 & \cdots & (n-1)^5 & n^5 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1^{2n-1} & \cdots & (n-1)^{2n-1} & n^{2n-1} \end{pmatrix}. \tag{3.5}$$

The determinant of (3.5) is zero. To see this, expand down column $2n$ of (3.5) and expand down the last column of every subsequent minor. Choose a nonzero entry of the last column of (3.5) and consider its minor. We expand down its last column and repeat this $n-1$ times. In the last step, choose a nonzero entry and consider its minor. Its last column is necessarily a column of zeros. Thus, the determinant of (3.5) is zero, and it follows that $\det(\mathbf{P}_0)=0$.

We now have a formula for each weight given $p$. Employing Equations (3.3) and (3.4) we have

$$\begin{aligned} C_{j\geq 1} &= \frac{\det(\mathbf{P}_{j\geq 1})}{\det(\mathbf{P})} \\ &= \frac{(-1)^{j+1}\frac{n^4(n+1)^n}{j(n-j)!(n+j)!}\prod_{i=1}^{n-1} i^4(n-i)^{2i}(i+1)^i(2n-i+1)^i}{n^2(n+1)^n\prod_{i=1}^{n-1} i^2(n-i)^{2i}(i+1)^i(2n-i+1)^i} \\ &= \frac{(-1)^{j+1}}{j}\frac{(n!)^2}{(n-j)!(n+j)!} \\ &= \frac{(-1)^{j+1}}{j}\frac{n!}{j!(n-j)!}\frac{j!n!}{(n+j)!} \end{aligned}$$

$$= (-1)^{j+1} \frac{\binom{n}{j}}{j \binom{n+j}{j}}.$$

Thus, given an order of error $p = 2n$, or equivalently a number of nodes $p + 1$, the corresponding symmetric difference formula takes the form

$$f'(x) = \frac{1}{h} \sum_{j=-p/2}^{p/2} C_j f(x + jh) + O(h^p),$$

where

$$C_j = \begin{cases} (-1)^j \dfrac{\binom{n}{|j|}}{|j| \binom{n+|j|}{|j|}}, & j = -n, -n+1, \ldots, -1 \\ 0, & j = 0 \\ (-1)^{j+1} \dfrac{\binom{n}{j}}{j \binom{n+j}{j}} & j = 1, 2, \ldots, n. \end{cases}$$

Now that we have a formula for the weights in a general central difference approximation, we consider their limits as $p \to \infty$.

**Lemma 3.1.** $\displaystyle\lim_{n\to\infty} \frac{\binom{n}{j}}{\binom{n+j}{j}} = 1.$

*Proof.* By induction. When $j = 1$, the statement is true since

$$\frac{\binom{n}{1}}{\binom{n+1}{1}} = \frac{n}{n+1} \to 1.$$

Let $k \in \mathbb{N}$ be given and suppose the statement is true for $j = k$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{\binom{n}{k+1}}{\binom{n+(k+1)}{k+1}} &= \lim_{n\to\infty} \left( \frac{n!}{(k+1)!(n-k-1)!} \right) \left( \frac{(k+1)!n!}{(n+k+1)!} \right) \\ &= \lim_{n\to\infty} \left( \frac{(n-k)}{(n+k+1)} \right) \left( \frac{n!n!}{(n-k)!(n+k)!} \right) \\ &= \lim_{n\to\infty} \frac{(n-k)}{(n+k+1)} \lim_{n\to\infty} \frac{\binom{n}{k}}{\binom{n+k}{k}} \\ &= 1^2 = 1. \end{aligned}$$

Thus $\lim_{n\to\infty} \frac{\binom{n}{j}}{\binom{n+j}{j}} = 1$ for all $j \in \mathbb{N}$. ∎

By the lemma we have

$$\lim_{n\to\infty} C_j = \begin{cases} \lim_{n\to\infty} (-1)^j \frac{\binom{n}{|j|}}{|j| \binom{n+|j|}{|j|}} = \frac{(-1)^j}{|j|}, & j = -n, -n+1, \ldots, -1 \\ \lim_{n\to\infty} 0 = 0, & j = 0 \\ \lim_{n\to\infty} (-1)^{j+1} \frac{\binom{n}{j}}{j \binom{n+j}{j}} = \frac{(-1)^{j+1}}{j}, & j = 1, 2, \ldots, n. \end{cases}$$

As recorded by Fornberg [5], the table below produces the weights and their limits for central difference formulas given their order of error $p$—or equivalently the number of nodes employed in the approximation—and position index $j$.

| **Weights** | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Order$(p)\backslash j$ | | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | |
| 2 | | | | | $-1/2$ | $0$ | $1/2$ | | | | |
| 4 | | | | $1/12$ | $-2/3$ | $0$ | $2/3$ | $-1/12$ | | | |
| 6 | | | $-1/60$ | $3/20$ | $-3/4$ | $0$ | $3/4$ | $-3/20$ | $1/60$ | | |
| 8 | | $1/280$ | $-4/105$ | $1/5$ | $-4/5$ | $0$ | $4/5$ | $-1/5$ | $4/105$ | $-1/280$ | |
| $\vdots$ | $\cdots$ | $1/4$ | $-1/3$ | $1/2$ | $-1$ | $0$ | $1$ | $-1/2$ | $1/3$ | $-1/4$ | $\cdots$ |

## 4. CONCLUSION

We derived an explicit formula for weights in symmetric difference approximations to the first order derivative by employing Cramer's rule and Vandermonde type determinants. Considering these formulas, we calculated the limits of the weights as the number of nodes in an approximation diverges to infinity. In further work, it would be natural to consider the approximation to any derivative, instead of just the first order derivative, as well as to consider skewed finite difference approximations in order to see what formulas we obtain in these cases.

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