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Non-Normality Points of β X\X

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NON-NORMALITY POINTS OF $\beta X \setminus X$

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Abstract

We seek conditions implying that $(\beta X\backslash X)\backslash\{y\}$ is not normal. Our main theorem: Assume GCH and all uniform ultrafilters are regular. If X is a locally compact metrizable space without isolated points, then $(\beta X\backslash X)\backslash\{y\}$ is not normal for all $y \in \beta X\backslash X$. In preparing to prove this theorem, we generalize the notions "uniform", "regular", and "good" from set ultrafilters to z-ultrafilters. We discuss non-normality points of the product of a discrete space and the real line. We topologically embed a nonstandard real line into the remainder of this product space.

keywords non-normality point, butterfly point, regular z-ultrafilter

1. Introduction

Theorem 1.1. *Assume GCH and that all uniform ultrafilters are regular. If X is a crowded, locally compact metrizable space, then both $\beta X\backslash\{y\}$ and $(\beta X\backslash X)\backslash\{y\}$ are not normal for all $y \in \beta X\backslash X$.*

Our proof uses ideas from [1] in the context of [15] and [19].

Theorem 1.2. (Beslagic and van Douwen [1]) *Assume GCH. If X is a discrete space, then both $(\beta X\backslash X)\backslash\{y\}$ and $\beta X\backslash\{y\}$ are not normal for all $y \in \beta X\backslash X$.*

Theorem 1.3. (Logunov [15] and Terasawa [19], independently) *If X is a crowded metrizable space space, then $\beta X\backslash\{y\}$ is not normal for all $y \in \beta X\backslash X$.*

Let us compare Theorem 1.1 and Theorem 1.3. The latter uses no extra axioms and applies to all crowded metrizable spaces. The former has the additional conclusion $(\beta X\backslash X)\backslash\{y\}$ is not normal.

We explain some of the topological terminology we have used freely already. In proving Theorem 1.1, it is natural to generalize the notion "regular" from set ultrafilters to z-ultrafilters. Having made that generalization, we decided to generalize "uniform" and "good" as well. A recent overview of ultrafilters is [11]. The reference [3] is quite detailed, but is sometimes out-of-date.

Before considering non-normality points of $\beta X \backslash X$ for general metrizable X, we discuss points of βX when X is the product of a discrete space and the real line. The product structure of X helps us visualize the closed sets used in various proofs that $\beta X \backslash \{y\}$ and $(\beta X \backslash X) \backslash \{y\}$ are not normal. The case of a countably complete free ultrafilter on the discrete factor is an interesting contrast to the case of a countably incomplete on the discrete factor. We also topologically embed a nonstandard real line into βX.

We show that a metrizable space X has a π-base $\mathcal{B} = \{\mathcal{B}_n : n \in \omega\}$ such that each $\mathcal{B}_n$ is locally finite pairwise disjoint and every open cover of X is densely refined by a locally finite pairwise disjoint subset of $\mathcal{B}$. Define Ξ to be the family

$$\{\xi \subset \mathcal{B} : \xi \text{ is locally finite disjoint and } \mathrm{cl} \bigcup \xi = X\}.$$

For y a free z-ultrafilter on X, we define a partial order $<_y$ on Ξ, and we discuss cofinal subsets of $(\Xi, <_y)$. From such a cofinal set, we define a nested sequence $\{H_\gamma : \gamma < \theta\}$ of closed subsets of $\beta X \backslash X$ whose intersection is $\{y\}$. In two cases, X is locally compact and X is κ^ω-like, we define sets $\mathcal{L}$ that split the H's. The goal of all this machinery is to embed the nonnormal space $NU(\theta)$ (the nonuniform ultrafilters on θ) into $(\beta X \backslash X) \backslash \{y\}$ as a closed subset.

2. Topological Spaces

All spaces X are Tychonoff, and hence have a Stone-Čech compactification βX. We consider a point of βX to be a z-ultrafilter on X. We consider X to be a subspace of βX by identifying a point x of X with the z-ultrafilter $\hat{x}$, the collection of all zero sets of X of which x is an element. The basic open sets of βX are

$$B(U) = \{y \in \beta X : (\exists Z \in y) Z \subseteq U\}$$

where U varies over all open sets of X.

For a space X, we use $C^*(X)$ to denote the ring of bounded, continuous, functions from X to $\mathbb{R}$. We say that X is C^**-embedded* in Y if every f in $C^*(X)$ extends to an F in $C^*(Y)$. The only (up to a homeomorphism fixing X) compact space in which X is dense and C^*-embedded is βX. In this case we denote by βf the unique extension of f.

A space is called *crowded* if it has no isolated points. We use the letters κ, λ, θ, etc. to denote infinite cardinals and the discrete spaces of that cardinality.

Recall that in metrizable spaces many of the global cardinal functions are the same ([6], Theorem 4.1.1.5). In particular, $w(X)$ the weight of X equals $L(X)$ the Lindelöf number of X equals $e(X)$ the extent of X.

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \omega$$
$$L(X) = \min\{\kappa : \text{ every open cover of } X \text{ has a subcover of size } \kappa\} + \omega$$
$$e(X) = \sup\{|E| : E \text{ is closed discrete in } X\} + \omega$$

We say that the extent is attained in X if there is a closed discrete subset E of X such that $|E| = e(X)$. For example, if X is the unit interval, $e(X)$ is not attained. Note that $e(X) = \omega$ because for every $n \in \omega$, there is a closed discrete set of size n, but there is no infinite closed discrete subset of X because X is compact. If X is metrizable, $e(X) > \omega$, and the extent is not attained, then X must have the special form described in the next lemma.

Lemma 2.1. (Fitzpatrick, Gruenhage, Ott [8]) *Let κ be an uncountable cardinal and let X be a metrizable space in which $e(X) = \kappa$ is not attained. Let K be the set of points x of X such that every neighborhood of x has extent κ. Then*

(1) *κ is a singular cardinal of cofinality ω.*
(2) *K is a nonempty, compact, nowhere dense subset of X.*
(3) *If U is an open subset of X such that $\operatorname{cl} U \cap K = \emptyset$, then $e(U) < \kappa$.*

Let κ be an infinite cardinal. We say a metrizable space X is *κ^ω-like* if X is nowhere locally compact and $w(U) = \kappa$ for every nonempty open set U in X. For example, the irrationals and the rationals are ω^ω-like, but the real line is not. We denote the product of ω copies of the discrete space of cardinality κ by κ^ω. Of course, κ^ω is κ^ω-like.

Corollary 2.2. *For an infinite cardinal κ, every open subset of an κ^ω-like metrizable space has extent attained.*

Proof. Let V be a nonempty open subset of X. Every neighborhood of every point in V has weight, and therefore extent, κ. If $\kappa = \omega$, then since X is nowhere locally compact, V has a closed discrete subset of size κ. If $\kappa > \omega$ we apply Lemma 2.1 to V. If the extent of V was not attained, then by (2), $K = V$ is a compact nowhere dense subset of itself, a contradiction. $\square$

Lemma 2.3. *Let X be a κ^ω-like metrizable space and let Z be a subset of X with $w(Z) = \lambda < \kappa$. There is a λ^ω-like closed subset Y of X containing Z.*

Proof. Set $Z_1 = Z$. Given Z_n with $L(Z_n) = \lambda$, choose $\mathcal{V}_n \in [\tau(X)]^\lambda$ such that $Z_n \subset \mathcal{V}_n^*$ and $\operatorname{diam} V < 1/n$ for all $V \in \mathcal{V}$. Choose Z_{n+1} to satisfy $Z_n \subseteq Z_{n+1}$, $|Z_{n+1} \setminus Z_n| \leq \lambda$ (hence $L(Z_{n+1}) = \lambda$), and for all $V \in \mathcal{V}_n$ there is $E \in [V \cap Z_{n+1}]^\lambda$ which closed discrete (hence $w(V \cap Z_{n+1}) = \lambda$). Set $Y_0 = \bigcup_{n \in \mathbb{N}} Z_n$; note that $w(Y) \leq \lambda$ because $\{V \cap Y_0 : (\exists n)\, V \in \mathcal{V}_n\}$ is a base for Y_0.

Let $y \in W$ open in Y_0. There are $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $y \in V \cap Z^{n+1} \subseteq W$. Then $w(W) \geq w(V \cap Z_{n+1}) = \lambda$. Finally, set $Y = \operatorname{cl} Y_0$. $\square$

Logunov [15], Terasawa [19], and Beslagic and VanDouwen [1] describe their results in terms of non-normality points and butterfly points. We say that y is a *non-normality point* of Y if $Y \setminus \{y\}$ is not normal. We say that y is a *butterfly point* of Y iff there are closed subsets H_0 and H_1 of Y such that $H_0 \cap H_1 = \{y\}$ and y is not isolated in either H_0 or H_1. Expressed differently, y is a *butterfly point* of Y iff $\{y\} = \operatorname{cl}(H_0 \setminus \{y\}) \cap \operatorname{cl}(H_1 \setminus \{y\}$. When we use these terms in this paper, Y will be of the form βX or $\beta X \setminus X$ and y will be a point of the remainder $\beta X \setminus X$.

Lemma 2.4. *(a) Let X be a locally compact space. A non-normality point of $\beta X \setminus X$ is a non-normality point of βX. (b) Let X be a space. If y is a butterfly point of $\beta X \setminus X$, then y is a non-normality point of βX*

Proof. (a) X is locally compact iff $\beta X \setminus X$ is closed in βX.

(b) Let H_0 and H_1 be sets showing that y is a butterfly point. Note that H_0 and H_1 are disjoint closed subsets of $\beta X \setminus \{y\}$. If $\beta X \setminus \{y\}$ were normal, there would be a continuous function $f : \beta X \setminus \{y\} \to [0,1]$ such that $H_0 \subset f^{\leftarrow}\{0\}$ and $f^{\rightarrow} H_1 = \{1\}$. Because X is C^*-embedded in βX, the continuous function $f|X$ has a continuous extension $F : \beta X \to [0,1]$. By continuity $0 = F(y) = 1$. Contradiction. $\square$

We can use a subspace of the Tychonoff plank to distinguish non-normality points from butterfly points.

Example 2.5. Let X be $\omega_1 \times (\omega + 1)$. Then βX is homeomorphic to $(\omega_1 + 1) \times (\omega + 1)$. The sets $H_0 = \{(\omega_1, n) : n \in N$ and n is even $\}$ and $H_1 = \{(\omega_1, n) : n \in N$ and n is odd $\}$ show that $(\omega_1 + 1, \omega + 1)$ is a butterfly point of both βX and $\beta X \setminus X$. In contrast, $(\omega_1 + 1, \omega + 1)$ is a non-normality point of βX, but not of $\beta X \setminus X$. Observe that H_0 and H_1 show that βX is not normal in an indirect way. There are disjoint open subsets U_0 and U_1 of βX containing H_0 and H_1, respectively. However, $\mathrm{cl}_{\beta X} U_0$ and $\mathrm{cl}_{\beta X} U_1$ both meet the long bottom edge in a large set.

3. Uniform z-Ultrafilters

Definition 3.1. Let p be an ultrafilter on a set I. We say that p is *uniform* if $|E| = |I|$ for all $E \in p$. Let y be a z-ultrafilter on a metrizable space X. We say that y is *uniform* if $w(Z) = w(X)$ for all $Z \in y$.

It is easy to see that on an infinite sets there are uniform ultrafilters, and on an uncountable set there are free nonuniform ultrafilters. In the proof of Theorem 1.1, we will use the fact that if $\theta = \kappa^+$, then $NU(\theta)$ is not normal.

Lemma 3.2. *Let $NU(\theta)$ denote the subspace of $\beta\theta$ of non-uniform ultrafilters. That is, $NU(\theta) = \{y \in \beta\theta : (\exists Z \in Y)|Z| < \theta\}$.*

(1) [16] *If θ is regular and not a strong limit cardinal – in particular, if $\theta = \kappa^+$, then $NU(\theta)$ is not normal.*
(2) [16] *If θ is singular then $NU(\theta)$ is not normal.*
(3) [14] *The space $NU(\theta)$ is normal if and only if θ is weakly compact.*

In contrast to uncountable sets, there are metrizable spaces of uncountable weight with no uniform z-ultrafilters.

Lemma 3.3. *Suppose y is a free z-ultrafilter on a metrizable space X. If $Z \in y$ is such that $w(Z)$ is minimum, then $e(Z)$ is attained.*

Proof. Let y be a free z-ultrafilter on a metrizable space X and let $Z \in y$ be such that $\kappa = w(Z)$ is minimum. Suppose $e(Z) = \kappa$ is not attained. If $e(Z) = \omega$ and $e(Z)$ is not attained, then Z is compact and y is not free. So we may assume that $e(Z) > \omega$. Let K be the compact set from Lemma 2.1 (2). Since y is free, $K \notin y$. Let $Z' \in y$ be such that $Z' \cap K = \emptyset$ and $Z' \subset Z$. Since Z is normal, there is an open set U containing Z' such that $\mathrm{cl}\, U \cap K = \emptyset$. By Lemma 2.1

(3), $e(U) = w(U) < \kappa$. Hence $w(Z') < \kappa$, which is a contradiction to $w(Z)$ being minimum.

□

Corollary 3.4. *A metrizable space with extent not attained has no uniform z-ultrafilters.*

4. Regular z-Ultrafilters

For X a space, let $C(X)$ be the family of continuous functions from X to $\mathbb{R}$. $C(X)$ is a partially ordered commutative ring. If M is a maximal ideal, then the quotient ring $C(X)/M$ is a totally ordered field. In fact $C(X)/M$ is real-closed – meaning that every positive element of $C(X)/M$ is a square and every polynomial (in one indeterminant) with coefficients from $C(X)/M$ of odd degree has a zero in $C(X)/M$. Every $y \in \beta X$ determines a maximal ideal, $M_y = \{f \in C(X) : \beta f(y) = 0\}$. Equivalently, $f \in M_y$ iff $Z(f) = \{x \in X : f(x) = 0\} \in y$.

The notion of regular ultrafilter appears implictly in papers from the mid 1950's, for example [7].

