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On the Axiomatic Systems of Steenrod Homology Theory of Compact Spaces

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Abstract

On the category of compact metric spaces an exact homology theory was defined and its relation to the Vietoris homology theory was studied by N. Steenrod [St]. In particular, the homomorphism from the Steenrod homology groups to the Vietoris homology groups was defined and it was shown that the kernel of the given homomorphism are homological groups, which was called weak homology groups [St], [Ed].

1 2

[1] [St] Steenrod, N. E. Regular cycles of compact metric spaces. Ann. of Math. (2) 41, (1940). 833–851

[2] [Ed] David A. Edwards and Harold M. Hastings. *Čech Theory: Its past, present,*

Abstract

The Steenrod homology theory on the category of compact metric pairs was axiomatically described by J.Milnor. In [Mil] the uniqueness theorem is proved using the Eilenberg-Steenrod axioms and as well as relative homeomorphism and clusres axioms. J. Milnor constructed the homology theory on the category Top^2_C of compact Hausdorff pairs and proved that on the given category it satisfies nine axioms - the Eilenberg-Steenrod, relative homeomorphis and cluster axioms (see theorem 5 in [Mil]). Besides, using the construction of weak homology theory, J.Milnor proved that constructed homology theory satisfies partial continuity property on the subcategory Top^2_{CM} (see theorem 4 in [Mil]) and the universal coefficient formula on the category Top^2_C (see Lemma 5 in [Mil]).

3

[3] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

Abstract

On the category of compact Hausdorff pairs, different axiomatic systems were proposed by N. Berikashvili [B1], [B2], H.Inasaridze and L. Mdzinarishvili [IM], L. Mdzinarishvili [M] and H.Inasaridze [I], but there was not studied any connection between them.

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[4] [B1] Berikashvili, N. A. Steenrod-Sitnikov homology theory on the category of compact spaces. (Russian) Doklady Akad. Nauk SSSR 22, No 2 (1980), 544-547.

[5] [B2 Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

[6] [IM] Inasaridze, Kh. N.; Mdzinarishvili, L. D. On the connection between continuity and exactness in homology theory. (Russian) Soobshch. Akad. Nauk Gruzin. SSR 99 (1980), no. 2, 317–320.

[7] [M] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

[8] [I] Inassaridze, Hvedri. On the Steenrod homology theory of compact spaces. Michigan Math. J. 38 (1991), no. 3, 323–338

Abstract

The paper studies this very problem. In particular, in the paper it is proved that any homology theory in Inasaridze sense is the homology theory in the Berikashvili sense, which itself is the homology theory in the Mdzinarishvili sense. On the other hand, it is shown that if a homology theory in the Mdzinarishvili sense is exact functor of the second argument, then it is the homology in the Inasaridze sense.

Homological functor

A sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in Z}$ of covariant functors $\bar{H}_n : Top^2_C \to Ab$ is called homological [M], [ES], if:

1_H) for each object $(X, A) \in Top^2_C$ and $n \in Z$ there exists a ∂-homomorphis

$$\partial : \bar{H}_n(X, A) \to \bar{H}_{n-1}(A) \tag{1}$$

$(\bar{H}_n(A) \equiv \bar{H}_{n-1}(A, \emptyset)$, wehere $\emptyset$ is the empty set);

2_H) the diagram

$$\begin{array}{ccc} \bar{H}_n(X, A) & \to & \bar{H}_{n-1}(A; G) \\ \downarrow f_* & & \downarrow (f_{|A})_* \\ \bar{H}_n(Y, B) & \to & \bar{H}_{n-1}(B; G) \end{array} . \tag{2}$$

is commutative for each continuous mapping $f : (X, A) \to (Y, B)$.
9 10

[9] [M] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

[10] [ES] Eilenberg, Samuel; Steenrod, Norman. Foundations of algebraic topology. Princeton University Press, Princeton, New Jersey, 1952

Homology theory of Eilenberg-Steenrod sense

A homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in Z}$ defined on the category Top^2_C is called homology theory in the Eilenberg-Steenrod sense if it satisfies homotopy, excision, exactness and dimension axioms [ES]. It is known that up to an isomorphism such a homology theory is unique on the subcategory $\mathscr{P}ol^2$ of compact polyhedral pairs [ES] and it is denoted by $H_* = \{H_n\}_{n \in Z}$, but it is not unique on the category Top^2_C of compact Hausdorff pairs.

