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Climbing the Branches of the Graceful Tree Conjecture

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DERIVATION OF THE (CLOSED-FORM) PARTICULAR SOLUTION OF THE POISSON'S EQUATION IN 3D USING OSCILLATORY RADIAL BASIS FUNCTION

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Abstract. Partial differential equations (PDEs) are useful for describing a wide variety of natural phenomena, but analytical solutions of these PDEs can often be difficult to obtain. As a result, many numerical approaches have been developed. Some of these numerical approaches are based on the particular solutions. Derivation of these particular solutions are challenging. This work is about how the Laplace operator can be written in a more convenient form when it is applied to radial basis functions and then use this form to derive the (closed-form) particular solution of the Poisson's equation in 3D with the oscillatory radial function in the forcing term.

1. Introduction

Radial basis functions (RBFs) have increasingly been used for solving partial differential equations (PDEs). To that point, the effectiveness and simplicity of the Kansa method [5] has led to the further development of RBF collocation methods such as the method of approximate particular solutions (MAPS) [3]. In the MAPS, it is assumed that the solution to the PDEs can be approximated with the span of the particular solutions, it is then critical that we derive, first, the particular solutions for some simple PDE with the RBFs in the forcing term. There are some other RBFs based methods which require particular solutions such as method of fundamental solutions coupled with the method of particular solutions (MFS-MPS), localized method of particular solutions (LMPS), and fast method of particular solutions (FMPS) etc [6]. It is not only in these meshless methods but also to successfully solve the nonhomogeneous equation, the dual reciprocity method (DRM) [8] which is a boundary element method, particular solutions are needed.

Traditionally, many of these RBFs have been chosen as non-oscillatory in nature (e.g. multiquadric and Gaussian), but it is not necessary to restrict our choice of basis functions to those that are non-oscillatory [7]. Thus, in this work, we illustrate how to derive particular solution of the Poisson's equation in 3D with the oscillatory radial function (O-RBFs) in the forcing term.

Consider the following class of the O-RBFs defined by

$$\phi_d(r) = \frac{J_{d/2-1}(cr)}{(cr)^{d/2-1}}, \quad c > 0,\ d = 1, 2, 3, ... \tag{1.1}$$

where c is the shape parameter and $J_\nu(r)$ signifies the J Bessel function of the first kind of order ν [4]. The (closed-form) particular solutions of the Poisson's equation in 2D by taking the family of the oscillatory RBFs as (1.1) in the forcing term has been derived in [7].

Without loss of generality, in this paper, we have only used one of the O-RBFs from the above family. In section 2, we derive the (closed-form) particular solutions of the Poisson's equation in 3D by taking $\phi_3(r)$ in the forcing term. Note that, by using the Rayleigh's formula [1], $\phi_3(r)$ can be expressed as

$$\phi_3(r) = \sqrt{\frac{2}{\pi}}\frac{\sin(cr)}{cr}. \tag{1.2}$$

There are several approaches to define what constitute a closed-form [2]. The author also said in [2],

" ... a closed-form is "that which looks 'fundamental' to the requisite consumer" ..."

Definitely, these derived particular solutions satisfy one of the approaches but as in [2], we left the reader to determine either our derived particular solutions are closed form or not so we decided to write closed-form in the parentheses.

2. Derivation of the (closed-form) particular solution

Before we derive the particular solution, let us obtain a more convenient form of the Laplace operator when it is applied to radial basis functions.

Lemma 2.1. For any radial basis function $\Phi(r)$,

$$\Delta\Phi(r) = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi(r), \tag{2.1}$$

where Δ is a Laplace operator and r is the radial distance in 3D.

Proof. Since Δ is a Laplace operator in 3D.

$$\Delta\Phi(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(r).$$

Suppose $r := r(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, where (x_0, y_0, z_0) is any fixed point in 3D. Then, we obtain

$$\frac{\partial^2}{\partial x^2}\Phi(r) = \frac{d^2\Phi(r)}{dr^2}\frac{(x-x_0)^2}{r^2} + \frac{d\Phi(r)}{dr}\frac{r^2-(x-x_0)^2}{r^3}.$$

Similarly, we can obtain the expressions for $\frac{\partial^2}{\partial y^2}\Phi(r)$, $\frac{\partial^2}{\partial z^2}\Phi(r)$. Combining these expressions together we obtain

$$\Delta\Phi(r) = \frac{d\Phi(r)}{dr}\frac{2}{r} + \frac{d^2\Phi(r)}{dr^2}.$$

Notice that,

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi(r) = \frac{d\Phi(r)}{dr}\frac{2}{r} + \frac{d^2\Phi(r)}{dr^2}. \tag{2.2}$$

Hence we proved (2.1). ∎

Now, we derive the particular solution of the Poisson's equation in 3D by taking (1.2) in the forcing term.

Theorem 2.1. Consider the Poisson's equation

$$\Delta\Phi(r) = \sqrt{\frac{2}{\pi}}\frac{\sin(cr)}{cr}, \tag{2.3}$$

where Δ is a Laplace operator in 3D and r is the radial distance in 3D. Then,

$$\Phi(r) = \begin{cases} -\sqrt{\frac{2}{\pi}}\frac{\sin(cr)}{c^3 r}, & r \neq 0, \\ -\sqrt{\frac{2}{\pi}}\frac{1}{c^2}, & r = 0. \end{cases} \tag{2.4}$$

Proof. By using Lemma 2.1, we can write (2.3) as

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi(r) = \sqrt{\frac{2}{\pi}}\frac{\sin(cr)}{cr}. \tag{2.5}$$

Then,

$$r^2\frac{d}{dr}\Phi(r) = \int \sqrt{\frac{2}{\pi}}\frac{r^2\sin(cr)}{cr}dr. \tag{2.6}$$

Now, using the direction integration we obtain

$$\Phi(r) = \sqrt{\frac{2}{\pi}}\left(-\frac{\sin(cr)}{c^3 r} - \frac{D_1}{r}\right) + D_2, \tag{2.7}$$

where D_1 and D_2 are the integration constants. This is the general solution of the Poisson's equation (2.3). We notice that the solution (2.7) has a singularity at $r = 0$ due to $\frac{\sin(cr)}{r}$, which is removable.

The particular solution $\Phi(r)$ that we need for the numerical methods must be defined for $r \geq 0$. For that, we choose the integration constants $D_1 = 0$ and $D_2 = 0$ and define

$$\begin{aligned} \Phi(0) &= \lim_{r\to 0}\Phi(r), \\ &= \lim_{r\to 0}\sqrt{\frac{2}{\pi}}\left(-\frac{\sin(cr)}{c^3 r}\right) \\ &= -\sqrt{\frac{2}{\pi}}\frac{1}{c^2}. \end{aligned}$$

Then, we obtain the required particular solution. ∎

3. CONCLUSION

In this paper, we have derived a (closed-form) particular solution to the Poisson's equation in 3D by taking one of the O-RBFs in the forcing term. In the future, we will be deriving the particular solutions by taking a more general class of the O-RBFs. We will numerically validate these particular solutions by using one of the numerical methods, such as MAPS, for solving elliptic PDEs.

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