Theorem 4.1. *(*[10] Section 12.7*) Let κ be an infinite cardinal. There is a maximal ideal M in $C(\kappa)$ such that $|C(\kappa)/M| > \kappa$. In fact, no set of power at most κ is cofinal in the ordered field $C(\kappa)/M$. If $2^\kappa = \kappa^+$, then $\mathrm{cf}(C(\kappa)/M) = |C(\kappa)/M| = 2^\kappa$.*

Proof. Because κ is infinite, there is a bijection $\alpha \mapsto a_\alpha$ from κ to $[\kappa]^{<\omega}$. For each $\alpha \in \kappa$, set $Z_\alpha = \{\gamma \in \kappa : \alpha \in a_\gamma\}$. By construction, $\{Z_\alpha : \alpha \in \kappa\}$ has the finite intersection property – if $a = a_\gamma \in [\kappa]^{<\omega}$, then $\gamma \in \bigcap\{Z_\alpha : \alpha \in a\}$. Extend $\{Z_\alpha : \alpha \in a\}$ to a z-ultrafilter y, and set $M = \{f \in C(\kappa) : f^{\leftarrow}\{0\} \in y\}$.

Given $B = \{g_\alpha : \alpha < \kappa\} \subset C(\kappa)$, define

$$f(\gamma) = 1 + \max\{g_\alpha(\gamma) : \alpha \in a_\gamma\}.$$

The maximum exists because α_γ is finite, and f is continuous because κ is discrete. Let $g_\alpha \in B$ be arbitrary. For every $\gamma \in Z_\alpha$

$$g_\alpha(\gamma) \leq \max\{g_{\alpha'} : \alpha' \in a_\gamma\} < f(\gamma).$$

□

Definition 4.2. Let y be a z-ultrafilter on a space X. We say that y is *κ-regular* if there is a subset $\mathcal{Z}$ of y such that $\mathcal{Z}$ is locally finite and $|\mathcal{Z}| = \kappa$. $\mathcal{Z}$ is called a *regularizing* family. We say that y is *regular* if y is $w(X)$-regular.

Proposition 4.3. *If y is a countably incomplete z-ultrafilter on a space X, then y is ω-regular.*

Proof. Let $\mathcal{Z} = \{Z_n : n \in \omega\}$ be a nested decreasing sequence from y with empty intersection. Then $\mathcal{Z}$ is locally finite and $|\mathcal{Z}| = \omega$. □

Proposition 4.4. *Let y be a z-ultrafilter on a space X. The cardinality of locally finite subfamilies of y is bounded above by $\min\{w(\tilde{Z}) : \tilde{Z} \in y\}$ Hence, if y is a regular z-ultrafilter, then y is a uniform z-ultrafilter.*

Proof. Let $\mathcal{B}$ be a base for $\tilde{Z} \in y$, and $\mathcal{Z}$ a locally finite subfamily of y. There is $\mathcal{B}' \subseteq \mathcal{B}$ such that each $B \in \mathcal{B}'$ meets at most finitely many $Z \in \mathcal{Z}$. For each $Z \in \mathcal{Z}$, there is $B(Z) \in \mathcal{B}'$ such that $Z \cap B(Z) \neq \varnothing$ because $Z \cap \tilde{Z} \neq \varnothing$. Because the map $Z \mapsto B(Z)$ is finite-to-one, $|\mathcal{Z}| \leq |\mathcal{B}|$. □

Theorem 4.5. *Let y be a κ-regular z-ultrafilter on a paracompact space X. For every $H \in [C(X)]^{\leq\kappa}$, there are $f \in C(X)$ and $\{Z_h : h \in H\} \subset y$ satisfying $h(x) < f(x)$ for all $h \in H$ and $x \in Z_h$. That is, the cofinality of $C(X)/M_y$ is greater than κ.*

Proof. Let $\mathcal{Z} = \{Z_h : h \in H\}$ be a regularizing family of y. Because $\mathcal{Z}$ is locally finite, for every $x \in X$ there are $b(x) \in [H]^{<\omega}$ and $W(x)$ an open set containing x such that $\{h \in H : W(x) \cap Z_h \neq \varnothing\} = b(x)$. Set $W_b = \bigcup\{W(x) : b(x) = b\}$, then $\mathcal{W} = \{W_b : b \in [H]^{<\omega}\}$ is an open cover of X. For each b, define $f_b : W_b \to \mathbb{R}$ by $f_b(x) = \max\{h(x) : h \in b\} + 1$. Note that f_b is bounded and continuous.

Because X is paracompact, there is $\{\varphi_b : b \in [H]^{<\omega}\}$, a locally finite partition of unity subordinate to $\mathcal{W}$ ([6], Theorem 5.1.9.) In more detail,
(1) each φ_b is a continuous function from X to $[0,1]$,
(2) $\varphi_b^{\leftarrow}(0,1] \subseteq W_b$ for all b,
(3) there is an open cover $\mathcal{V}$ of X such that each $V \in \mathcal{V}$ meets only finitely many $\varphi_b^{\leftarrow}(0,1]$, and
(4) $\sum \varphi_b(x) = 1$ for all $x \in X$.

For all $x \in X$, set $f(x) = \sum \varphi_b(x) \cdot f_b(x)$. Note that f is continuous because for each $V \in \mathcal{V}$, $f|V$ is a finite sum of continuous functions.

Let $h \in H$ be arbitrary. Observe that if $x \in Z_h$ and $\varphi_b(x) > 0$, then $x \in Z_h \cap W_b \neq \varnothing$, hence $h \in b$. It follows that for such x and b

$$h(x) \leq \max\{g(x) : g \in b\} < \max\{g(x) : g \in b\} + 1 = f_b(x).$$

Now fix x and h, but let b vary.

$$h(x) = 1 \cdot h(x) = \sum \varphi_b(x) \cdot h(x) < \sum \varphi_b(x) \cdot f_b(x) = f(x).$$

□

5. GOOD z-ULTRAFILTERS

In the proof of Theorem 4.5, the definition $f(x) = \max\{h(x), k(x)\} + 1$ yields $h(x) < f(x)$ for all $x \in X$. We assigned the index h to $Z \in \mathcal{Z}$ without considering the values of h on Z. In Theorem 5.4, we want a function f satisfying $l(x) < f(x) < h(x)$. The definition $f(x) = \frac{1}{2}(l(x) + h(x))$ yields the desired inequalities only when $l(x) < h(x)$. We must be careful that this inequality holds for $x \in Z_h \cap Z_l$. This line of reasoning leads to good ultrafilters.

Definition 5.1. We say that a linearly ordered set $(R, <)$ is an η_α-set if whenever K is a subset of R of cardinality less than $\aleph_\alpha$ and H, L are subsets of K satisfying $h \in H$ and $l \in L$ implies $l < h$, then there is an $f \in R$ which satisfies $l < f$ for all $l \in L$ and $f < h$ for all $h \in H$.

Definition 5.2. Let y be a countably incomplete ultrafilter on a space X. We say that y is *κ^+-good* if for every monotone function $F : [\kappa]^{<\omega} \to y$, there is a locally finite, multiplicative function $G : [\kappa]^{<\omega} \to y$ which refines F. In more detail, we say that y is *κ^+-good* if for every function $F : [\kappa]^{<\omega} \to y$ such that

$F(c) \subseteq F(b)$ whenever $b \subseteq c \in [\kappa]^{<\omega}$,

there is a function $G : [\kappa]^{<\omega} \to y$ satisfying

(1) $\{G(b) : b \in [\kappa]^{<\omega}\}$ is locally finite,
(2) $G(b) = \bigcap\{G(\{\gamma\}) : \gamma \in b\}$ for all $b \in [\mu]^{<\omega}$, and
(3) $G(b) \subseteq F(b)$ for all $b \in [\mu]^{<\omega}$.

Observe that if y is κ^+-good and $\mu < \kappa$, then y is μ^+-good. Also observe that if y is κ^+-good then y is κ-regular.

Proposition 5.3. *If y is a countably incomplete z-ultrafilter on a space X, then y is ω_1-good.*

Proof. From Proposition 4.3, there is $\mathcal{Z} = \{Z_n : n \in \omega\}$, a locally finite subfamily of y. Given F as in the hypothesis of ω_1-good, for each $n \in \omega$ set

$$G(\{n\}) = Z_n \cap \bigcap\{F(b) : \max b = n\}.$$

For $b \in [\omega]^{<\omega}$, set $G(b) = \bigcap\{G(\{n\}) : n \in b\}$. □

An ancestor of the next result is 13.8 of [10]: if y is a countably incomplete z-ultrafilter on a space X, Then $C(X)/\mathcal{M}_y$ is an η_1-set.

Theorem 5.4. *Let $\kappa = \aleph_\alpha$. If y is a countably incomplete κ^+-good z-ultrafilter on a paracompact space X, then $C(X)/\mathcal{M}_y$ is an η_α-set.*

Proof. Let $K = L \cup H$ be a subset of $C(X)$ of cardinality less than κ such that for all $l \in L$ and $h \in H$ there is $F(\{l,h\}) \in y$ satisfying $l(x) < h(x)$ for all $x \in F(\{l,h\})$. For $b \in [K]^{<\omega}$ such that $b \cap L = \varnothing$ or $b \cap H = \varnothing$, set $F(b) = X$. For other $b \in [K]^{<\omega}$, set

$$F(b) = \bigcap\{F(\{l,h\}) : l \in b \cap L, h \in b \cap H\}.$$

Since b is finite, $F(b)$ is a member of y. Notice that the function $F : [K]^{<\omega} \to y$ satisfies $F(c) \subset F(b)$ whenever $b \subset c$. Since y is countably incomplete and κ^+-good, there is $G : [K]^{<\omega} \to y$ such that $G(b) \subset F(b)$, $G(b) = \bigcap\{G(\{k\}) : k \in b\}$, and $\{G(b) : b \in [K]^{<\omega}\}$ is locally finite. For each $x \in X$ let $W(x)$ be an open neighborhood of x such that

$$\{k : W(x) \cap G(\{k\}) \neq \varnothing\} = b(x) \in [K]^{<\omega}.$$

For $b \in [K]^{<\omega}$, set $W_b = \bigcup\{W(x) : b(x) = b\}$, then $\mathcal{W} = \{W_b : b \in [K]^{<\omega}\}$ is an open cover of X. For each $b \in [K]^{<\omega}$, define $f_b : W_b \to \mathbb{R}$ by

$$f_b(x) = \begin{cases} \max\{l(x) : l \in L \cap b\} + 1 & \text{if } H \cap b = \varnothing \\ \min\{h(x) : h \in H \cap b\} - 1 & \text{if } L \cap b = \varnothing \\ \frac{1}{2}\max\{l(x) : l \in L \cap b\} & \\ \qquad + \frac{1}{2}\min\{h(x) : h \in H \cap b\} & \text{otherwise} \end{cases}$$

Because X is paracompact, there is $\{\varphi_b : b \in [K]^{<\omega}\}$, a locally finite partition of unity subordinate to $\mathcal{W}$. For all $x \in X$, set $f(x) = \sum \varphi_b(x) \cdot f_b(x)$. Note that f is continuous. We now show that $l(x) < f(x)$ for all $x \in G(\{l\})$ and $l \in L$ and therefore $[l] <_y [f]$. Let $l \in L$ be arbitrary. Suppose that $x \in G(\{l\})$ and $\phi_b(x) > 0$. Then $x \in W_b$ and hence $W_b \cap G(\{l\}) \neq \varnothing$. Therefore $l \in b \cap L$. If $H \cap b = \varnothing$ then

$$l(x) \leq \max\{g(x) : g \in L \cap b\} < \max\{l(x) : l \in L \cap b\} + 1 = f_b(x).$$

If $H \cap b \neq \varnothing$ then

$$l(x) < \frac{1}{2}(\max\{g(x) : g \in L \cap b\} + \min\{h(x) : h \in H \cap b\}) = f_b(x).$$

Now fix x and h, but let b vary.

$$l(x) = 1 \cdot l(x) = \sum \varphi_b(x) \cdot l(x) < \sum \varphi_b(x) \cdot f_b(x) = f(x)$$

A similar argument shows that for $h \in H$, $f(x) < h(x)$ for all $x \in G(\{h\})$ and therefore $[f] <_y [h]$. $\square$

The existence of good ultrafilters and the existence of not good ultrafilters can be proved without extra axioms of set theory. Keisler [12] defined κ^+-good ultrafilters and showed, assuming GCH, that they exist on a set of cardinality κ. Kunen [13] proved without extra axioms that there are κ^+-good ultrafilters on κ. Let p be a κ-regular ultrafilter on κ and let q be a λ-regular ultrafilter on λ. If $\lambda < \kappa$, then the iterated ultrafilter

$$\{E \subseteq \kappa \times \lambda : \{\beta \in \lambda : \{\alpha \in \kappa : (\alpha, \beta) \in E\} \in p\} \in q\}$$

is κ-regular, but not κ^+-good.

6. Nonregular Ultrafilters

Regular set ultrafilters exist – for every infinite cardinal κ, we found a κ-regular ultrafilter on the set κ in the proof of Theorem 4.1. It is harder to find nonregular set ultrafilters. The most familiar example of a uniform, non-regular ultrafilter is a κ-complete free z-ultrafilter q on a discrete space of measurable cardinality κ. Because every infinite subset of q has nonempty intersection, q is not even ω-regular.

If q were the only example, we might wonder if all uniform countably incomplete ultrafilters are regular. We introduce product filters to give other examples of nonregular ultrafilters.

Definition 6.1. Let p be a z-ultrafilter on a space X_0 and let u be a z-ultrafilter on a space X_1. Let $p \times u$ be the z-filter on $X = X_0 \times X_1$ generated by rectangles $A \times Z$, where $A \in p$ and $Z \in u$.

Lemma 6.2. *Let p be a z-ultrafilter on a space X_0 and let u be a z-ultrafilter on a space X_1. Set $\lambda = |\{Z \subset X_1 : Z$ is a zero set$\}|$.*

(1) *If p and u are both countably incomplete, then $p \times u$ is not a z-ultrafilter.*
(2) *If p is λ^+-complete, then $p \times u$ is a z-ultrafilter.*
(3) *If p is λ^+-complete and u is countably incomplete, then $p \times u$ is a countably incomplete z-ultrafilter which is not λ^+-regular. Hence $p \times u$ is not regular.*

Proof. (1) Let $\{A_n : n \in \omega\} \subset p$ be nested with empty intersection. Let $\{Z_n : n \in \omega\} \subset u$ be nested with empty intersection. Because the complement of a z-set is the union of countably many z-sets, there is an increasing sequence $\{A'_n : n \in \omega\}$ of z-sets of X_0 satisfying $A_n \cap A'_n = \varnothing$ for all $n \in \omega$, and $\bigcup\{A_n : n \in \omega\} = X_0$. We investigate T, a z-set of $X_0 \times X_1$.

$$T = \bigcup\{A'_n \times Z_n : n \in \omega\}$$

Let $A \times B$ be a rectangle of $X_0 \times X_1$. Suppose that $A \times B \subset T$. Let $b \in B$. There is an n such that $b \notin Z_n$. Hence $A \subseteq A'_n$ and $A \notin p$. Similarly, if $(A \times B) \cap T = \varnothing$, then $B \notin u$. In conclusion, T shows that $p \times u$ is not a z-ultrafilter.