11

[11] [ES] Eilenberg, Samuel; Steenrod, Norman. Foundations of algebraic topology. Princeton University Press, Princeton, New Jersey, 1952

Milnor axioms

The Steendor homology theory on the category of compact metric pairs was first axiomatically described by J. Milnor [Mil]. He proved the uniqueness theorem using the Eilenberg-Steenrod axioms and additionally two more - relative homeomorphism and clustes axioms:

RH (relative homeomorphism axiom): if $f : (X, A) \to (Y, B)$ is a map in Top^2_{CM} which carries $X - A$ homeomorphically onto $Y - B$, then

$$f_* : \bar{H}_n(X, A) \to \bar{H}_n(Y, B) \tag{3}$$

is an isomorphism.

CL(cluster axiom): if X is the union of countable many compact subsets $X_1, X_2, \ldots$ which intersect pairwise at a single point $*$, and which have diameters tending to zero, then $\bar{H}_n(X, *)$ is naturally isomorphic to the direct product of the groups $\bar{H}_n(X_i, *)$.

12

[12] [ES] Eilenberg, Samuel; Steenrod, Norman. Foundations of algebraic topology. Princeton University Press, Princeton, New Jersey, 1952

Theorem (see theorem 3 in [Mil]):

Given two homology theories $\bar{H}^M_*$ and $\bar{H}_*$ on the category Top^2_{CM}, both satisfying the nine axioms (the Eilenberg-Steenrod, relative homeomorpism and cluster axioms), any coefficient isomorpism $\bar{H}^M_0(*) \approx \bar{H}_0(*) \approx G$ extends uniquely to an equivalence between the two homology theories.

13

[13] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

Milnor homology

In [Mil] J. Milnor constructed the homology theory $\bar{H}_*^M$ on the category Top_C^2 of compact Hausdorff pairs and gave its several properties. In particular [Mil] the following is proved:

Theorem (see theorem 5 in [Mil]):

The homology theory $\bar{H}_*^M$, defined on the category Top_C^2 of compact Hausdorff pairs, satisfies the nine axioms (the Eilenberg-Steenrod axioms as well as relative homeomorphism and clusre axioms).

14

[14] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

Theorem (see theorem 4 in [Mil]):

Let $\bar{H}_*^M$ be a homology theory satisfying the nine axioms (the Eilenberg-Steenrod axioms as well as relative homeomorphism and cluster axioms), and let $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$ be an inverse system of compact metric spaces with inverse limit X. Then there is an exact sequence

$$0 \to \varprojlim{}^1 \bar{H}_{n+1}^M(X_i) \to \bar{H}_n^M(X) \to \varprojlim \bar{H}_n^M(X_i) \to 0 \tag{4}$$

for each integer n.

15

[15] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

Universal coefficient formula

Theorema(see lemma 5 in [Mil]):

The homology theory $\bar{H}_*^M$ is related to the Čech cohomology theory by a split exact sequence

$$0 \to Ext(\check{H}^{n+1}(X, A); G) \to \bar{H}_n^M(X, A; G) \to Hom(\check{H}^n(X, A); G) \to 0.$$

16

[16] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

Axiomatic system on the category of compact Hausdorff pairs

As we see the uniqueness theorem was proved only on the category Top^2_{CM} of compact metric pairs [Mil] and therefore, the problem was open for the category Top^2_C.

[17][Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

[18][B1] Berikashvili, N. A. Steenrod-Sitnikov homology theory on the category of compact spaces. (Russian) Doklady Akad. Nauk SSSR 22, No 2 (1980), 544-547

[19][B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Axiomatic system on the category of compact Hausdorff pairs

As we see the uniqueness theorem was proved only on the category Top^2_{CM} of compact metric pairs [Mil] and therefore, the problem was open for the category Top^2_C.