(2) Let T be an arbitrary z-set of of $X_0 \times X_1$. For each $\alpha \in X_0$, set $T_\alpha = \{y \in X_1 : (\alpha, y) \in T\}$. Because p is λ^+-complete, there are Z, a z-set of X_1, and $A \in p$ such that $T_\alpha = Z$ for all $\alpha \in A$. Then $T \in p \times u$ if $Z \in u$, and T is disjoint from $A \times Z \in p \times u$ if $Z \notin u$. Hence $p \times u$ is an ultrafilter.

(3) $p \times u$ is a z-ultrafilter which is countably incomplete because u is countably incomplete. Let $\mathcal{E} = \{E_\gamma : \gamma < \lambda^+\} \subset p \times u$, and for each γ let $A_\gamma \times Z_\gamma \subseteq E_\gamma$. There are Z, a z-set of X_1 and $W \in [\lambda^+]^{\lambda^+}$ such that $Z_\gamma = Z$ for all $\gamma \in W$. Because p is λ^+-complete, there is $\alpha \in \bigcap\{A_\gamma : \gamma \in W\}$. Then $\{\alpha\} \times Z \subset \bigcap\{E_\gamma : \gamma \in W\} \neq \varnothing$. $\square$

The examples of nonregular ultrafilters presented so far an on sets of measurable cardinality or larger. Is it possible to have a nonregular ultrafilter on a set smaller than the first measurable cardinal? Yes – sort of. All known constructions non-regular uniform ultrafilter on a small cardinal start with (at least) a measurable cardinal, and then force the measurable cardinal to be small.

Definition 6.3. Let UR(κ) be the assertion that every uniform ultrafilter on a set of cardinality κ is κ-regular. Let UR assert that UR(κ) holds for every infinite κ. Informally, we read UR as every uniform ultrafilter is regular.

Recall that UR implies that there are no measurable cardinals. Like the assumption that there are no measurable cardinals UR is safe. The assumption of Theorem 1.1, GCH + UR is a consequence of V=L. Hence UR does not imply that ZFC is consistent. On the other hand, we will show that $\neg$UR does imply that ZFC is consistent. In fact, it is plausible to conjecture that $\neg$UR is equiconsistent with "there exists a measurable cardinal". We proceed by cases to sketch a "near proof" of this conjecture.

Case 1. *κ is singular cardinal.* Prikry [17] started with a measurable cardinal κ and forced it to have cofinality ω. In the extension, κ is a singular cardinal which carries a uniform, non-regular ultrafilter. In the other direction, Donder [5] showed that if a singular cardinal κ carries a uniform, non-regular ultrafilter, then κ is measurable in some inner model.

Case 2. *κ is a successor cardinal.* We say that θ is a stationary limit of measurables if θ is regular and $\{\kappa \in \theta : \kappa \text{ is measurable }\}$ is stationary in θ. Deiser and Donder [4] showed both directions of an equiconsistency theorem. Starting with a stationary limit of measurables, they force to get a successor cardinal which carries a uniform, non-regular ultrafilter. Conversely given a model with a successor cardinal which carries a uniform, non-regular ultrafilter, they find an inner model with a stationary limit of measurables.

Case 3. *κ is a regular limit cardinal.* Of course, a measurable cardinal is a regular limit cardinal which carries a uniform, non-regular ultrafilter. In the other direction, assuming that there is no inner model with a measurable cardinal, Donder [5] showed that every uniform ultrafilter on κ is λ-regular. Rephrasing, for every $\lambda < \kappa$ there is a regularizing family of cardinality λ. To complete the proof of equivalence, it is necessary to find a regularizing family of cardinality κ.

There is a consolation prize. A regular limit cardinal is itself a large cardinal. So assuming that UR is false entails assuming that ZFC is consistent.

We can show that not regular z-ultrafilters exist without extra axioms. For example, the unique free z-ultrafilter on the ordinal space ω_1 is uniform, but not ω-regular. However, the ordinal space ω_1 is not paracompact, a fortiori, not metrizable. The next result shows that the assumption UR(κ) implies that on certain metrizable spaces of weight κ, uniform z-ultrafilters are κ-regular.

Lemma 6.4. *Assume* UR(κ). *That is, every uniform ultrafilter p on a set of cardinality κ is κ-regular. Let X be a metrizable space of weight κ which is locally compact. Then every uniform z-ultrafilter y on X is κ-regular.*

Proof. Let $\mathcal{C}$ be the collection of open subsets of X that have compact closure. Because X is locally compact, $\mathcal{C}$ covers X. Let $\mathcal{R}$ be a locally finite open refinement of $\mathcal{C}$.

We claim that $|R| = \kappa$. Since y is free, X is not compact and therefore $\mathcal{R}$ cannot be finite. Hence if $\kappa = \omega$ then $|\mathcal{R}| = \kappa = \omega$. Suppose that $\kappa < \omega$. Let $\mathcal{B}$ be a base for X of cardinality κ. Because $\mathcal{R}$ is locally finite, $|\mathcal{R}| \leq |\mathcal{B}| = \kappa$. In the other direction, if $R \in \mathcal{R}$, then $L(R) = \omega$. Hence $\kappa = L(\bigcup \mathcal{R}) \leq |\mathcal{R}| \cdot \omega$. By the same argument, for all $\mathcal{S} \in [\mathcal{R}]^{<\kappa}$ and $Z \in y$, we have $Z \nsubseteq \bigcup \mathcal{S}$ because y is a uniform z-ultrafilter.

For each $Z \in y$, set $\mathcal{U}(Z) = \{U \in \mathcal{R} : U \cap Z \neq \emptyset\}$. Observe that $p^0 = \{\mathcal{U}(Z) : Z \in y\} \cup \{X \setminus \mathcal{S} : \mathcal{S} \in [\mathcal{R}]^{<\kappa}\}$ has the finite intersection property, and extend to a uniform ultrafilter p on $\mathcal{R}$.

Because p is κ-regular, there is a point finite collection $\{\mathcal{U}_\alpha : \alpha \in \kappa\} \subset p$. For each α, set $Z_\alpha = \operatorname{cl} \bigcup \mathcal{U}_\alpha$. We claim that $Z_\alpha \in y$. Let $Z \in y$ be arbitrary and let $\alpha \in \kappa$. The collections $\mathcal{U}_\alpha$ and $\mathcal{U}(Z)$ are both members of p, so $\mathcal{U}_\alpha \cap \mathcal{U}(Z) \neq \varnothing$. Let $U \in \mathcal{U}_\alpha \cap \mathcal{U}(Z)$. Then $U \cap Z \neq \varnothing$ and therefore $\mathcal{U}_\alpha^* \cap Z \neq \varnothing$. Hence $Z_\alpha \cap Z \neq \varnothing$, so $Z_\alpha \in y$.

We have shown that $\{Z_\alpha : \alpha \in \kappa\}$ is a subset of y; we must show that it is locally finite. Because $\mathcal{R}$ is locally finite, for each $x \in X$ there is an open set V such that $x \in V$ and $\{U \in \mathcal{R} : V \cap U \neq \varnothing\}$ is finite. Then $\{\alpha \in \kappa : (\exists U \in \mathcal{U}_\alpha) V \cap U \neq \varnothing\}$ is finite, and we are done. $\square$

In the result above the hypothesis "X is locally compact" can be replaced with the cumbersome "Let X have a cover $\mathcal{C}$ of open sets of weight less than λ, for some regular cardinal λ less than or equal to κ".

7. Examples on Products

In this section, we discuss the remainder $\beta X \setminus X$, where X is the product of a discrete space κ and the real line $\mathbb{R}$. First, recall that the projections $\pi_0 : X \to \kappa$ and $\pi_1 : X \to \mathbb{R}$ extend to maps $\beta\pi_0 : \beta X \to \beta\kappa$ and $\beta\pi_1 : \beta X \to \beta\mathbb{R}$. Combining these maps, we obtain a map $\pi : \beta X \to \beta\,\kappa \times \beta\mathbb{R}$.

We can characterize these maps.

$$\begin{aligned}
\beta\pi_0(y) = p &\Leftrightarrow \{A \times \mathbb{R} : A \in p\} \subset y \\
\beta\pi_1(y) = u &\Leftrightarrow \{\kappa \times Z : Z \in u\} \subseteq y \\
\pi(y) = (p, u) &\Leftrightarrow \{A \times Z : A \in p \wedge Z \in u\} \subseteq y \\
&\Leftrightarrow p \times u \subseteq y
\end{aligned}$$

The next remark follows from the above characterization and Lemma 6.2.

Remark 1. Let p be an ultrafilter on κ, and let u be a z-ultrafilter on $\mathbb{R}$. Then $\pi^{\leftarrow}\{(p,u)\}$ is a singleton if p is $(2^{\omega})^{+}$-complete, and $\pi^{\leftarrow}\{(p,u)\}$ is not a singleton if p and u are both countably incomplete.

We can visualize $\beta\pi_0^{\leftarrow}\{p\}$ as a vertical line over $p \in \beta\theta$, and $\beta\pi_1^{\leftarrow}\{u\}$ as a horizontal line from $u \in \beta\mathbb{R}$. Next we describe a thinner horizontal line from $r \in \mathbb{R}$. For each $r \in \mathbb{R}$ and $p \in \beta\kappa$, let $e_p(r)$ be the z-ultrafilter generated by

$$\{\{(\alpha, r) : \alpha \in A\} : A \in p\}$$

Set $H_r = \{e_p(r) : p \in \beta\kappa\}$. It is easy to verify that $\beta\pi_0(e_p(r)) = p$ and $\beta\pi_1(e_p(r)) = r$ for all $p \in \beta\kappa$ and $r \in \mathbb{R}$. Moreover, H_r is closed in βX because the map $p \mapsto e_p(r)$ is a homeomorphism.

We will review several constructions of non-normality points, and explain how they can be visualized. First, Blaszczyk and Symanski [2] showed, within ZFC, that for every discrete space λ, there are non-normality points y in $\beta\lambda\backslash\lambda$. The first step is finding a disjoint open family $\{V_\alpha : \alpha \in \kappa\}$ in $\beta\lambda\backslash\lambda$. For each α, choose a point $s_\alpha \in V_\alpha$, and let y be an accumulation point. We mimic their construction in $X = \kappa \times \mathbb{R}$. We visualize V_α as $\{\alpha\} \times \mathbb{R}$, s_α as $(\alpha, 0)$, and y as $e_p(0)$. Then the two closed sets that they use are (analogous to) the horizontal line H_0 and the vertical line $\beta\pi_0^{\leftarrow}\{p\}$.

Second, Beslagic and van Douwen showed, assuming GCH, that for every κ, every $p \in \beta\kappa\backslash\kappa$ is a nonnormailty point. Their method is to construct a sequence $\{t_\gamma : \gamma \in \kappa^+\}$ in $\beta\kappa\backslash\kappa$ with two properties: (1) the sequence converges to p (that is, p is the only complete accumulation point of the full sequence), and (2) each proper initial segment is C^*-embedded in $\beta\kappa\backslash\kappa$ (that is, each proper initial segment has the most possible accumulation points). They extend the map $g : \gamma \mapsto t_\gamma$ to a map $\beta g : \beta\kappa^+ \to (\beta\kappa\backslash\kappa)$. Next, $\beta\kappa^+$ is partitioned into $U(\kappa^+)$, the uniform ultrafilters on κ^+, and $NU(\kappa^+)$, the nonuniform ultrafilters on κ^+. Malyhin [16] showed that $NU(\kappa^+)$ is not normal. The map βg is a homeomorphism on $NU(\kappa^+)$ and takes every $u \in U(\kappa^+)$ to y. Putting all these ideas together, $(\beta\kappa\backslash\kappa)\backslash\{p\}$ is not normal because is has a closed subset homeomorphic to the not normal space $NU(\kappa^+)$. We visualize this construction as occurring inside the horizontal line H_0.

Logunov [15] and Terasawa [19] consider a crowded metrizable space X and a point y of the remainder. Without extra axioms, they construct two sequences $\{t^0_\gamma : \gamma < \theta\}$ and $\{t^1_\gamma : \gamma < \theta\}$ such that $H_0 = \{t^0_\gamma : \gamma < \theta\} \cup \{y\}$ and $H_1 = \{t^1_\gamma : \gamma < \theta\} \cup \{y\}$ show that y is a butterfly point.

If we follow their construction with $X = \kappa \times \mathbb{R}$ and $y = e_p(0)$, we can arrange that the sequences are subsets of the vertical line $\beta\pi_0^{\leftarrow}\{p\}$.

Here is how we visualize our proof of Theorem 1.1. Following [15] and [19], we work within the vertical line $\beta\pi_0^{\leftarrow}\{p\}$, to construct a sequence $\{t_\gamma : \gamma \in \kappa^+\}$ with the two properties of the Beslagic-van Douwen construction. With no extra axioms, we cannot control θ, the length of the convegent sequence. We will see in Example 7.1 that $\theta = \omega$ is possible! We make the assumption UR to ensure that $\theta > \kappa$, and we assume GCH so that the 2^κ many tasks can be arranged in a list of length $\theta = \kappa^+$.

Before we implement this vision, we return to examples on $X = \kappa \times \mathbb{R}$. We will show that $(\beta X \backslash X)\backslash\{y\}$ is not normal in certain situations. The cases $y = e_q(0)$, where q is a κ-complete free ultrafilter on κ is quite different from the case $y = e_p(0)$, where p is a countably incomplete free ultrafilter on κ.

Example 7.1. Let q be a κ-complete ultrafilter on κ. Then the restriction of $\beta\pi_1$ to $\beta\pi_0^{\leftarrow}\{q\}$ is a homeomorphism. Set $y = e_q(0)$. There are two disjoint closed subsets of $(\beta X \backslash X)\backslash\{y\}$ which cannot be separated by a continuous real-valued function.

From the remark above, we see that $r \mapsto e_q(r)$ is a set bijection. Let V be open in $\mathbb{R}$. Then $e_q(r) \in B(\kappa \times V)$ iff $r \in V$.

For continuity in the other direction, assume that $e_q(r) \in B(U)$. It means that U is open in $\kappa \times \mathbb{R}$ and that there is $A \in q$ such that $A \times \{r\} \subset U$. For each $\alpha \in A$, set $U_\alpha = \{s \in \mathbb{R} : (\alpha, s) \in U\}$. Because q is $(2^\omega)^+$-complete, there is V open in $\mathbb{R}$ and $A' \in q$ such that $u_\alpha = V$ for all $\alpha \in A'$. Then $e_q(s) \in B(U)$ for all $s \in V$.