The axiomatic description of the Steenrod homology theory on the category Top^2_C of compact Hausdorff pairs was given by Berikashvili [B1], [B2].

17 18 19

[17] [Mil] Milnor, John. On the Steenrod homology theory. Mimeographed Note, Princeton, 1960

[18] [B1] Berikashvili, N. A. Steenrod-Sitnikov homology theory on the category of compact spaces. (Russian) Doklady Akad. Nauk SSSR 22, No 2 (1980), 544-547

[19] [B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Berikashvili axioms

Theorem:

If a homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n\in Z}$ defined on the category Top_C^2 of compact Hausdorff pairs satisfies the Eilenberg-Steenrod axioms and the following A, B and C axioms, then it is naturally isomorphic to the Chogoshvili homology theory.

Berikashvili axiomatic system

Axiom A:

The projection $(X, A) \to (X/A, *)$ induces an isomorphism $\bar{H}_n(X, A) \approx \bar{H}_*(X/A, *)$.

Berikashvili axiomatic system

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The projection $(X,A) \to (X/A,*)$ induces an isomorphism $\bar{H}_n(X,A) \approx \bar{H}_*(X/A,*)$.

Axiom B:

For the inverse spectrum of pairs $\{(S^n_\alpha,*),\pi_{\alpha\beta}\}$, where S^n_α is a finite cluster of n-dimension spheres and $\pi_{\alpha\beta}$ maps each sphere of the cluster either to the fixed piont or homeomorphically to a cluster sphere, there holds the equality

$$\bar{H}_*(\varprojlim\{(S^n_\alpha,*),\pi^\alpha_\beta\}) \approx \varprojlim\{\bar{H}_*(S^n_\alpha,*),(\pi_{\alpha\beta})_*\}, \qquad n \in Z. \tag{5}$$

Berikashvili axioms

Axiom C:

The natural homomorphism

$$\varinjlim \bar{H}_n(|\mathscr{N}(X)|_p) \to \bar{H}_n(X), \qquad n \in Z, \tag{6}$$

induced by the mapping $\omega : |\mathscr{N}(X)| \to X$, were $|\mathscr{N}(X)|$ is the limit of the inverse spectrum of realizations of complexes of the spectrum $\mathscr{N}(X) = \{\mathscr{N}_\lambda(X), \pi_{\lambda\mu}\}$ ($\mathscr{N}_\lambda(X)$ is the nerve of a finite closed covering obtained from a finite partitioning of X [**?**]) and $|\mathscr{N}_\lambda(X)|_p = \varprojlim\{|\mathscr{N}_\lambda(X)|^p, \pi_{\lambda\mu}\} \subset |\mathscr{N}(X)|$ (K^p denotes the p-skeleton of the complex K), is an isomorphisms.

In [B2] Berikashvili proposed new C_1 and C_2 axioms and the universal coefficient formula as one more new axiom as well:

[20][B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Berikashvili axioms

In [B2] Berikashvili proposed new C_1 and C_2 axioms and the universal coefficient formula as one more new axiom as well:

Axiom C_1:

If a continuous map $f : X \to Y$ induces an isomorphism $f^* : \check{H}^n(Y; Z) \to \check{H}^n(X; Z)$ for $n < p$, then for $n < p - 1$ a homomorphism $f_* : \bar{H}_n(X; Z) \to \bar{H}_n(Y; Z)$ is an isomorphism as well.

[20] [B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

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Axiom C_2:

If a continuous map $f : X \to Y$ is surjective and $\bar{H}_n(f^{-1}(y), *) = 0$ for each $y \in Y$ and $n < p$, then for $n < p$ a homomorphism $f_* : \bar{H}_n(X; Z) \to \bar{H}_n(Y; Z)$ is an isomorphism.

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[20] [B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Berikashvili axioms

Axiom D:

For each pairs (X, A) there exists a functorial exact sequence

$$0 \to Ext(\check{H}^{n+1}(X, A); G) \to \bar{H}_n(X, A; G) \to Hom(\check{H}^n(X, A); G) \to 0, \tag{7}$$

where $G = \bar{H}_0(*)$.

Uniqueness theorem

Theorem (see theorem 4.4 [B2]):

The Steenrod-Sitnikov homology theory defined on the category Top^2_C of compact Hausdorff pairs with coefficients any module G uniquely is characterized by the Eilenberg-Steenrod axioms with one of the following 4 systems of axioms: 1) A, B, C axioms; 2)axiom D; 3) A, B, C_1 axioms; 4) A, B, C_2 axioms for the finite generated abelian group.

[21]

[21] [B2] Berikashvili, N. A. Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37

Inasaridze-Mdzinarishvili axiomatic system

In [IM] H. Inasaridze and L. Mdzinarishvili gave one more different axiomatic system using the modified form of the continuity axiom, as it is called, partial continuity axiom:

[22] [IM] Inasaridze, Kh. N.; Mdzinarishvili, L. D. On the connection between continuity and exactness in homology theory. (Russian) Soobshch. Akad. Nauk Gruzin. SSR 99 (1980), no. 2, 317–320

In [IM] H. Inasaridze and L. Mdzinarishvili gave one more different axiomatic system using the modified form of the continuity axiom, as it is called, partial continuity axiom:

PC (partial continuity axiom):

Let (X, A) be the inverse limit of inverse system $\{(X_\lambda, A_\lambda), p_{\lambda,\lambda'}, \Lambda\}$ of compact polyhedra, then for each integer n there is a functorial exact sequence

$$0 \to \varprojlim{}^1 H_{n+1}(X_\lambda, A_\lambda) \to \bar{H}_n(X, A) \to \varprojlim H_n(X_\lambda, A_\lambda) \to 0. \tag{8}$$

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[22] [IM] Inasaridze, Kh. N.; Mdzinarishvili, L. D. On the connection between continuity and exactness in homology theory. (Russian) Soobshch. Akad. Nauk Gruzin. SSR 99 (1980), no. 2, 317–320

Extension of homology theory

Extension of homology theory

Homological sequence $\bar{H}_*$ defined on the category Top^2_C is called extension of homology theory H_* (which is unique up to an isomorphism) defined on the category $\mathscr{P}ol^2$, if on the subcategory $\mathscr{P}ol^2$ it is equivalent to H_* [M].

23

[23] [M]Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

Nontrivial extension of homology theory

Nontrivial exension

The homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in Z}$ defined on the category Top^2_C is called a nontrivial external extension of the homology theory $H_* = \{H_n\}_{n \in Z}$ defined on the category $\mathscr{P}ol^2$, if the following 1_{NT}, 2_{NT}, 3_{NT} and 4_{NT} conditions are fulfilled:

1_{NT}

1_{NT}) $\bar{H}_*$ is an extension of the homology theory H_*;

Nontrivial extension of homology theory

2_{NT}

2_{NT}) the exact sequence

$$0 \to \varprojlim{}^1 H_{n+1}(X_\lambda, A_\lambda) \to \bar{H}_n(X, A) \to \varprojlim H_n(X_\lambda, A_\lambda) \to 0 \qquad (9)$$

holds for any object $(X, A) \in Top^2_C$, any inverse system $\{(X_\lambda, A_\lambda), p_{\lambda,\lambda'}, \Lambda\}$ of compact polyhedra such that $(X, A) = \varprojlim\{(X_\lambda, A_\lambda), p_{\lambda,\lambda'}, \Lambda\}$ and $n \in Z$;