Consider the two closed sets $H_0 = \{(e_q(\frac{1}{n}) : n \in \mathbb{N} \wedge n \text{ is even }\}$ and $H_1 = \{(e_q(\frac{1}{n}) : n \in \mathbb{N} \wedge n \text{ is odd }\}$. Clearly, $\mathrm{cl}_{\beta X} H_0 \cap \mathrm{cl}_{\beta X} H_1 = \{e_q(0)\}$, so $e_q(0)$ is a butterfly point, and hence $\beta X \backslash \{e_q(0)\}$ is not normal. However, H_0 and H_1 can be separated by disjoint open sets. Is $(\beta X \backslash X)\backslash\{e_q(0)\}$ normal? We use ideas from Example 2.5.

Suppose that $H_0 \subseteq U_0$ open and $H_1 \subseteq U_1$ open. For each n, find $A_n \in q$ such that $A_n \times \{\frac{1}{n}\} \subseteq B(V_n) \subseteq U_e$ for $e = 0, 1$ as appropriate. Let $\alpha \in \bigcap_{n\in\omega} A_n$. Then $(\alpha, 0) \in \mathrm{cl}_{\beta X} U_0 \cap \mathrm{cl}_{\beta X} U_1$, hence $\beta X \backslash \{e_q(0)\}$ is not normal. To show that $(\beta X \backslash X)\backslash\{e_q(0)\}$ is not normal, let $q' \neq q$ be a countably closed ultrafilter on κ such that $\bigcap_{n\in\omega} A_n \in q'$. Then $e_{q'}(0) \in \mathrm{cl}_{\beta X} U_0 \cap \mathrm{cl}_{\beta X} U_1$.

Observe that the embedding of $n \mapsto e_q(\frac{1}{n+1})$ of discrete ω into $(\beta X \backslash X) \backslash \{e_q(0)\}$ extends to an embedding of $NU(\omega)$ onto a closed subset of $(\beta X \backslash X) \backslash \{e_q(0)\}$. However, because $NU(\omega)$ is ω, a normal space, we cannot conclude that y is a non-normality point of $\beta X \backslash X$ from this argument. The axiom UR ensures that a situation like this example does not occur in the proof of Theorem 1.1. In fact, the weaker axiom "no measurable cardinals" forbids this situation. If a metrizable (more generally, a paracompact) space X has cardinality less than the first measurable cardinal, then X is realcompact. If X is realcompact, then every closed subset of $\beta X \backslash X$ includes a subspace homeomorphic to $\beta\omega$. In a strong way, there are no convergent sequences in the remainder of a realcompact space.

The next example is a prototype for Theorem 1.1. We assume that the ultrafilter p is κ^+-good so that it is easier to describe the sequence $\{t_\gamma : \gamma < \kappa^+\}$.

Example 7.2. Let p be a countably incomplete ultrafilter on κ. Then $\{e_p(r) : r \in \mathbb{R}\}$ is discrete in βX. Set $y = e_q(0)$. Assume $2^\kappa = \kappa^+$ and that p is κ^+-good. Then there is a κ^+ sequence converging to $e_p(0)$. Every proper initial segment of that sequence is C^*-embedded. Hence $(\beta X \backslash X) \backslash \{y\}$ contains a closed subset homeomorphic to the nonnormal space $NU(\kappa^+)$.

Let $\{A_n : n \in \omega\} \subset p$ be nested with empty intersection. Define $j : \kappa \to \mathbb{R}$ by $j(\alpha) = \frac{1}{n}$ iff $\alpha \in A_n \backslash A_{n+1}$. Let y' be the z-ultrafilter on X generated by

$$\{\{(\alpha, j(\alpha)) : \alpha \in A\} : A \in p\}.$$

Then $\beta\pi_0(y') = p = \beta\pi_0(e_p(0))$ and $\beta\pi_1(y') = 0 = \beta\pi_1(e_p(0))$, but $y' \neq e_p(0)$ because $\kappa \times \{0\} \in e_p(0)$ and $\kappa \times \{0\} \notin y'$.

We use j to show that $\{e_p(r) : r \in \mathbb{R}\}$ is a discrete subspace of βX. Fix $r \in \mathbb{R}$. Set $U = \{(\alpha, s) : \alpha \in \kappa \text{ and } r - j(\alpha) < s < r + j(\alpha)\}$, an open subset of X. Then $B(U) \cap \{e_p(r) : r \in \mathbb{R}\} = \{e_p(r)\}$.

The sequence $\{t_\gamma : \gamma < \theta\}$ converging to $e_p(0)$ might start with $t_n = e_p(1/n)$, but because $e_p(0)$ is not in the closure of that countable set, we must continue. We would like to think of j as an infinitesmal, and continue the sequence with $t_\omega = y' = e_p(j)$. Ultrapowers of $\mathbb{R}$ provide the machinery to make this idea precise.

The ultrapower $\mathcal{R}$ is the family of equivalence classes of the relation $\sim$ on the product $\mathbb{R}^\kappa$, defined by,

$$f \sim g \text{ iff } \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in p.$$

For $f \in \mathcal{R}^\kappa$ let us define a point $e_p(f) \in \beta X$ to be the z-ultrafilter generated by $\{(\alpha, f(\alpha)) : \alpha \in A\}$ for $A \in p$. (If $\mathbf{r}$ is the constant function whose value is always r, then $e_p(\mathbf{r})$ coincides with the previously defined $e_p(r)$). Observe that $e_p(f) = e_p(g)$ iff $f \sim g$. Hence there is a function $e_p : \mathcal{R} \to \beta X$ defined by $[f] \mapsto e_p(f)$. Let us call the range $e_p(\mathcal{R})$.

Algebraic operations $+$, $\cdot$, etc. transfer from $\mathbb{R}$ to $\mathcal{R}$, as does the linear order $<$. The open intervals of $\mathcal{R}$ generate a topology on $\mathcal{R}$. Above we showed that $e_p : \mathcal{R} \to e_p(\mathcal{R})$ is a bijection; let us show that it and its inverse are continuous.

Suppose that $[f] \in B(U) \cap e_p(\mathcal{R})$ for some U open in X. Set $U_\alpha = \{r \in \mathbb{R} : (\alpha, r) \in U\}$. Then $A = \{\alpha \in \kappa : f(\alpha) \in U_\alpha\} \in p$. Choose $l : \kappa \to \mathbb{R}$ and $h : \kappa \to \mathbb{R}$ which satisfy $f(\alpha) \in (l(\alpha), h(\alpha)) \subseteq U_\alpha$ for all $\alpha \in A$. Then $([l], [h])$ is an open interval of $\mathcal{R}$, and $[f] \in ([l], [h]) \subseteq B(U) \cap e_p(\mathcal{R})$.

Conversely, given an open interval $([l], [h])$ of $\mathcal{R}$, choose representatives l and u, then set $U = \{(\alpha, r) : l(\alpha) < r < h(\alpha)\}$. Then $B(U) \cap e_p(\mathcal{R}) = ([l], [h])$.

We seek a monotone decreasing sequence $\{t_\gamma : \gamma \in \theta\}$ from $e_p(\mathcal{R})$ converging to $e_p(0)$. What is the value of θ? By reciprocals, θ is the cofinality of $\mathcal{R}$. Always, $\mathrm{cf}(\mathcal{R}) \leq |\mathbb{R}^\kappa| = 2^\kappa$. If p is κ-regular, then $\kappa < \mathrm{cf}(\mathcal{R})$. Hence if GCH + UR, then $\mathrm{cf}(\mathcal{R}) = \kappa^+$.

Let $\tilde{T} = \{\tilde{t}_\gamma : \gamma < \kappa^+\}$ be a monotone decreasing sequence in $\mathcal{R}$ converging to 0. By the above paragraph, a proper initial segment $\tilde{S}$ of $\tilde{T}$ with no last element has no greatest lower bound in $\mathcal{R}$. So $\mathrm{cl}_{\mathcal{R}} \tilde{T} = \tilde{T} \cup \{0\}$. When we embed $\mathcal{R}$ into the larger space βX, then $e_p(\tilde{S})$ has accumulation points because βX is compact. Assuming that p is κ^+-good, we will show that every small (cardinality less than κ^+) subset of $e_p(\mathcal{R})$ is C^*-embedded in βX. Then $\mathrm{cl}_{\beta X} e_p(\tilde{T}) \cong NU(\kappa^+ \cup \{0\})$, and $(\beta X \setminus X)\{y\}$ will be not normal because it contains a closed subset homeomorphic to the not normal space $NU(\kappa^+)$.

Let $e_p(S)$ be a subset of $e_p(\mathcal{R})$ with $|S| = \mu < \kappa^+$. Let $g_0 : S \to \mathbb{R}$ be a bounded continuous function. We will find a continuous function $g_3 : \beta X \to \mathbb{R}$ such that $g_3 | S = g_0$.

Choose representatives $\{s_\gamma : \gamma < \mu\} \subset \mathbb{R}^\kappa$. For $b \in [\mu]^\omega$ set $F(b) = \{\alpha \in \kappa : |\{s_\gamma(\alpha) : \gamma \in b\}| = |b|\}$. In words, if $\gamma \neq \gamma'$, then $s_\gamma(\alpha) \neq s_{\gamma'}(\alpha)$. Note that $F : [\mu^+]^{<\omega} \to p$ satisfies the hypothesis of μ^+-good. (Definition 5.2) Hence there is a locally finite, multiplicative $G : [\mu^+]^{<\omega} \to p$ refining F.

Set $W = \{(\alpha, s_\gamma(\alpha)) : \alpha \in \kappa \wedge \alpha \in G(\gamma) \wedge \gamma \in \mu\}$. Because G is locally finite, W meets every vertical line in a finite set. Hence W is a closed discrete subset of X and every function from W to $\mathbb{R}$ is continuous. Because $G(b) \subset F(b)$, the definition $g_1(\alpha, s_\gamma(\alpha)) = g_0([s_\gamma])$ is unambiguous.

Because W is a closed subset of the metrizable, hence normal space X, there is a bounded continuous $g_2 : X \to \mathbb{R}$ extending g_1. By the properties of βX there is a continuous $g_3 : \beta X \to \mathbb{R}$ extending g_2. Because $[s_\gamma] \in \mathrm{cl}_{\beta X}\{(\alpha, s_\gamma(\alpha)) : \alpha \in \kappa \wedge \alpha \in G(\gamma)\}$, we get $g_3([s_\gamma]) = g_0([s_\gamma])$ for all $\gamma \in \mu$, and we have found the required extension.

8. Pi Bases

We will use locally finite pairwise disjoint collections, ξ, of open sets in our constructions. The collections will come from an appropriate pi-base. Following Terasawa we use ξ^* to denote $\bigcup \xi$. Observe that such a collection, ξ, is locally finite maximal disjoint if and only if ξ^* is dense in X.

Proposition 8.1. *[Terasawa] Let X be a metrizable space without isolated points. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$$

such that

1. *$\mathcal{B}_n$ is a locally finite, maximal disjoint family of nonempty open sets;*
2. *$\mathcal{B}_n$ refines $\mathcal{B}_{n-1}$;*
3. *for each $B \in \mathcal{B}_{n-1}$, there are three sets $B^{(i)} \in \mathcal{B}_n$, $i = 0, 1, 2$, such that $\mathrm{cl}\, B^{(i)} \subset B$ and $\mathrm{cl}\, B^{(i)} \cap \mathrm{cl}\, B^{(i)} = \varnothing$ for $i \neq j$;*
4. *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.*

Suppose $y \in \beta X \backslash X$. Terasawa remarks that the π-base in Proposition 8.1 can be easily modified so that

(#) $$y \notin \mathrm{cl}_{\beta X} B \text{ for all } B \in \mathcal{B}.$$

This property of $\mathcal{B}$, however, was not necessary in his proof that $\beta X \backslash \{y\}$ is not normal; the butterfly sets did not need to be subsets of $\beta X \backslash X$. To show that $(\beta X \backslash X) \backslash \{y\}$ is not normal, our construction will require closed subsets of $\beta X \backslash X$. The following propositions define a π-base, $\mathcal{B}$, for two types of metric spaces. For X locally compact, $(\#)$ is true for $\mathcal{B}$ for any $y \in \beta X \backslash X$. For X κ^ω-like, given $y \in \beta X \backslash X$, we construct $\mathcal{B}$ so that $(\#)$ is satisfied.

We say that a π-base, $\mathcal{B}$, for a crowded metric space is *nice* if it satisfies (1), (2) and (4) in Proposition 8.1. We use the properties of a nice π-base to construct locally finite collections in Section 9. In the sections after 9 we use a nice π-base with the additional properties (3) and $(\#)$.

Proposition 8.2. *Let X be a locally compact metric space without isolated points. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$$

such that

(1) *$\mathcal{B}_n$ is a locally finite, maximal disjoint family such that $\mathrm{cl}_X B$ is compact for each $B \in \mathcal{B}$.*
(2) *$\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ and $|\{B' \in \mathcal{B}_{n+1} : B' \subset B\}| = 4$ for all $B \in \mathcal{B}_n$.*
(3) *For $B \in \mathcal{B}_n$ there are $B^0, B^1 \in \mathcal{B}_{n+1}$ such that $\mathrm{cl}\, B^0 \cap \mathrm{cl}\, B^1 = \varnothing$ and $\mathrm{cl}\, B^0, \mathrm{cl}\, B^1 \subset B$.*
(4) *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.*

Proof. Let $\kappa = w(X)$. Let $\mathcal{O}$ be an open cover of X consisting of sets U such that $\mathrm{cl}\, U$ is compact. Let $\mathcal{B}_0'$ be a locally finite open refinement of cardinality at most κ. In fact, it must be that $|\mathcal{B}_0'| = \kappa$. Otherwise, if $|\mathcal{B}_0'| < \kappa$ then $w(X) \leq |\mathcal{B}_0'| \cdot \omega = |\mathcal{B}_0'| < \kappa$ since $w(B) = \omega$ for each $B \in \mathcal{B}_0'$. Well order $\mathcal{B}_0'$ as $\{B_\alpha' : \alpha \in \kappa\}$. Define $B_\alpha = B_\alpha' \backslash \bigcup_{\gamma < \alpha} \mathrm{cl}\, B_\gamma$ and set $\mathcal{B}_0 = \{B_\alpha : \alpha \in \kappa\}$. Note that each B_α is open and $\mathcal{B}_0$ is locally finite since $\mathcal{B}_0'$ is locally finite. Furthermore, $\mathrm{cl}\, B_\alpha$ is compact since $\mathrm{cl}\, B_\alpha \subset \mathrm{cl}\, B_\alpha'$.

Fix $\alpha \in \kappa$. Since $\operatorname{cl} B_\alpha$ is compact and metric, there is a countable base for $\operatorname{cl} B_\alpha$. Let $\mathcal{A}_\alpha = \{A_i \subset \operatorname{cl} B_\alpha : i \in \omega\}$ be such a base such that $A_0 = B_\alpha$ and A_i is open with respect to $\operatorname{cl} B_\alpha$. Let $\mathcal{W}^0_\alpha = \{B_\alpha\}$. Assume we have defined for each i such that $0 < i \leq n$ a collection $\mathcal{W}^i_\alpha$ of open subsets of B_α such that:

i) $\mathcal{W}^i_\alpha$ is a pairwise disjoint finite collection such that $\operatorname{cl}(\bigcup \mathcal{W}^i_\alpha) = \operatorname{cl} B_\alpha$.

ii) $\mathcal{W}^i_\alpha$ refines $\mathcal{W}^{i-1}_\alpha$ and $|\{B' \in \mathcal{W}^i_\alpha : B' \subset B\}| = 4$ for all $B \in \mathcal{W}^{i-1}_\alpha$.

iii) For $B \in \mathcal{W}^{i-1}_\alpha$ there are $B^0, B^1 \in \mathcal{W}^i_\alpha$ such that $\operatorname{cl} B^0 \cap \operatorname{cl} B^1 = \varnothing$ and $\operatorname{cl} B^0, \operatorname{cl} B^1 \subset B$.

iv) For each $B \in \mathcal{W}^i_\alpha$, either $B \subset A_i$ or $B \subset B_\alpha \setminus \operatorname{cl} A_i$.

Fix $W \in \mathcal{W}^n_\alpha$.

Case 1. $W \cap A_{n+1} = \varnothing$ or $W \setminus A_{n+1} = \varnothing$.

Because X has no isolated points, we can find B^0 and B^1, non-empty open subsets of W, such that $\operatorname{cl} B^0 \cap \operatorname{cl} B^1 = \varnothing$ and $\operatorname{cl} B^0 \cup \operatorname{cl} B^1 \subsetneq W$. Then let B^2 and B^3 be non-empty open subsets of W such that $B^2 \cup B^3$ is dense in $W \setminus (\operatorname{cl} B^0 \cup \operatorname{cl} B^1)$.

Case 2. $W \cap A_{n+1} \neq \varnothing$ and $W \setminus A_{n+1} \neq \varnothing$.

If $W \subset \operatorname{cl}(A_{n+1})$ then $W \cap A_{n+1}$ is an open (in X) dense subset of W. We can choose B^0 and B^1, non-empty open subsets of $W \cap A_{n+1}$, such that $\operatorname{cl} B^0 \cap \operatorname{cl} B^1 = \varnothing$ and $\operatorname{cl} B^0 \cup \operatorname{cl} B^1 \subsetneq W \cap A_{n+1}$. Then let B^2 and B^3 be non-empty open subsets of $W \cap A_{n+1}$ such that $B^2 \cup B^3$ is dense in $W \setminus (\operatorname{cl} B^0 \cup \operatorname{cl} B^1)$.

If $W \setminus \operatorname{cl}(A_{n+1}) \neq \varnothing$ then we let B^0 be a non-empty open subset of W such that $\operatorname{cl} B^0 \subsetneq W \cap A_{n+1}$ and let
$B^2 = (W \cap A_{n+1}) \setminus \operatorname{cl} B^0$. Then let B^1 be a non-empty open subset of W such that $\operatorname{cl} B^1 \subsetneq W \setminus A_{n+1}$ and let $B^3 = W \setminus (\operatorname{cl} A_{n+1} \cup \operatorname{cl} B^1)$. Again, since X has no isolated points, this can be done.

Set $\mathcal{W}^{n+1}_\alpha = \{B^i : i = 0, 1, 2, 3\}$. By construction, $\mathcal{W}^{n+1}_\alpha$ has properties (i) - (iv). Let $\mathcal{B}_n = \bigcup_{\alpha \in \kappa} \mathcal{W}^n_\alpha$. Properties (i)-(iii) for $\mathcal{W}^n_\alpha$ imply properties (1)-(3) for $\mathcal{B}_n$. It remains to show that (4) holds. Let $\mathcal{U}$ be an open cover of X. Fix $\alpha \in \kappa$. If $B_\alpha \subset U$ for some $U \in \mathcal{U}$ then let $\mathcal{V}_\alpha = \mathcal{W}^0_\alpha = \{B_\alpha\}$. Otherwise, consider $\mathcal{O} = \{A_i \in \mathcal{A}_\alpha : A_i \subset U$ for some $U \in \mathcal{U}\}$. The collection $\mathcal{O}$ is an open (with respect to $\operatorname{cl} B_\alpha$) cover of the compact set $\operatorname{cl} B_\alpha$. Let $\{A_{i_k} : k = 1, \ldots, m\}$ be a finite subcover and let $n = \max\{i_k : k = 1, \ldots, m\}$. Then, $\mathcal{W}^n_\alpha$ has the property that for all $W \in \mathcal{W}^n_\alpha$, $W \subset A_i$ or $W \subset B_\alpha \setminus \operatorname{cl} A_i$ for all $i \le n$. So, for each $W \in \mathcal{W}^n_\alpha$ there exist $k \in \{1, \ldots, m\}$ and $U \in \mathcal{U}$ such that $W \subset A_{i_k} \subset U$. Let $\mathcal{V}_\alpha = \mathcal{W}^n_\alpha$.

Now, let $\mathcal{V} = \bigcup_{\alpha \in \kappa} \mathcal{V}_\alpha$. Since $\mathcal{V}_\alpha = \mathcal{W}^n_\alpha$, it is finite. Moreover, since $\bigcup \mathcal{V}_\alpha \subset B_\alpha$ and $\mathcal{B}_0$ is locally finite, $\mathcal{V}$ is locally finite. Since $\operatorname{cl}(\bigcup \mathcal{W}^n_\alpha) = B_\alpha$ and $\operatorname{cl}(\bigcup \mathcal{B}_0) = X$, $\operatorname{cl}(\bigcup \mathcal{V}) = X$. Finally, $\mathcal{V}$ refines $\mathcal{U}$ by construction. □

Proposition 8.3. *Let κ be an infinite cardinal and let X be an κ^ω-like metric space. Let y be a free z-ultrafilter on X. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$$

such that

(1) *$\mathcal{B}_n$ is a locally finite, maximal disjoint family of nonempty open sets;*
(2) *$\mathcal{B}_n$ refines $\mathcal{B}_{n-1}$;*
(3) *$|\mathcal{B}_0| = \kappa$ and for each $B \in \mathcal{B}_{n-1}$, there are sets $B^{(\eta)} \in \mathcal{B}_n$, $\eta \in \kappa$, such that $\operatorname{cl} B^{(\eta)} \subset B$ and $\operatorname{cl} B^{(\eta)} \cap \operatorname{cl} B^{(\eta')} = \varnothing$ for $\eta \neq \eta'$;*
(4) *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$;*
(5) *$\operatorname{cl} B \notin y$ for all $B \in \mathcal{B}$.*

Proof. Every open subset, U, of X has extent κ and by Lemma 2.2, this extent is attained. Since X is a metric space, any closed discrete subset is separated by a locally finite collection of open sets whose closures are pairwise disjoint. Therefore, in any nonempty open set U, one can construct a locally finite maximal disjoint collection of size κ with that property that $\operatorname{cl} V \subset U$ for κ many V in the collection. We will use this fact to modify Terasawa's π-base into one of the form in the statement of this proposition. Let $\mathcal{B}' = \bigcup \{\mathcal{B}'_n : n \in \omega\}$ be a π-base as in Lemma 8.1. Since y is free, $\mathcal{U} = \{U \text{ open} : y \notin cl_{\beta X} U\}$ covers X. Let $\mathcal{B}_0$ be any locally finite maximal disjoint collection of cardinality κ of nonempty open sets refining $\mathcal{B}'_0$ and $\mathcal{U}$. For each $B \in \mathcal{B}_0$ let $\mathcal{V}(B)$ be a locally finite maximal disjoint collection of size κ of open subsets of B with that property that $\operatorname{cl} B' \subset B$ for κ many $B' \in B$. Then, $\mathcal{V}_1 = \bigcup \{\mathcal{V}(B) : B \in \mathcal{B}_0\}$ is a locally finite maximal disjoint collection in X. The collection $\mathcal{B}_1 = \{V \cap B : V \in \mathcal{V}_1, B \in \mathcal{B}'_1\} \setminus \{\varnothing\}$ satisfies (1), (2), (3) and (5). Suppose for each n such that $1 \le n < k$ we have defined a collection $\mathcal{B}_n$ that refines $\mathcal{B}'_n$ and satisfies (1), (2), (3) and (5). For each $B \in \mathcal{B}_{k-1}$ let $\mathcal{V}(B)$ be a locally finite maximal disjoint collection of size κ of open subsets of B with that property that $\operatorname{cl} B' \subset B$ for κ many $B' \in B$. Set $\mathcal{V}_k = \bigcup\{V(B) : B \in \mathcal{B}_{k-1}\}$ and $\mathcal{B}_k = \{V \cap B : V \in \mathcal{V}_k, B \in \mathcal{B}'_k\} \setminus \{\varnothing\}$. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$. For each $n \in \omega$, $\mathcal{B}_n$ is locally finite and densely refines $\mathcal{B}'_n$. Therefore, if $B \in \mathcal{B}'$, there is a locally finite collection in $\mathcal{B}$ that densely refines B. Hence $\mathcal{B}$ is a π-base for X and has property (4). □

9. Locally finite collections and cofinalities

Let X be a crowded metric space with a nice π-base $\mathcal{B}$. Let Ξ be the collection of maximal pairwise disjoint, locally finite collections, $\xi \subset \mathcal{B}$.

Remark 2. For each $B, B' \in \mathcal{B}$, if $B \cap B' \neq \varnothing$ then either $B = B'$, $B \subsetneq B'$ or $B' \subsetneq B$.

Remark 3. If $\xi, \eta \in \Xi$ and $B \in \xi$, then since both ξ^* and η^* are dense in X, because of Remark 1, there is $B' \in \eta$ such that either $B = B'$, $B \subsetneq B'$ or $B' \subsetneq B$.

Fix a free z-ultrafilter y on X. Let $\mathcal{N}_y = \{X \cap O : y \in O, O \text{ is open in } \beta X\}$. The collection $\mathcal{N}_y$ is a free open filter on X. We write $\hat{\mathcal{N}_y}$ for the collection of open subsets, U, of X that are dense in some $N \in \mathcal{N}_y$; that is $N \subset \operatorname{cl} U$. Using $\hat{\mathcal{N}_y}$, we define a strict partial order, $<_y$, on Ξ. For $\xi, \eta \in \Xi$ let $L(\xi, \eta) = \{B \in \xi : B' \subsetneq B \text{ for some } B' \in \eta\}$. Define $\xi <_y \eta$ if $L(\xi, \eta)^* \in \hat{\mathcal{N}_y}$.

Lemma 9.1. *Suppose $y \in \beta X \setminus X$ is a regular z-ultrafilter. Any subset $\{\xi_\gamma : \gamma \in \lambda\}$ of Ξ where $\lambda \leq w(X)$ is bounded.*

Proof. Let $\kappa = w(X)$ and let $\{\xi_\gamma : \gamma \in \kappa\} \subset \Xi$. We construct $\xi \in \Xi$ such that $\xi_\gamma <_y \xi$ for all $\gamma \in \kappa$. Let $\{Z_\gamma : \gamma \in \kappa\} \subset y$ be a locally finite subcollection of y. Since X is paracompact, there is a locally finite collection $\mathcal{W} = \{W_\gamma : \gamma \in \kappa\}$ of open subsets of X such that $Z_\gamma \subset W_\gamma$ for all $\alpha \in \kappa$ (see [6] Remark 5.1.19). Note that $W_\gamma \in \mathcal{N}_y$. For each $x \in X$ let $F_x = \{\gamma : x \in \operatorname{cl} W_\gamma\}$ and set $U^0_x = X \setminus \bigcup\{\operatorname{cl} W_\gamma : \gamma \notin F_x\} = X \setminus \operatorname{cl} \bigcup\{W_\gamma : \gamma \notin F_x\}$. For $\gamma \in F_x$ let $\mathcal{C}(x, \gamma) = \{B \in \xi_\gamma : x \in \operatorname{cl} B\}$ and set $\mathcal{C}_x = \bigcup\{\mathcal{C}(x, \gamma) : \gamma \in F_x\}$. Define $U_x = U^0_x \setminus \bigcup\{\operatorname{cl} B : B \in \xi_\gamma \setminus \mathcal{C}_x, \gamma \in F_x\}$. Since ξ_γ is locally finite, U_x is an open neighborhood of x. Choose a finite set of points, $E_x \subset X$, distinct from x such that $|E_x \cap B| \geq 1$ for each $B \in \mathcal{C}_x$. Let $V_x = U_x \backslash E_x$. For $B, B' \in \mathcal{B}$, observe that if $B \subset V_x$, $\gamma \in F_x$, $B' \in \xi_\gamma$ and $B \cap B' \neq \varnothing$ then $B \subsetneq B'$. The collection $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of X. Let $\xi \in \Xi$ be a maximal locally finite collection refining $\mathcal{V}$. Suppose $\gamma \in \kappa$. We will show that $L(\xi_\gamma, \xi)^*$ contains $W_\gamma \cap \xi^* \cap \xi_\gamma^*$, and is therefore dense in W_γ, and hence $\xi_\gamma <_y \xi$.

Let $x' \in W_\gamma \cap \xi^* \cap \xi_\gamma^*$. So, there are $x \in X$, $B \in \xi$, and $B' \in \xi_\gamma$ such that $B \subset V_x$ and $x' \in B \cap B'$. Since $V_x \cap W_\gamma \neq \varnothing$ it must be that $\gamma \in F_x$. Following a previous observation, $B \subsetneq B'$. Hence $x' \in L(\xi_\gamma, \xi)^*$. $\square$

We now discuss cofinal sequences in Ξ. Suppose y is a regular z-ultrafilter on a space X with weight κ. We can use Lemma 9.1 to construct a $<_y$ increasing sequence $\{\xi_\gamma : \gamma \in \kappa^+\}$ in Ξ. If we assume that $2^\kappa = \kappa^+$, since $|\Xi| = 2^\kappa$ we can also arrange that $\{\xi_\gamma : \gamma \in \kappa^+\}$ is cofinal in Ξ. We define $\{\xi_\gamma : \gamma \in \kappa^+\}$ by induction. Since $2^\kappa = \kappa^+$ we may write Ξ as $\{\zeta_\gamma : \gamma \in \kappa^+\}$. Apply Lemma 9.1 to define ξ_γ greater than $\{\xi_\alpha : \alpha < \gamma\} \cup \{\zeta_\gamma\}$.