Nontrivial extension of homology theory

3_{NT}

3_{NT}) The commutative diagram

$$\begin{array}{ccc} \varprojlim^1 \bar{H}_{n+1}(X_\lambda, A_\lambda) & \rightarrow & \bar{H}_n(X, A) \\ \downarrow \varprojlim^1 \tilde{f}_* & & \downarrow f_* \\ \varprojlim^1 \bar{H}_{n+1}(Y_\gamma, B_\gamma) & \rightarrow & \bar{H}_n(Y, B) \end{array}, \tag{10}$$

holds for any continuous mapping $f : (X, A) \rightarrow (Y, B)$ from Top^2_C, where $\tilde{f}_* : \{H_n(X_\lambda, A_\lambda), (p_{\lambda,\lambda'})_*, \Lambda\} \rightarrow \{H_n(Y_\gamma, B_\gamma), (q_{\gamma,\gamma'})_*, \Gamma\}$ is mapping of the inverse systems;

4_{NT}

4_{NT}) $\bar{H}_*$ satisfies the exactness axiom.

Nontrivial extension

Theorem (see theorem 1.2 in [M]):

if $\bar{H}_*$ is a nontrivial external extension of the homology theory H_* to the category Top^2_C, then it is a theory in the Eilenberg-Steenrod sense.

Theorem (see theorem 1.3 in [M]):

if $\bar{H}_*$ is a nontrivial external extension of the homology theory H_*, defined on the category Top^2_C, then it is a homology theory in the Milnor sense.

Corollary:

On the category Top^2_{CM} of compact metric pairs the $\bar{H}_*$ is a nontrivial external extension if and only if it is the homology theory in the Milnor sense.

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[24] [M] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

Nontrivial extension

Theorem (see theorem 1.5 in [M]):

if $\bar{H}_*$ is a nontrivial external extension of the homology theory H_* defined on the category Top^2_C, then it is a homology theory in the Berikashvili sense.

Corollary (see corollary 1.4 in [M]):

Any nontrivial external extension of the homology theory H_* defined on the category Top^2_C is isomorphic to the Steenrod homology theory.

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[25] [M] Mdzinarishvili, L. D. On homology extensions. Glas. Mat. Ser. III 21(41) (1986), no. 2, 455–482

In the paper [I] H. Inasaridze described exact bifunctor homology theory using the continuity property for the infinitely divisible abelian groups. In particular, [**?**] the following is proved:

Theorem (see theorem 1 in [I]):

There exists one and only one exact bifunctor homology theory on the category Top^2_C of compact Hausdorff pairs with coefficients in the category of abelian groups (up to natural equivalence) which satisfies the axioms of homotopy, excision, dimension, and continuity for every infinitely divisible group.

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[26] [I] Inassaridze, Hvedri. On the Steenrod homology theory of compact spaces. Michigan Math. J. 38 (1991), no. 3, 323–338

Different axiomatic system of Steenrod homology theory

Therefore, for the Steenrod homology theory there are different axiomatic systems, but it is not known what the relation between them is and which one is the minimal one in the axiomatic sense. In the second part we will study this problem.

Homology theories

In this paper we will say that a homological sequence $\bar{H}_*$ defined on the category Top^2_C of compact Hausdorff pairs is:
1) a homology theory in the **Berikashvili sense** if it satisfies the axioms of homotopy, excision, exactness, dimension and axiom D (The Universal Coefficient Formula);
2) a homology theory in the **Mdzinarishvili sense** if it is a nontrivial external extension;
3) a homology theory in the **Inasaridze sense** if it is the exact functor of the second argument and satisfies the axioms of homotopy, excision, exactness, dimension and continuity for every infinitely divisible group.

Main Theorems

Theorem 1:

If $\bar{H}_*$ is a homology theory in the Inasaridze sense, defined on the category Top^2_C of compact Hausdorff pairs, then it is the homology theory in the Berikashvili sense.

Theorem 2:

If $\bar{H}_*$ is a homology theory in the Berikashvili sense, defined on the category Top^2_C of compact Hausdorff pairs, then it is a homology theory in the Mdzinarishvili sense.

Theorem 3:

If a homology theory $\bar{H}_*$ in the Mdzinarishvili sense, defined on the category Top^2_C of compact Hausdorff pairs, is an exact functor of the second argument, then it is a homology theory in the Inasaridze sense.

Thank You!