The reader may have noticed that we did not define equivalence classes on Ξ. The rest of this section is a digression showing the use of an open ultrafilter and an equivalence relation on Ξ to investigate the cofinality of Ξ relative to y. We also compare this cofinality to the cofinality of $C(X)/\mathcal{M}_y$.

Extend $\mathcal{N}_y$ to an open ultrafilter, Ω, on X. Define $\xi =_\Omega \eta$ if $(\xi \cap \eta)^* \in \Omega$ and $\xi \prec \eta$ if $L(\xi, \eta)^* \in \Omega$. Then $\prec$ is a linear order on $\Xi / =_\Omega$, and Lemma 9.1 along with $2^\kappa = \kappa^+$ implies $\operatorname{cf}(\Xi / =_\Omega) = \kappa^+$.

Given a z-ultrafilter y, the cofinalities of $\Xi / =_\Omega$ and $C(X)/\mathcal{M}_y$ are both bounded above by 2^κ. If y is regular, they are both bounded below by κ^+. Hence if y is regular and we assume GCH, they are equal because they are both κ^+. On the other hand, in Example 7.1 they are equal because they are both ω. If y is a remote point we can directly show that they are equal, without extra axioms of set theory (Proposition 9.3).

Definition 9.2. Let y be a z-ultrafilter on a metric space X. We say that y is a *remote point* if $Z \notin y$ whenever Z is a nowhere dense subset of X.

In the case that y is a remote point, $\Omega = \mathcal{N}_y$ is already a free open ultrafilter on X; there is no need to extend. Consequently $<_y$ is a linear order on $\Xi / =_\Omega$.

Proposition 9.3. *Let X be a crowded metric space and let y be a remote point in βX. Then,* $\mathrm{cf}(\Xi / =_\Omega) = \mathrm{cf}(C(X)/\mathcal{M}_y)$.

Proof. Let $\mathcal{B}$ be a nice π-base. For each $B \in \mathcal{B}$ and $x \in X$ define $g_B(x) = \frac{1}{n} \cdot \min\{d(x, X \backslash B), 1\}$ where n is such that $B \in \mathcal{B}_n$. For each $\xi \in \Xi$ and $x \in X$ define $g_\xi(x) = \sum_{B \in \xi} g_B(x)$. The function g_ξ is well defined since ξ is pairwise disjoint and continuous since ξ is locally finite. We now show that $\xi <_{\mathcal{O}} \eta \iff g_\eta <_y g_\xi$. For $B, B' \in \mathcal{B}$, we have that $B \subsetneq B' \iff g_B(x) < g_{B'}(x)$ for all $x \in B \iff g_B(x) < g_{B'}(x)$ for some $x \in B$. Suppose $\xi <_{\mathcal{O}} \eta$. Let $\mathcal{U} = \{B \in \eta : \exists B' \in \xi, B \subsetneq B'\}$, then $\mathcal{U}^* \in \mathcal{O}$. Since $\mathcal{O}$ extends $\mathcal{O}_y$, it must be that $\mathrm{cl}\mathcal{U}^* \in y$. However, since $\mathrm{cl}\mathcal{U}^* \backslash \mathcal{U}^*$ is nowhere dense, there is $Z \in y$ such that $Z \subset \mathcal{U}^*$. Hence $g_\eta(x) < g_\xi(x)$ for all $x \in Z$ and therefore $g_\eta <_y g_\xi$. Now suppose $g_\eta <_y g_\xi$ and let $Z \in y$ be such that $g_\eta(x) <_y g_\xi(x)$ for all $x \in Z$. If $B \in \eta$ is such that $B \cap Z \neq \emptyset$ then there is $x \in B$ such that $g_\eta(x) < g_\xi(x)$. It follows that there is $B' \in \xi$ such that $B \subsetneq B'$. So, $\{B \in \eta : B \cap Z \neq \emptyset\} = \mathcal{U}$ where $\mathcal{U} = \{B \in \eta : \exists B' \in \xi, B \subsetneq B'\}$. Since ξ^* is dense in X, $Z \subset \mathrm{cl}\mathcal{U}^*$ and hence $\mathrm{cl}\mathcal{U}^* \in y$. As before, $\mathrm{cl}\mathcal{U}^* \backslash \mathcal{U}^* \notin y$, so there is $Z' \in y$ such that $Z' \subset \mathcal{U}^*$. Therefore $\mathcal{U}^* \in \mathcal{O}$ and $\xi <_{\mathcal{O}} \eta$. This shows that $g_{(\cdot)}$ is an order reversing map from $\Xi/\theta_{\mathcal{O}}$ to $C(X)/\mathcal{M}_y$.

Let $C^+(X) = \{f \in C(X) : \mathbf{0} <_y f\}$. To see that $\mathrm{cf}(\Xi/\theta_{\mathcal{O}}) = \mathrm{cf}(C(X)/\mathcal{M}_y)$ we argue that for each $\xi \in \Xi$ there is $f \in C^+(X)$ such that $f <_y g_\xi$ and for each $f \in C^+(X)$ there is $\xi \in \Xi$ such that $g_\xi <_y f$. Let $\xi \in \Xi$. The zero set of g_ξ is exactly $X \backslash \xi^*$, which is nowhere dense. So, there is $Z \in y$ such that $g_\xi(x)$ is positive for all $x \in Z$. The continuous function $f = \frac{1}{2} g_\xi$ satisfies $0 < f(x) < g_\xi(x)$ for all $x \in Z$. Hence $\mathbf{0} <_y f <_y g_\xi$. Now suppose that $f \in C^+(X)$. Let $Z(f) = \{x \in X : f(x) = 0\}$. Since $\mathbf{0} <_y f$, there is $Z \in y$ such that $Z \cap Z(f) = \emptyset$. Define $U_0 = f^{\leftarrow}[(\frac{1}{2}, \infty)]$ and for each $n \in \mathbb{N}$ define $U_n = f^{\leftarrow}[(\frac{1}{n+2}, \frac{1}{n})]$. The collection $\mathcal{W} = \{X \backslash Z\} \cup \{U_n : n \in \omega\}$ is an open cover of X. Let $\eta \in \Xi$ be a dense refinement of $\mathcal{W}$. For each $B \in \eta$ such that $B \cap Z \neq \emptyset$, there is $n_B \in \omega$ such that $B \subset U_n$ and therefore $f(x) > \frac{1}{n_B+3}$ for all $x \in B$. Let $m(B)$ be the maximum of $n_B + 3$ and the $n \in \omega$ for which $B \in \mathcal{B}_n$. For each $B \in \eta$ such that $B \cap Z \neq \emptyset$, define $\mathcal{V}(B) = \{B' \in \mathcal{B}_{m(B)} : B' \subset B\}$. Since $\mathcal{V}(B) \subset \mathcal{B}_{m(B)}$ it is a locally finite collection in X. Let $\mathcal{V} = \bigcup\{\mathcal{V}(B) : B \in \eta, B \cap Z \neq \emptyset\}$. Then, $\xi = \mathcal{V} \cup \{B \in \eta : B \subset X \backslash Z\}$ is an element of Ξ. If $x \in B' \in$

$\mathcal{V}(B)$, then $g_{B'}(x) \leq \frac{1}{m_B} \leq \frac{1}{n_B+3} < f(x)$. Therefore $g_\xi(x) < f(x)$ for all $x \in \mathcal{V}^*$. Since $Z \subset \operatorname{cl}\mathcal{V}^*$ and $\operatorname{cl}\mathcal{V}^* \setminus \mathcal{V}^* \notin y$, there must be $Z' \in y$ such that $Z' \subset Z$ and $Z' \subset \mathcal{V}^*$. Therefore $g_\xi <_y f$. $\square$

10. H'S AND $\mathcal{L}$'S

Suppose y is a z-ultrafilter on a crowded metric space X with weight κ. Following Logonov [15] and Terasawa [19], in this section we use a cofinal sequence from Ξ to define a sequence of closed sets intersecting to y.

Suppose $\{\xi_\gamma : \gamma \in \theta_y\}$ is a cofinal $<_y$-increasing sequence in Ξ. We note now that $\theta_y \leq 2^\kappa$ and make extra assumptions on θ_y later. Without loss of generality we may assume that $\xi_\gamma \cap \mathcal{B}_0 = \varnothing$; replace ξ_γ with $(\xi_\gamma \setminus \mathcal{B}_0) \cup \{B \in \mathcal{B}_1 : \exists B' \in \xi_\gamma \cap \mathcal{B}_0, B \subset B'\}$. Let $\mathcal{N}_\gamma = \{\mathcal{U} \subset \xi_\gamma : \mathcal{U}^* \in \hat{\mathcal{N}_y}\}$ and let

$$H_\gamma = \bigcap \{\operatorname{cl}_{\beta X}(\mathcal{U}^*) : \mathcal{U} \in \mathcal{N}_\gamma\}.$$

If $\mathcal{U}^*$ and $\mathcal{V}^*$ are dense in N and N' from $\mathcal{N}_y$, then $\mathcal{U}^* \cap \mathcal{V}^*$ is dense in $N \cap N'$ which is also in $\mathcal{N}_y$. Hence, $\mathcal{N}_\gamma$ is a filter on ξ_γ. Every $U \in \hat{\mathcal{N}_y}$ is dense in some $N \in \mathcal{N}_y$, the trace of a neighborhood of y on X. Therefore, $y \in \operatorname{cl}_{\beta X} U$ for all $U \in \hat{\mathcal{N}_y}$. Hence $y \in H_\gamma$ for all $\gamma \in \theta_y$.

Claim. *For each $\gamma \in \theta_y$, $H_\gamma \subset \beta X \setminus X$.*

Proof. By Lemma 8.3 5) for any $B \in \xi_\gamma$, since $y \notin cl_{\beta X} B$ it must be that $\xi_\gamma \setminus \{B\} \in \mathcal{N}_\gamma$. Fix $x \in X$. Since ξ_γ is locally finite, $\mathcal{U} = \{B \in \xi_\gamma : x \in \operatorname{cl}_{\beta X} B\}$ is finite and hence $\xi_\gamma \setminus \mathcal{U} \in \mathcal{N}_\gamma$. Also, $x \notin \operatorname{cl}_{\beta X}(\xi_\gamma \setminus \mathcal{U})^*$ and therefore $x \notin H_\gamma$. $\square$

Claim. *If $\gamma' < \gamma$ then $H_\gamma \subset H_{\gamma'}$.*

Proof. Let $\gamma' < \gamma$ and let $\mathcal{U} \in \mathcal{N}_{\gamma'}$. We will show that $H_\gamma \subset cl_{\beta X}\mathcal{U}^*$. Since $\gamma' < \gamma$, $\xi_{\gamma'} <_y \xi_\gamma$ and therefore $L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Hence $\mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Since $\mathcal{U}, L(\xi_{\gamma'}, \xi_\gamma) \subset \xi_{\gamma'}$ we have that $\mathcal{U}^* \cap L(\xi_{\gamma'}, \xi_\gamma)^* = (\mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma))^*$. Let $\mathcal{W} = \mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma)$ and let $\mathcal{V} = \{V \in \xi_\gamma : V \cap U \neq \varnothing \text{ for some } U \in \mathcal{W}\}$. Since ξ_γ^* is dense in X, $cl_X\mathcal{V}^* \supset \mathcal{W}^*$. Furthermore, $V \in \xi_\gamma$, $U \in L(\xi_{\gamma'}, \xi_\gamma)$ and $V \cap U \neq \varnothing$ implies that $V \subset U$. Therefore $\mathcal{V}^* \subset \mathcal{W}^*$ and hence $cl_X\mathcal{V}^* = cl_X\mathcal{W}^*$. Since $\mathcal{W}^* \in \hat{\mathcal{N}_y}$ and $\mathcal{V}^*$ is dense in $\mathcal{W}^*$ we have that $\mathcal{V} \in \mathcal{N}_\gamma$. Therefore, $H_\gamma \subset cl_{\beta X}\mathcal{V}^* = cl_{\beta X}\mathcal{W}^* \subset cl_{\beta X}\mathcal{U}^*$. $\square$

Claim. $\bigcap\{H_\gamma : \gamma \in \theta_y\} = \{y\}$.

Proof. We have seen that $y \in \bigcap\{H_\gamma : \gamma \in \theta_y\}$. Let $O' \in \tau_y$. We will find $\gamma \in \theta_y$ such that $H_\gamma \subset O'$. Let $W', U' \in \tau_y$ be such that

$$\mathrm{cl}_{\beta X} W' \subset W' \subset \mathrm{cl}_{\beta X} U \subset O.$$

Let $O = O' \cap X, U = U' \cap X$ and $W = W' \cap X$. So, $cl_X W \subset U \subset cl_X U \subset O$. Let $V = X \setminus cl_X W$. Then $\{U, V\}$ is an open cover of X. By Proposition 8.2 there is $\xi \in \Xi$ that refines $\{U, V\}$. Let $\gamma \in \theta_y$ be such that $\xi <_y \xi_\gamma$. Note that $W \in \mathcal{N}_y$. Since $\xi <_y \xi_\gamma$ we have $L(\xi, \xi_\gamma)^* \in \hat{\mathcal{N}_y}$ and $W \cap L(\xi, \xi_\gamma)^* \in \hat{\mathcal{N}_y}$. Let $\hat{W} = W \cap L(\xi, \xi_\gamma)^*$ and let $\mathcal{V} = \{B \in \xi_\gamma : B \cap \hat{W} \neq \varnothing\}$. Since ξ_γ^* is dense in X and $\hat{W}$ is open, $cl_X \mathcal{V}^* \supset \hat{W}$. Hence $\mathcal{V}^* \in \hat{\mathcal{N}_y}$. On the other hand, if $B \in \mathcal{V}$ then $B \cap L(\xi, \xi_\gamma)^* \neq \varnothing$ and therefore $B \subset B'$ for some $B' \in \xi$. Since ξ refines $\{U, V\}$, either $B \subset B' \subset U$ or $B \subset B' \subset V$. Since $B \cap W \neq \varnothing$, it cannot be the case that $B \subset V$. Therefore $B \subset U$ and hence $\mathcal{V}^* \subset U$ and $cl_X \mathcal{V}^* \subset cl_X U \subset O$. Then, since X is normal, $cl_{\beta X} \mathcal{V}^* \subset O'$. Since $\mathcal{V} \subset \xi_\gamma$ and $\mathcal{V}^* \in \hat{\mathcal{N}_y}$ we have that $H_\gamma \subset cl_{\beta X} \mathcal{V}^* \subset O'$ as desired. $\square$

Continuing with the cofinal sequence $\{\xi_\gamma : \gamma < \theta_y\}$ we define a pair of locally finite collections, $\mathcal{L}_\gamma^0$ and $\mathcal{L}_\gamma^1$ from $\mathcal{B}$ such that $\mathrm{cl}(\mathcal{L}_\gamma^0)^* \cap \mathrm{cl}(\mathcal{L}_\gamma^0)^* = \varnothing$. In our induction, we must do θ_y many tasks, and each step of the induction can have at most κ predecessors. Now we assume $2^\kappa = \kappa^+$ to get $\theta_y \leq \kappa^+$. The constructions of the $\mathcal{L}$'s for the two types of spaces are not the same. However, in either case, the pairs will be used for the same purpose; to 'split' the H_γ's.

10.1. **X locally compact.** We are able to arrange the cofinal sequence of collections $\{\xi_\gamma : \gamma \in \theta_y\}$ as "step functions" which makes the definition of the $\mathcal{L}'s$ easier than in the κ^ω-like case. List $\mathcal{B}_0 = \{B_{\alpha,\varnothing} : \alpha \in \kappa\}$ and $\mathcal{B}_i = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in {}^i 4\}$ such that $B_{\alpha,\sigma} \subset B_{\alpha,\sigma'}$ if σ extends σ'. We may assume that for $\alpha \in \kappa$ and $\sigma \in {}^i 4$, $\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} \cap \mathrm{cl}_X B_{\alpha,\sigma^\frown 1} = \varnothing$ and $\mathrm{cl}_X B_{\alpha,\sigma^\frown 0}, \mathrm{cl}_X B_{\alpha,\sigma^\frown 1} \subset B_{\alpha,\sigma}$. Notice that the collections ξ from Ξ that have the property that $B_{\alpha,\sigma}, B_{\alpha,\sigma'} \in \xi$ implies $|\sigma| = |\sigma'|$ form an unbounded set in Ξ. To see this, let $\xi' \in \Xi$ and let $n(\alpha) = \max\{|\sigma| : B_{\alpha,\sigma} \in \xi'\} + 1$. Then the collection $\xi = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in {}^{n(\alpha)} 4\}$ has the property that $\xi >_y \xi'$ since $L(\xi', \xi) = \xi'$.

Therefore, we may assume that $\{\xi_\gamma : \gamma \in \theta_y\}$ is a sequence of collections that have the property that for each $\gamma \in \theta_y$ and $\alpha \in \kappa$ if

$B_{\alpha,\sigma}, B_{\alpha,\sigma'} \in \xi_\gamma$ then $|\sigma| = |\sigma'|$. For each $\gamma \in \theta_y$ define the function $n(\gamma,\cdot) : \kappa \to \omega$ such that $\xi_\gamma = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in {}^{n(\gamma,\alpha)}4\}$. Notice that for any $\gamma' < \gamma < \theta_y$ the set $L(\xi_\gamma, \xi_{\gamma'})^*$ is dense in $\{B_\alpha : \alpha \in S\}^*$ for non-empty set $S \subset \kappa$.

Defining the $\mathcal{L}^i_\gamma$'s

For $\gamma \in \theta_y$ and $i = 0, 1$ define $\mathcal{L}^i_\gamma = \{B_{\alpha,\sigma^\frown i} : \alpha \in \kappa, \sigma \in {}^{n(\gamma,\alpha)}4\}$.

Claim. *For all* $\gamma \in \theta_y$, $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \varnothing$.

Proof. For each $\alpha \in \kappa$ and $\sigma \in {}^i4$, $\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} \cap \mathrm{cl}_X B_{\alpha,\sigma^\frown 1} = \varnothing$. Also, $B_{\alpha,\sigma} \cap B_{\alpha,\beta} = \varnothing$ for $\sigma \neq \beta \in {}^{n(\gamma,\alpha)}4$, and for $i = 0, 1$ we have $\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \subset B_{\alpha,\sigma}$ and $\mathrm{cl}_X B_{\alpha,\beta^\frown i} \subset B_{\alpha,\beta}$. Therefore

$$\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \cap \mathrm{cl}_X B_{\alpha,\beta^\frown j} = \varnothing$$

for $i, j = 0, 1$. So,

$$\bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4\} \cap \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4\} = \varnothing.$$

Now, since $\{B_{\alpha,\varnothing} : \alpha \in \kappa\}$ is a locally finite family and since $\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \subset B_{\alpha,\varnothing}$ for each $\sigma \in \bigcup_{n\in\omega} {}^n4$ and $i = 0, 1$, we have that

$$\mathrm{cl}_X(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_X(\bigcup \mathcal{L}^1_\gamma)$$

$$= \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4, \alpha \in \kappa\} \cap \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4, \alpha \in \kappa\}$$

$$= \varnothing.$$

Finally, since $\mathrm{cl}_X(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_X(\bigcup \mathcal{L}^1_\gamma) = \varnothing$ we have that $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \varnothing$. □

Since $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \varnothing$, y can be in at most one of $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma)$ or $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma)$. Without loss of generality, assume $y \notin \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma)$ for each $\gamma \in \theta_y$.

Consider a finite collection $\{\xi_{\gamma_i} : i \in m\} \subset \{\xi_\gamma : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq m$ and let $U(i,j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i,j)^* \in \hat{\mathcal{N}}_y$ for each $i < j$ and hence $U = \bigcap\{U(i,j)^* : i < j \leq m\} \in \hat{\mathcal{N}}_y$. For any $B \in \xi_{\gamma_0}$ such that $B \cap U \neq \varnothing$ we have that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_j : B' \subset B\}$ whenever $0 < j < i \leq m$.

A special case of the following claim, in particular when Φ is constant, is proven in [[19], Lemma 3 and [15], Proposition 6].

Claim 10.1. *For any $\rho < \theta_y$ and $\Phi : D \subset [\rho, \theta_y) \to 2$, the collection $\{H_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \in D\}$ has nonempty intersection.*

Proof. Let $\rho < \theta_y$ and $\Phi : D \to 2$ for some $D \subset [\rho, \theta_y)$. We will show that $\{\mathrm{cl}_{\beta X}\,\mathcal{U}^* : \mathcal{U} \in \mathcal{N}_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \geq \rho\}$ has the finite intersection property. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{N}_\rho$ and let $\gamma_1, \ldots, \gamma_m \in D$ be such that $\gamma_m \geq \cdots \geq \gamma_1 \geq \rho$. Since $\mathcal{N}_\rho$ is a filter, $\mathcal{U} = \bigcap\{\mathcal{U}_i : 1 \leq i \leq n\} \in \mathcal{N}_\rho$ and therefore $V = \mathcal{U}^* \in \hat{\mathcal{N}_y}$. For $i < j \leq m$, let $U(i,j)^* = L(\xi_{\gamma_i}, \xi_{\gamma_j})$ and notice that $U = \bigcap\{U(i,j)^* : i < j \leq m\} \in \hat{\mathcal{N}_y}$. Let $B_{\alpha,\sigma} \in \xi_\rho$ be such that $B_{\alpha,\sigma} \subset V$ and $B_{\alpha,\sigma} \cap U \neq \varnothing$. As noted before, $\{B \in \gamma_i : B \subset B_{\alpha,\sigma}\}$ refines $\{B \in \gamma_j : B \subset B_{\alpha,\sigma}\}$ whenever $0 < j < i \leq m$. Define $\sigma' \in {}^{n(\gamma_m,\alpha)+1}4$ as follows: $\sigma'|_{n(\rho,\alpha)} = \sigma$, $\sigma'(n(\gamma_i,\alpha)+1) = \Phi(\gamma_i)$ for each $1 \leq i \leq m$ and $\sigma'(k) = 0$ otherwise. Then, $B_{\alpha,\sigma'} \subset B_{\alpha,\sigma}$, since σ' extends σ hence $B_{\alpha,\sigma'} \subset \mathcal{U}^*$. Furthermore, $B_{\alpha,\sigma} \subset \bigcup \mathcal{L}_{\gamma_i}^{\Phi(\gamma_i)}$ since σ' extends $\sigma'|_{n(\gamma_i,\alpha)+1} = \sigma'|_{n(\gamma_i,\alpha)}{}^\frown\Phi(\gamma_i)$ and $B_{\alpha,\sigma'|_{n(\gamma_i,\alpha)}{}^\frown\Phi(\gamma_i)} \in \mathcal{L}_{\gamma_i}^{\Phi(\gamma_i)}$. □

10.2. **X κ^ω-like.** Consider a finite collection $\{\xi_{\gamma_i} : i \in n\} \subset \{\xi_\gamma : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq n$ and let $U(i,j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i,j)^* \in \hat{\mathcal{N}_y}$ for each $i < j$ and hence $U = \bigcap\{U(i,j)^* : i < j \leq n\} \in \hat{\mathcal{N}_y}$. It is tempting to assume that, as in the locally compact case, $\{B \in \xi_{\gamma_0} : B \subset \mathrm{cl}\,U\} \neq \varnothing$. However, there may not exist $B \in \xi_{\gamma_0}$ such that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_j : B' \subset B\}$ whenever $0 < j < i \leq n$.

Defining the $\mathcal{L}_\gamma^i$'s We define $\{\mathcal{L}_\gamma^i : i \in 2, \gamma \in \theta_y\}$ by induction on $\gamma \in \theta_y$.

Let $P = \{p : \mathrm{dom}(p) \in [\theta_y]^{<\omega}, \mathrm{ran}(p) \subset 2\}$. Let $\gamma_p = \max(\mathrm{dom}(p))$ and $n(p) = |p|$. Define $p|_i$ to be the function p restricted to the first i elements of $\mathrm{dom}(p)$. We say $B \in \mathcal{B}$ and $p \in P$ are *aligned* if for each $\gamma \in \mathrm{dom}(p)$ and $B' \in \xi_\gamma$ such that $B' \cap B \neq \varnothing$, $B' \subsetneq B$. We will define $\mathcal{L}(B,p)$ for each B and p and set

$$\mathcal{L}_\gamma^i = \bigcup\{\mathcal{L}(B,p) : \gamma_p = \gamma \text{ and } p(\gamma) = i\}.$$

If B and p are not aligned, set $L(B,p) = \varnothing$.

Stage $\gamma = 0$: There are two $p \in P$ with $\mathrm{dom}(p) = \{0\}$, namely $p^0 = \{(0,0)\}$ and $p^1 = \{(0,1)\}$. Notice that $B \in \mathcal{B}$ is aligned with p^0 or p^1 if there exists $B' \in \xi_0$ such that $B' \subsetneq B$ and that there are κ

such B. List as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$, all pairs (B, p) such that $p = p^0$ or $p = p^1$ and B is aligned with p, so that each (B, p) appears in the list κ times. We will define a sequence $\{L(\nu) : \nu \in \kappa\}$ and for each p and B aligned with p, we will set $\mathcal{L}(B, p) = \{L(\nu) : (B, p) = (B_\nu, p_\nu)\}$.

Suppose we have defined $L(\mu) \in \mathcal{B}$ for each $\mu < \nu$ such that $L(\mu) \subsetneq V_\mu \subsetneq B_\mu$ where V_μ is some element of ξ_0. Also assume that if $L(\mu), L(\mu') \subset V \in \xi_0$, then $\mu = \mu'$. We now define $L(\nu)$. For each $V \in \xi_0$ such that $V \cap B_\nu \neq \varnothing$ there is $\eta \in \kappa$ such that $V \subset B^\eta_\nu$. Furthermore, since ξ^*_0 is dense in X, for each $\eta \in \kappa$ there is $V \in \xi_0$ such that $V \subset B^\eta_\nu$. For each $\mu < \nu$, $L(\mu)$ is contained in an element V of ξ_0 and $|\nu| < \kappa$. Therefore, there are κ many $\eta \in \kappa$ such that for all $\mu < \nu$, $B^\eta_\nu \cap L(\mu) = \varnothing$. So, let η_0 be one such η and choose $L(\nu) \in \mathcal{B}$ so that $L(\nu) \subsetneq V_\nu \subset B^{\eta_0}_\nu \subsetneq B_\nu$ for some $V_\nu \in \xi_0$.

For $p = p^0$ or p^1 and each B aligned with p, set

$$\mathcal{L}(B, p) = \{L(\nu) : (B, p) = (B_\nu, p_\nu)\}.$$

Let

$$\mathcal{L}^i_0 = \bigcup\{\mathcal{L}(B, p) : p = p^i \text{ and } B \text{ aligned with } p\}.$$

Notice that if $L(\nu), L(\mu) \subset B' \in \xi_0$ then $\nu = \mu$. So, since ξ_0 is locally finite, $\mathrm{cl}(\bigcup \mathcal{L}^0_0)$ is disjoint from $\mathrm{cl}(\bigcup \mathcal{L}^1_0)$. Since each (B, p) is listed κ times, $|\{\nu : L(B_\nu, p_\nu) \subsetneq B\}| = \kappa$. Consequently, $|\{\eta \in \kappa :$ there is $L \in \mathcal{L}(B, p), L \subset B^\eta\}| = \kappa$.

Induction Hypothesis Let B and p be aligned such that $\gamma_p \leq \gamma$ and $n(p) > 1$. Then, for κ many $\eta \in \kappa$, there is a sequence $\{L_i : 0 \leq i < n(p), L_i \in \mathcal{L}(B, p|_i)\}$ such that:

$$L_{n(p)-1} \subset L_{n(p)-2} \subset \cdots \subset L_0 \subset B^\eta \subset B.$$

Also, for each $\gamma' < \gamma$, $\mathrm{cl}(\bigcup \mathcal{L}^0_{\gamma'})$ is disjoint from $\mathrm{cl}(\bigcup \mathcal{L}^1_\gamma)$.

Stage γ: Consider all (B, p) such that $\gamma_p = \gamma$ and B is aligned with p. We have assumed $2^\kappa = \kappa^+$. So, $\gamma < \kappa^+$ and hence there are $\leq \kappa$ many p with $\gamma_p = \gamma$. Therefore, we can list the collection of such (B, p) as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$ such that each (B, p) appears κ times. Assume we have defined $L(\mu) \in \mathcal{B}$ for each $\mu < \nu$ such that such that $L(\mu) \subsetneq V_\mu \subsetneq B_\mu$ where V_μ is some element of ξ_γ. Also assume that if $L(\mu), L(\mu') \subset V \in \xi_\gamma$, then $\mu = \mu'$. Let $\eta \in \kappa$ be such that there is $\{L_i : 0 \leq i < n(p_\nu), L_i \in \mathcal{L}(B_\nu, p_\nu|_i)\}$ such that:
$L_{n(p_\nu)-1} \subset L_{n(p_\nu)-2} \subset \cdots \subset L_0 \subset B^\eta_\nu \subset B_\nu$. Since we have defined $L(\mu)$ for $|\nu| < \kappa$ many μ, by the inductive hypothesis we may also assume that η is such that $B^\eta_\nu \cap L(\mu) = \varnothing$ for all $\mu < \nu$.

Let $V \in \xi_\gamma$ be such that $L_{n(p_\nu)-1} \cap V \neq \varnothing$. Let $L(\nu)$ be an element of $\mathcal{B}$ such that

$$L(\nu) \subsetneq (V \cap L_{n(p_\nu)-1}) \subset L_{n(p_\nu)-2} \subset \cdots \subset L_0 \subset B_\nu^\eta \subset B_\nu.$$

Set $\mathcal{L}(B,p) = \{L(\nu) : (B_\nu, p_\nu) = (B,p)\}$ and observe that

$$\left(\bigcup \mathcal{L}(B,p)\right) \cap \bigcap \left\{ \bigcup \mathcal{L}(B, p|_i) : i < n(p) \right\} \neq \varnothing.$$

Now, set $\mathcal{L}_\gamma^i = \bigcup\{\mathcal{L}(B,p) : \gamma_p = \gamma \text{ and } p(\gamma) = i\}$. This concludes stage γ.

For each p and B aligned with p, we have that

$$(\bigcup \mathcal{L}(B,p)) \cap \bigcap \{\bigcup \mathcal{L}(B,p|_i) : i < n(p)\} \neq \varnothing.$$

Therefore, if $\mathrm{dom}(p) \setminus \{\gamma_p\} = \{\gamma_i : 1 \leq i < n(p)\}$, then $\bigcap\{\mathcal{L}_{\gamma_i}^{p(\gamma_i)} : i < n(p)\} \cap B \neq \varnothing$.

Claim 10.2. *For any $\rho < \theta_y$ and $\Phi : D \subset [\rho, \theta_y) \to 2$, the collection $\{H_\rho\} \cup \{cl_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \in D\}$ has nonempty intersection.*

Proof. Let $\rho < \theta_y$ and $\Phi : D \to 2$ for some $D \subset [\rho, \theta_y)$. We will show that $\{cl_{\beta X}\mathcal{U}^* : \mathcal{U} \in \mathcal{N}_\rho\} \cup \{cl_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \geq \rho\}$ has the finite intersection property. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{N}_\rho$ and let $\gamma_1, \ldots, \gamma_m \in D$ be such that $\gamma_m > \cdots > \gamma_1 > \rho$. For each $i \leq m$, $L(\xi_\rho, \xi_{\gamma_i}) \in \mathcal{N}_\rho$ since $\xi_{\gamma_i} > \xi_\rho$. Hence, $\mathcal{U} = \bigcap\{\mathcal{U}_i : 1 \leq i \leq n\} \cap \bigcap\{L(\xi_\rho, \xi_{\gamma_i}) : 1 \leq i \leq m\} \in \mathcal{N}_\rho$. Let p be the function Φ restricted to $\{\gamma_i : 1 \leq i \leq m\}$. Note that if $B \in \mathcal{U}$ then B is aligned with p. From the previous construction we have that $\bigcap\{\bigcup \mathcal{L}_{\gamma_i}^{p(\gamma_i)} : i \leq m\} \cap B \neq \varnothing$.

$\square$

11. Theorems

Theorem 11.1. *Let X be a metric space of weight κ without isolated points that is either locally compact or κ^ω-like. Let $y \in \beta X \setminus X$. Suppose that $2^\kappa = \kappa^+$ and $[\theta_y]^{<\theta_y} = \theta_y$. Then there is a closed copy of $NU(\theta_y)$ in $(\beta X \setminus X) \setminus \{y\}$.*

Proof. We follow the argument found in [1] to embed $NU(\theta_y)$ into $(\beta X \setminus X) \setminus \{y\}$, using the $\mathcal{L}_\gamma$'s to play the role of the reaping sets.

The induction

Denote by θ_y the discrete space of size θ_y. We define an embedding, g, of θ_y into $\beta X \setminus X$ such that

(1) $y \in cl_{\beta X} g[A]$ if and only if $|A| = \theta_y$.
(2) If $A, B \in [\theta_y]^{<\theta_y}$ and $A \cap B = \varnothing$ then $cl_{\beta X} g[A] \cap cl_{\beta X} g[B] = \varnothing$.

Then, we extend g to $\beta g : \beta\theta_y \to \beta X \setminus X$ and prove that $U(\theta_y) = g^{\leftarrow}[\{y\}]$. Therefore $(\beta X \setminus X) \setminus \{y\}$ contains a closed copy of $NU(\theta_y)$ and is therefore not normal.

By assumption, we have that $|\theta_y|^{<\theta_y} = \theta_y$. List $\theta_y \cup \{(A, B) : A, B \in [\theta_y]^{<\theta_y}$ and $A \cap B = \varnothing\}$ as $\{T_\eta : \eta \in \theta_y\}$ in such a way that if $T_\eta = (A, B)$, then $\eta \geq \sup(A \cup B)$ and if $T_\eta \in \theta_y$, then $\eta \geq T_\eta$. For $\rho \in \theta_y$ let $D_\rho = \{\eta : T_\eta = (A, B)$ and $\rho \in A \cup B\} \cup \{\eta : \rho \in T_\eta\}$. Note that $D_\rho \subset [\rho, \theta_y)$.

For each $\rho \in \theta_y$ we define $\Phi_\rho : D_\rho \to 2$ and choose $g(\rho)$ to be any element of $\bigcap(\{H_\rho\} \cup \{cl_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi_\rho(\gamma)}) : \gamma \in D\})$. We define Φ_ρ by induction.

Let $\eta \in \theta_y$ and assume we have defined $\Phi_\rho|_{\eta \cap D_\rho}$. If $T_\eta \in \theta_y$, let $\Phi_\beta(\eta) = 0$ for all $\beta < T_\eta$. If $T_\eta = (A, B)$, let $\Phi_\beta(\eta) = 0$ for all $\beta \in A$ and let $\Phi_\beta(\eta) = 1$ for all $\beta \in B$. Let $K_\rho = \bigcap(\{H_\rho\} \cup \{cl_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi_\rho(\gamma)}) : \gamma \in D_\rho\}) = \varnothing$. By the Claims 10.1 and 10.2, $K_\rho \neq \varnothing$ for each $\rho \in \theta_y$, so we may choose $g(\rho) \in K_\rho$.

To show (1), let $A \subset \theta_y$ be such that $|A| < \theta_y$. There is $\gamma \in \theta_y$ such that $A \subset [0, \gamma)$. Let η be such that $T_\eta = \gamma$. Note, $\eta \geq \gamma$. For any $\rho < \gamma = T_\eta$, $\Phi_\rho(\eta) = 0$. So, for $\rho \in A$, $K_\rho \subset \mathcal{L}_\eta^0$. But, $y \notin cl_{\beta X}(\bigcup \mathcal{L}_\eta^0)$. Hence, $y \notin cl_{\beta X} g[A]$. For the other direction, let $A \subset \theta_y$ be such that $|A| = \theta_y$. Since θ_y is regular, A is unbounded in θ_y. Let $U \in \mathcal{N}$. There is $\gamma \in \theta_y$ such that $H_\gamma \subset U$. For $\rho \geq \gamma$, $g(\rho) \in H_\rho \subset H_\gamma \subset U$. Hence $y \in cl_{\beta X} g[A]$.

To show (2), let $A, B \in [\theta_y]^{<\theta_y}$ be such that $A \cap B = \varnothing$. Let η be such that $T_\eta = (A, B)$. Then, for each $\rho \in A$, $\Phi_\rho(\eta) = 0$ and for each $\rho \in B$, $\Phi_\rho(\eta) = 1$. Hence $g(\rho) \in K_\rho \subset cl_{\beta X}(\bigcup \mathcal{L}_\eta^0)$ for $\rho \in A$ and $g(\rho) \in K_\rho \subset cl_{\beta X}(\bigcup \mathcal{L}_\eta^1)$ for $\rho \in B$. But, $cl_{\beta X}(\bigcup \mathcal{L}_\eta^0) \cap cl_{\beta X}(\bigcup \mathcal{L}_\eta^1) = \varnothing$. Hence $cl_{\beta X} g[A] \cap cl_{\beta X} g[B] = \varnothing$. Note, (2) implies g is one-to-one.

Since θ_y is discrete, g is continuous. Extend g to $\beta g : \beta\theta_y \to \beta X \setminus X$.

Claim. *$\beta g[NU(\theta_y)]$ is a closed homeomorphic copy of $NU(\theta_y)$ in $(\beta X \setminus X) \setminus \{y\}$.*

Proof. To show $\beta g[NU(\theta_y)]$ is a closed subset of $(\beta X \setminus X) \setminus \{y\}$, we show that $\beta g[\beta\theta_y] \setminus \{y\} = \beta g[NU(\theta_y)]$. Let $y \in NU(\theta_y)$. There is $A \subset \theta_y$ such that $|A| < \theta_y$ and $A \in y$. Since βg is continuous, $g(y) \in cl_{\beta X} g[A]$. Hence, $g(y) \neq y$. Let $z \in \beta g[\beta\theta_y] \setminus \{y\}$. Let U be an open neighborhood of z such that $y \notin cl_{\beta X} U$. Since βg is continuous, $A' = U \cap \beta g[\theta_y] \neq \varnothing$. Let $A = g^{\leftarrow}[A']$. Since $y \notin U$, $|A| < \theta_y$. If $y \in g^{\leftarrow}(z)$ then $y \in cl_{\beta\theta_y} A$. Hence $y \in NU(\theta_y)$ and therefore $z \in \beta g[NU(\theta_y)]$.

It is left to be shown that βg is one-to-one on $NU(\theta_y)$. Let $y \neq y' \in NU(\theta_y)$. There are $A, B \in [\theta_y]^{<\theta_y}$ such that $A \cap B = \varnothing$ and $y \in cl_{\beta\theta_y} A$ and $y' \in cl_{\beta\theta_y} B$. By continuity, $g(y) \in cl_{\beta X} g[A]$ and $g(y') \in cl_{\beta X} g[B]$. But, by (2) $cl_{\beta X} g[A] \cap cl_{\beta X} g[B] = \varnothing$. Hence $g(y) \neq g(y')$. $\square$

The claim shows that $NU(\theta_y)$ is embedded as a closed subset of $(\beta X \setminus X) \setminus \{y\}$, and therefore the theorem is proved. $\square$

Theorem 11.2. *($2^\kappa = \kappa^+$) Let X be a metric space of weight κ without isolated points that is either locally compact or κ^ω-like. Any regular z-ultrafilter is a non-normality point of $\beta X \setminus X$.*

Proof. Since y is regular, by lemma 9.1 $\theta_y > \kappa$. By the hypothesis, $\theta_y = \kappa^+ = 2^\kappa$ and hence θ_y is regular and not a strong limit. By 3.2, $NU(\theta_y)$ is not normal. Hence, by Theorem 11.1, y is a non-normality point of $\beta X \setminus X$. $\square$

Corollary 11.3. *Suppose GCH+UR. Let X be a locally compact metric space. Then each $y \in \beta X \setminus X$ is a non-normality point of $\beta X \setminus X$.*

Proof. We have seen that if $y \in \beta X \setminus X$ is uniform then it is a non-normality point of $\beta X \setminus X$. Suppose that $y \in \beta X \setminus X$ is not uniform. That is, there exists $Z \in y$ such that $w(Z) < w(X)$. Let $Z \in y$ be such that $\lambda = w(Z)$ is minimum. Then y is a uniform z-ultrafilter on the set Z and by UR, it is regular. However, it may be the case that Z has isolated points or is not locally compact. We aim to find a locally compact closed subset Y of X with weight λ without isolated points such that $Z \subset Y$. There is a cover of Z consisting of sets $\mathrm{cl}\, B$ from a subcollection, $\mathcal{Z}$, of $\mathcal{B}_0$ of size λ. Let $Y = \bigcup\{\mathrm{cl}\, B : B \in \mathcal{Z}\}$. Since $\mathcal{B}_0$ is locally finite, Y is closed. Each $B \in \mathcal{Z}$ has no isolated points and has compact closure, so Y has no isolated points and is locally compact.

So, $y \in \mathrm{cl}_{\beta X} Y$. Since X is normal and Y is closed, Y is C^*-embedded in X. Therefore, $\beta Y = \mathrm{cl}_{\beta X} Y$ and $y|_Y$ is uniform on Y. So, by the theorem, y is a non-normality point of the set $\mathrm{cl}_{\beta X} Y \setminus Y$ and hence is a non-normality point of $\beta X \setminus X$. $\square$

12. Questions

Gillman's question [9], which started research in this area, is still not completely answered.

Question 12.1. *Let X be $\mathbb{N}$. Let y be any point of $\beta X\backslash X$. Without extra axioms of set theory, is $(\beta X\backslash X)\backslash\{y\}$ not normal? If yes, what if X is any discrete space? If yes, what if X is any metrizable space?*

There are many ways that our work can be extended. For example,

Question 12.2. *Assume GCH. For every crowded metrizable space X and every $y \in X\backslash X$, is $(\beta X\backslash X)\backslash\{y\}$ not normal?*

Our next question is a bit vague. Can known results be combined nicely? For example, here is a flawed proof that GCH implies that for every metrizable space X, every $y \in \beta X\backslash X$ is a butterfly point. Let I be the set of isolated points of X. Then $X\backslash I$ is closed. If $X\backslash I \notin y$, then there is a subset J of I, closed in X with $J \in y$. Apply Beslagic-van Douwen to $y \in \beta J\backslash J$. If $X\backslash I \in y$, then apply Logunov or Terasawa to $y \in \beta(X\backslash I)\backslash(X\backslash I)$. The flaw is that points of $X\backslash I$ can become isolated by discarding I, and so Logunov or Terasawa does not apply to $X\backslash I$. We can continue up the Cantor-Bendixon derivatives, but what to do after a limit stage? Let X be the ordinal $w^\omega = \sup\{\omega^n : n \in \omega\}$ with the order topolgy. Can the argument of this paragraph be improved to show that every $y \in \beta X\backslash X$ is a butterfly point?

If the space X is partitioned into a compact set K and a theorem applies to $X\backslash K$, then the splitting argument works. For example, let X be the hedgehog with κ spines. If $y \in \beta X\backslash X$, then there is $Z \in y$ with the special point 0 not in Z. Then Z is a closed, locally compact subset of X, and we can apply our theorem to $y \in \beta Z\backslash Z$.

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